MODEL CHECKING IN THE PROPOSITIONAL MU-CALCULUS

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Abstract: In this paper we address the problem of automatically verifying that a given finite state concurrent program meets a correctness assertion specified in the propositional Mu-Calculus $L\mu$ defined by Kozen in [Ko83]. The propositional Mu-Calculus provides a framework for handling fairness, past-tense temporal modalities, and extended temporal modalities such as those of [Wo81] in a uniform way. We give an efficient model checking algorithm for specifications given in a fragment, $L\mu_2$, of the propositional Mu-Calculus, where the alternation depth of least and greatest fixpoint operators is restricted to be at most 2. We show that $L\mu_2$ can succinctly encode (and is, in fact, strictly more expressive than) many of the commonly used logics of programs such as PDL ([FL79]), PDL-Δ ([St82]), CTL ([EC82], [CES83]), FCTL and GFCTL ([EL85]). In practice, we therefore have a small polynomial time model checker for a most useful portion of the propositional Mu-Calculus.

1. Introduction

Due to the rapid development of multiprocessor computer systems and communication networks, a tremendous amount of *concurrently programmed software* will be needed to take advantage of the fast, improved hardware. Because of the high degree of nondeterminism inherent in concurrency, software engineering techniques such as modular design and systematic debugging are not, by themselves, adequate for ensuring the development of error-free concurrent programs. *Program verification* thus plays an especially important role during the life cycle of concurrent program development. To improve the productivity of concurrent program development, *automatic verification tools* become more and more desirable. Recently Clarke, Emerson and Sistla ([CES83]) have suggested an automatic verification method for finite state concurrent systems: The global state graph of such a system can be viewed as a finite (Kripke) structure, and an efficient (small polynomial time) *model checking* algorithm is described for determining if a given structure is a model of a specification expressed in a propositional branching time temporal logic. Since the algorithm is very efficient, this method is readily mechanizable. Moreover, the approach is potentially of wide applicability since a large class of concurrent programming problems have finite state solutions, and their interesting properties can be specified in a propositional Temporal Logic (cf. [Va85], [LP85]).

Indeed, the role of Temporal Logic as a feasible approach to the specification and verification of concurrent systems is now widely accepted ([Pn84], [La83]). However, several recent developments indicate

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that enriching Temporal Logic can prove most advantageous. For example, it is known (cf. [La80], [EH83]) that the temporal logic CTL used in [CES83] does not permit reasoning under certain fairness assumptions. An extension of CTL, which we call Fair Computation Tree logic (FCTL), is thus introduced in [EL85]. Similarly, in [LPZ85] and [Pn84] it is argued that inclusion of past-tense temporal operators as well as the extended temporal operators of Wolper [Wo81] (e.g., "at all even moments, P holds") can facilitate the task of modular temporal reasoning about concurrent programs. Again, these sorts of temporal operators are not accommodated in the CTL logic of [CES83] (nor in the FCTL logic of [EL85].)

In this paper we investigate another way of extending model checking for CTL. We view CTL not as a sublanguage of the full branching time logic CTL* ([EH83]), but, rather, as a sublanguage of the *propositional Mu-Calculus* Lμ (cf. [Ko83], [Pr81], [EC80], [deB80]). The propositional Mu-Calculus provides a *least fixpoint* operator (μ) and a *greatest fixpoint* operator (ν) which make it possible to give *fixpoint characterizations* of the branching time modalities. Intuitively, the Mu-Calculus makes it possible to characterize the modalities in terms of recursively defined tree-like patterns. For example, the branching time temporal assertion EFp (along some computation path p will become true eventually) can be characterized as $\mu Z.p \vee EXZ$, the least fixpoint of the functional $p \vee EXZ$ where Z is an atomic proposition variable (intuitively ranging over sets of states) and EX denotes the existential nexttime operator.

The propositional Mu-Calculus provides a uniform framework for extending the CTL model checking approach of [CES83] to handle fairness constraints as in [EL85], plus past tense temporal operators similar to those in [LPZ85] (cf. [PW84]), and extended temporal operators like those in [Wo81]. For example, the temporal assertion $\overset{\infty}{EF}p$ ("along some computation path, p occurs infinitely often"), which is associated with fairness, can be characterized as $\nu Z_1.\mu Z_2.EX[(P \wedge Z_1) \vee Z_2]$. The extended temporal operator "along all paths, P holds of all even moments" (cf. [Wo81]) is captured by $\nu Z.P \wedge AXAXZ$. We can accommodate the (branching time) past tense through use of the converse relations.

