Chapter 5

Computability and Executability

5.1 The Turing machine

Definition 5.1 (Turing machine)**.** An *(Interactive) Turing machine* M is defined as a sixtuple $(\mathcal{S}, \mathcal{A}, \mathcal{D}, \rightarrow, \uparrow, \downarrow)$ where:

1. $\mathcal{S}$ is a finite set of states,
2. $\mathcal{A}$ is a finite alphabet,
3. $\mathcal{D}$ is a finite set of data,
4. $\rightarrow \subseteq \mathcal{S} \times (\mathcal{D} \cup \{\varepsilon\}) \times (\mathcal{A} \cup \{\tau\}) \times (\mathcal{D} \cup \{\varepsilon\}) \times \{L, R\} \times \mathcal{S}$ is a finite set of *transitions* or *steps*,
5. $\uparrow \in \mathcal{S}$ is the initial state,
6. $\downarrow \subseteq \mathcal{S}$ is the set of final states.

If $(s, d, a, e, M, t) \in \rightarrow$, we write $s \xrightarrow{d,a,e,M} t$, and this means that the machine, when it is in state s and reading symbol d on the tape, will execute input action a, change the symbol on the tape to e, will move one step left if $M = L$ and right if $M = R$ and thereby move to state t. It is also possible that d and/or e is ε: if d is ε, we are looking at an empty part of the tape, but, if the tape is nonempty, then there is a symbol immediately to the right or to the left; if e is ε, then a symbol will be erased, but this can only happen at an end of the memory string. The exact definitions are given below.

At the start of a Turing machine execution, we will assume the Turing machine is in the initial state, and that the memory tape is empty (denoted by the *blank* symbol $\Box$).

By looking at all possible executions, we can define the transition system of a Turing machine. To define a configuration, we need to know the contents of the memory tape, a string $x \in \mathcal{D}^*$, but also the present location. This location can be an element of the memory string, but can also be immediately to the left of the memory string, or immediately to the right of the memory string.

We indicate the present location by means of a bar, so if for instance 001 is the contents of the tape, then we have configurations of a state together with $\bar{\Box}001\Box$, $\Box\bar{0}01\Box$, $\Box 0\bar{0}1\Box$,$\Box 00\bar{1}\Box$ or $\Box 001\bar{\Box}$.

Definition 5.2. Let $M = (\mathcal{S}, \mathcal{A}, \mathcal{D}, \rightarrow, \uparrow, \downarrow)$ be a Turing machine. The *transition system* of M is defined as follows:

The set of states is $\{(s, \bar{\Box}), (s, \bar{\Box}xdy\Box), (s, \Box x\bar{d}y\Box), (s, \Box xdy\bar{\Box}) \mid s \in \mathcal{S}, d \in \mathcal{D}, x, y \in \mathcal{D}^*\}$, the alphabet is the same.

The transition relation is defined in a number of cases:

1. A symbol can be replaced by another symbol if the present location is not a blank. Moving right, there are two cases: there is another symbol to the right or there is a blank to the right.

   Whenever $s \xrightarrow{d,a,e,R} t$, then $(s, \Box x\bar{d}\Box) \xrightarrow{a} (t, \Box xe\bar{\Box})$ and $(s, \Box x\bar{d}fy\Box) \xrightarrow{a} (t, \Box xe\bar{f}y\Box)$ for all $f \in \mathcal{D}, x, y \in \mathcal{D}^*$.

   Similarly, there are two cases for a move left.

   Whenever $s \xrightarrow{d,a,e,L} t$, then $(s, \Box\bar{d}x\Box) \xrightarrow{a} (t, \bar{\Box}ex\Box)$ and $(s, \Box xf\bar{d}y\Box) \xrightarrow{a} (t, \Box x\bar{f}ey\Box)$ for all $f \in \mathcal{D}, x, y \in \mathcal{D}^*$.

2. To erase a symbol, it must be at the end of the string. For a move right, there are three cases.

   Whenever $s \xrightarrow{d,a,\varepsilon,R} t$, then $(s, \Box\bar{d}\Box) \xrightarrow{a} (t, \bar{\Box})$ and $(s, \Box xf\bar{d}\Box) \xrightarrow{a} (t, \Box xf\bar{\Box})$ and $(s, \Box\bar{d}fx\Box) \xrightarrow{a} (t, \Box\bar{f}x\Box)$ for all $f \in \mathcal{D}, x \in \mathcal{D}^*$.

   Similarly for a move left.

   Whenever $s \xrightarrow{d,a,\varepsilon,L} t$, then $(s, \Box\bar{d}\Box) \xrightarrow{a} (t, \bar{\Box})$ and $(s, \Box\bar{d}fx\Box) \xrightarrow{a} (t, \bar{\Box}fx\Box)$ and $(s, \Box xf\bar{d}\Box) \xrightarrow{a} (t, \Box x\bar{f}\Box)$ for all $f \in \mathcal{D}, x \in \mathcal{D}^*$.

3. To insert a new symbol, we must be looking at a blank. There are three cases for a move right.

   Whenever $s \xrightarrow{\varepsilon,a,d,R} t$, then $(s, \bar{\Box}) \xrightarrow{a} (t, \Box d\bar{\Box})$ and $(s, \bar{\Box}fx\Box) \xrightarrow{a} (t, \Box d\bar{f}x\Box)$ and $(s, \Box xf\bar{\Box}) \xrightarrow{a} (t, \Box xfd\bar{\Box})$ for all $f \in \mathcal{D}, x \in \mathcal{D}^*$.

   Similarly for a move left.

   Whenever $s \xrightarrow{\varepsilon,a,d,L} t$, then $(s, \bar{\Box}) \xrightarrow{a} (t, \bar{\Box}d\Box)$ and $(s, \Box xf\bar{\Box}) \xrightarrow{a} (t, \Box x\bar{f}d\Box)$ and $(s, \bar{\Box}fx\Box) \xrightarrow{a} (t, \bar{\Box}dfx\Box)$ for all $f \in \mathcal{D}, x \in \mathcal{D}^*$.

4. Finally, looking at a blank, we can keep it a blank. There are three cases for a move right.

   Whenever $s \xrightarrow{\varepsilon,a,\varepsilon,R} t$, then $(s, \bar{\Box}) \xrightarrow{a} (t, \bar{\Box})$ and $(s, \bar{\Box}fx\Box) \xrightarrow{a} (t, \Box\bar{f}x\Box)$ and $(s, \Box xf\bar{\Box}) \xrightarrow{a} (t, \Box xf\bar{\Box})$ for all $f \in \mathcal{D}, x \in \mathcal{D}^*$.

   Similarly for a move left.

   Whenever $s \xrightarrow{\varepsilon,a,\varepsilon,L} t$, then $(s, \bar{\Box}) \xrightarrow{a} (t, \bar{\Box})$ and $(s, \Box xf\bar{\Box}) \xrightarrow{a} (t, \Box x\bar{f}\Box)$ and $(s, \bar{\Box}fx\Box) \xrightarrow{a} (t, \bar{\Box}fx\Box)$ for all $f \in \mathcal{D}, x \in \mathcal{D}^*$.

The initial state is $(\uparrow, \bar{\Box})$. Whenever $s\downarrow$, then $(s, \bar{\Box})$ and $(s, \bar{\Box}x\Box)$ and $(s, \Box x\bar{\Box})$ are final states for all $x \in \mathcal{D}^* - \{\varepsilon\}$.