We show how model checking can be done when correctness assertions are formulated in the propositional Mu-Calculus Lμ. For the full Mu-Calculus Lμ our algorithm has exponential worst case time complexity. However, we consider model checking for restricted fragments $L\mu_k$ of Lμ where the depth of alternating nestings of μ's and ν's is bounded by k-1. For example, $L\mu_1$ consists of formulae obtained by composing modalities such as $\mu Z.P \vee EXZ$ (equivalent to EFP) and $\nu Z.P \wedge AXZ$ (equivalent to AGP) which contain only μ's or only ν's. The fragment $L\mu_2$ contains modalities such as $\nu Z_1.\mu Z_2.EX[(P \wedge Z_1) \vee Z_2]$ (equivalent to $\overset{\infty}{EF}P$) and $\mu Z_1.\nu Z_2.AX[(P \vee Z_1) \wedge Z_2]$ (equivalent to $\overset{\infty}{AG}P$, the dual of $\overset{\infty}{EF}P$), where one alternation of μ's inside ν's (or vice-a-versa) is allowed. We will show that the model checking problem for $L\mu_k$ can be solved in polynomial time (with degree of the polynomial = k+1).

We go on to show that these restricted fragments of the propositional Mu-Calculus have considerable expressive power. We establish that the propositional Mu-Calculus, even restricted to alternation depth 2, is adequate for formalizing most of the reasoning about concurrent programs that can be done in a propositional Temporal Logic by proving that it subsumes many of the commonly used logics of programs such as PDL ([FL79]), PDL-Δ ([St82]), CTL ([EC82], [CES83]), and FCTL ([EL85]). In particular, we show that PDL and CTL can be succinctly translated into $L\mu_1$ while PDL-Δ and FCTL can be succinctly translated into $L\mu_2$.[2] We thus have a model checker of small polynomial time complexity for a specification language able to handle almost any correctness assertion arising in practice including all of the examples cited above.

The rest of the paper is organized as follows: In section 2 the syntax and semantics of the propositional Mu-Calculus, $L\mu$, are defined. Section 3 describes how to do efficient model checking in the propositional Mu-Calculus $L\mu_k$, and analyzes the complexity of our model checking algorithm. In section 4 we present our results on translating program logics into $L\mu_1$ and $L\mu_2$. Finally, some concluding remarks are given in section 5.

2. Preliminaries

2.1. Syntax.

The formulae of the propositional Mu-Calculus $L\mu$ are:

(1) Atomic propositional constants P, Q, R, . . .
(2) Atomic propositional variables X, Y, Z, . . .
(3) $<A>p$, where A is a member of a set of program letters A, B, C, . . . and p is any formula.
(4) $\neg p$, the negation of formula p.
(5) $p \wedge q$, the conjunction of formulae p, q.
(6) $\mu X.p(X)$, where p(X) is any formula syntactically monotone in the propositional variable X, i.e., all free occurrences of X in p(X) fall under an even number of negations.

The set of formulae generated by the above rules forms the language $L\mu$. The other connectives are introduced as abbreviations in the usual way: $p \vee q$ abbreviates $\neg(\neg p \wedge \neg q)$, $[A]p$ abbreviates $\neg<A>\neg p$, $\nu X.p(X)$ abbreviates $\neg\mu X.\neg p(\neg X)$, etc. Intuitively, $\mu X.p(X)$ ($\nu X.p(X)$) stands for the least (greatest, resp.) fixpoint of p(X), $<A>p$ ($[A]p$) means p is true at some (every) next state t reachable by executing atomic program A, $\wedge$ means "and", etc. We use $|p|$ to denote the *length* (i.e., number of symbols) of p.[3]

We say that a formula q is a *subformula* of a formula p provided that q, when viewed as a sequence of symbols, is a substring of p. A subformula q of p is said to be *proper* provided that q is not p itself. A *top-level* (or *immediate*) subformula is a maximal proper subformula. We use SF(p) to denote the set of

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[2] CTL* can also be translated into $L\mu_2$. Because the translation is via PDL-Δ, however, it is not succinct.

[3] Alternatively, we can define $|p|$ as the size of the syntax tree for p; see section 4.

subformulae of p.

The fixpoint operators μ and ν are somewhat analogous to the quantifiers $\exists$ and $\forall$. Each occurrence of a propositional variable X in a subformula μX.p(X) (or νX.p(X)) of a formula is said to be *bound*. All other occurrence are *free*. By renaming variables if necessary we can assume that the expression μX.p(X) (or νX.p(X)) occurs at most once for each X.

A *sentence* (or *closed* formula) is a formula that contains no free propositional variables, i.e., every variable is bound by either μ or ν. A formula is said to be in *positive normal form* (PNF) provided that no variable is quantified twice and all the negations are applied to atomic propositions only. Note that every formula can be put in PNF by driving the negations in as deep as possible using DeMorgan's Laws and the dualities $\neg\mu$Y.p(Y) = νY.$\neg$p($\neg$Y), $\neg\nu$Y.p(Y) = μY.$\neg$p($\neg$Y) (This can at most double the length of the formula). *Subsentences* and *proper subsentences* are defined in the same way as subformulae and proper subformulae.

Let σ denote either μ or ν. If Y is a bound variable of formula p, there is a unique μ or ν subformula σY.p(Y) of p in which Y is quantified. Denote this subformula by σY. Y is called a *μ-variable* if σY=μY; otherwise, Y is called a *ν-variable*. A *σ-subformula* (*σ-subsentence*, resp.) is a subformula (subsentence) whose main connective is either μ or ν. We say that q is a *top-level* σ-subformula of p provided q is a proper σ-subformula of p but not a proper σ-subformula of any other σ-subformula of p. Finally, a *basic modality* is a σ-sentence that has no proper σ-subsentences.

2.2. Semantics.

We are given a set AP whose elements are called *atomic* (or *propositional*) constants and a set Π_0 whose elements are (*atomic*) program letters. Sentences of the propositional Mu-Calculus Lμ are interpreted with respect to a structure M = (S, $\mathcal{R}$, L), where

S is a nonempty *set of states*,
$\mathcal{R}$: $\Pi_0 \rightarrow 2^{S\times S}$ assigns a binary relation on S to each atomic program,
L: S $\rightarrow 2^{AP}$ is a *labelling* which assigns to each state a set of atomic propositions true in the state

The *size* of a structure M=(S, $\mathcal{R}$, L), written |M|, is defined to be $|S| + \Sigma_{A\in\Pi_0}|\mathcal{R}(A)|$, i.e., the sum of the number of states in S and the number of transitions in $\mathcal{R}$.

The power set of S, 2^S, may be viewed as the complete lattice (2^S, S, $\emptyset$, $\subseteq$, $\cup$, $\cap$). Intuitively, we identify a predicate with the set of states which make it true. Thus, *false* which corresponds to the empty set is the bottom element, *true* which corresponds to S is the top element, and implication ($\forall s\in S[P(s) \Rightarrow Q(s)]$) which corresponds to simple set-theoretic containment ($P \subseteq Q$) provides the partial ordering on the lattice.

Let τ: $2^S \rightarrow 2^S$ be given; then we say that τ is *monotonic* provided that $P \subseteq Q$ implies $\tau(P) \subseteq \tau(Q)$, τ is $\cup$-*continuous* provided that $P_0 \subseteq P_1 \subseteq P_2 \subseteq \ldots$ implies $\tau(\cup_i P_i) = \cup_i \tau(P_i)$, and that τ is $\cap$-*continuous* provided that $P_0 \supseteq P_1 \supseteq P_2 \supseteq \ldots$ implies $\tau(\cap_i P_i) = \cap_i \tau(P_i)$. Either type of continuity can easily be shown to imply monotonicity. Furthermore, if S is finite, then any monotonic functional is also both $\cup$-continuous and $\cap$-continuous.

A monotonic functional τ always has both a least fixpoint, $\mu X.\tau(X)$, and a greatest fixpoint $\nu X.\tau(X)$:

Theorem 2.1 [Tarski-Knaster]. Let τ: $2^S \rightarrow 2^S$ be a given monotonic functional. Then

(a). $\mu X.\tau(X) = \cap \{X : \tau(X) = X\} = \cap \{X : \tau(X) \subseteq X\}$,
(b). $\nu X.\tau(X) = \cup \{X : \tau(X) = X\} = \cup \{X : \tau(X) \supseteq X\}$,
(c). If τ is $\cup$-continuous then $\mu X.\tau(X) = \cup_i \tau^i(\mathit{false})$, and
(d). If τ is $\cap$-continuous then $\nu X.\tau(X) = \cap_i \tau^i[\mathit{true}]$.

A formula p with free variables X_1, ..., X_n is thus interpreted as a mapping p^M from $(2^S)^n$ to 2^S, i.e., it is interpreted as a predicate transformer. We write $p(X_1, \ldots, X_n)$ to denote that all free variables of p are among X_1, ..., X_n. A valuation $\mathcal{V} = (V_1, \ldots, V_n)$ is an assignment of subsets of S (V_1, ..., V_n) to free variables (X_1, ..., X_n). We use $p^M(\mathcal{V})$ to denote the value of p on the (actual) arguments V_1, ..., V_n (cf. [EC80], [Pr81], [Ko83]). The operator p^M is defined inductively as follows:

(1). $P^M(\mathcal{V}) = \{s: s \in S \text{ and } P \in L(s)\}$ for any atomic propositional constant $P \in AP$
(2). $X_i^M(\mathcal{V}) = V_i$
(3). $(p \wedge q)^M(\mathcal{V}) = p^M(\mathcal{V}) \cap q^M(\mathcal{V})$
(4). $(\neg p)^M(\mathcal{V}) = S \setminus (p^M(\mathcal{V}))$
(5). $(<A>p)^M(\mathcal{V}) = \{s: \exists t \in p^M(\mathcal{V}), (s,t) \in R(A)\}$
(6). $\mu X.p(X)^M(\mathcal{V}) = \cap\{S' \subseteq S: p(S')^M(\mathcal{V}) \subseteq S'\}$

Note that our syntactic restrictions on monotonicity ensure that least and greatest fixpoints are well-defined.

Usually we write M, $s \models p$ (M, $s \models p(\mathcal{V})$, resp.) instead of $s \in p^M$ ($s \in p^M(\mathcal{V})$) to mean that sentence (formula) p is true in structure M at state s (under valuation $\mathcal{V}$). When M is understood, we write simply $s \models p$.

2.3. Alternation Depth.

We introduce the notion of alternation depth of formulae in Lμ which plays an important role in the complexity analysis of our model checking algorithm. Although the intuition for alternation depth is quite obvious, it turns out that the formal definition is rather subtle. In the definition below we assume that f is in PNF. (For any formula f not in PNF, $\mathcal{A}(f)$ is defined as $\mathcal{A}(f')$, where $f' \equiv f$ is in PNF.)

Definition 2.2: The *alternation depth* of a formula f, $\mathcal{A}(f)$, is defined recursively as follows:

AD1. If f contains proper σ-subsentences p_1, ..., p_n, then $\mathcal{A}(f) = \max(\mathcal{A}(p_1), ..., \mathcal{A}(p_n), \mathcal{A}(f'))$, where f' is obtained from f by substituting fresh atomic propositional constants $P_1,...,P_n$, for $p_1,...,p_n$, resp., in f.

AD2. $\mathcal{A}(P) = 0$, for any atomic proposition constant P

AD3. $\mathcal{A}(Y) = 0$, for any atomic proposition variable Y

AD4. $\mathcal{A}(p \wedge q) = \max(\mathcal{A}(p), \mathcal{A}(q))$

AD5. $\mathcal{A}(p \vee q) = \max(\mathcal{A}(p), \mathcal{A}(q))$

AD6. $\mathcal{A}(\neg p) = \mathcal{A}(p)$

AD7. $\mathcal{A}(<A>p) = \mathcal{A}(p)$, for any program letter A

AD8. $\mathcal{A}([A]p) = \mathcal{A}(p)$, for any program letter A

AD9. $\mathcal{A}(\mu X.p) = \max(1, \mathcal{A}(p), 1+\mathcal{A}(\nu Y_1.q_1), ..., 1+\mathcal{A}(\nu Y_n.q_n))$, where $\nu Y_1.q_1$, ..., $\nu Y_n.q_n$ are the top-level ν-subformulae of p.

AD10. $\mathcal{A}(\nu X.p) = \max(1, \mathcal{A}(p), 1+\mathcal{A}(\mu Y_1.q_1), ..., 1+\mathcal{A}(\mu Y_n.q_n))$, where $\mu Y_1.q_1$, ..., $\mu Y_n.q_n$ are the top-level μ-subformulae of p.

Examples:

1. $\mathcal{A}(\mu Y.P \vee <A>Y) = 1$, by AD9
2. $\mathcal{A}(\mu Y.((\mu Z.P \vee [A]Z) \vee <A>Y)) = 1$, by AD1 then AD9
3. $\mathcal{A}(\mu Y.((\nu Z.P \wedge [A]Z) \vee <A>Y)) = 1$, by AD1 then AD9 and AD10
4. $\mathcal{A}(\nu Z_1.\neg\nu Z_2.[A]((\neg P \vee \neg Z_1) \wedge Z_2)) = \mathcal{A}(\nu Z_1.\mu Z_2.<A>((P \wedge Z_1) \vee Z_2))$, put it in PNF
   $= 1 + \mathcal{A}(\mu Z_2.<A>((P \wedge Z_1) \vee Z_2))$, by AD10
   $= 2$, by AD9
5. $\mathcal{A}(\nu Z_1.\mu Z_2.<A>((\nu Y_1.\mu Y_2.<A>((P \wedge Y_1) \vee Y_2)) \wedge Z_1) \vee Z_2))$
   $= \max\{\mathcal{A}(\nu Z_1.\mu Z_2.((<A>Q \wedge Z_1) \vee Z_2)), \mathcal{A}(\nu Y_1.\mu Y_2.<A>((P \wedge Y_1) \vee Y_2))\}$, by AD1
   $= 2$, by AD9 and AD10
6. $\mathcal{A}(\mu X.\nu Y.(P \vee ((\mu Z.(X \vee <A>Z)) \wedge <B>Y))) = 3$, by AD9 then AD10 and then AD9 again

Remark: One possibly helpful way to understand our definition of alternation depth is to use the notion of *active* variables introduced in [Ko83]. Let p_0 be a basic modality of Lμ. i.e., a sentence with no proper σ-subsentences. Draw the syntax tree for p_0 in the usual way; then, for each node corresponding to an occurrence of a σ-variable X, draw a directed arc from the node for X to the node for σX. (See Figure 2-1) We say that variable X is *active* in subformula q in sentence p_0 provided that there is a directed path from (the root node for) q to (the node for) σX. Then, for example, the formula $\mu X.\nu Y.(P \vee ((\mu Z.(X \vee <A>Z)) \wedge <B>Y))$ from above is of alternation depth 3 because it contains a chain of 2 consecutive "active" alternations of σ-variables: In the chain μX ... νY ... μZ, the alternation μX ... νY is active because X is active in νY (indeed, it occurs free in νY) while the alternation νY ... μZ is active because Y is active in μZ (even though Y does not occur free in μZ).

Definition 2.3: We denote by $L\mu_k$ the fragment of $L\mu$ restricted to alternation depth k.

3. Model Checking Algorithms.

In this section we develop a model checking algorithm for $L\mu$. We also show that the complexity of our algorithm for $L\mu_k$ is a polynomial of degree k+1 in both of the input structure and the input specification.

The algorithm of Figure 3-1 for model checking in the Mu-Calculus is a straightforward implementation of the semantics and the Tarski-Knaster Theorem (Theorem 2.1) for evaluating fixpoints.

We can improve the performance of the algorithm by observing that subsentences need only be evaluated once. Moreover, we can exploit the simultaneous monotonicity induced by σ-variables of the same type by using the following variant of the Tarski-Knaster Theorem:

Theorem 3.1: If $p(Y)$: $2^S \rightarrow 2^S$ is $\cup$-continuous, then $\mu Y.p(Y) = \bigcup_{i>0} p^i(Y_0)$ for any initial set $Y_0 \subseteq \mu Y.p(Y)$, and if $p(Y)$ is $\cap$-continuous then $\nu Y.p(Y) = \bigcap_{i>0} p^i(Y_0)$ for any initial set $Y_0 \supseteq \nu Y.p(Y)$. □

For example, consider $p = \mu Y_1.p_1(Y_1)$ where $p_1(Y_1)$ contains the subformula $\mu Y_2.p_2(Y_1, Y_2)$. Based on Theorem 3.1, to evaluate $\mu Y_1.p_1(Y_1)$, we successively compute $p_1(\mathit{false})$, $p_1^2(\mathit{false})$, (i.e., $p_1(p_1(\mathit{false}))$), $p_1^3(\mathit{false})$, ... until stabilization. In order to compute $p_1(\mathit{false})$ we must first evaluate the subformula $\mu Y_2.p_2(\mathit{false}, Y_2)$. We can then again use Theorem 3.1 to calculate $p_2(\mathit{false}, \mathit{false})$, $p_2^2(\mathit{false}, \mathit{false})$, (i.e., $p_2(\mathit{false}, p_2(\mathit{false}, \mathit{false}))$), $p_2^3(\mathit{false}, \mathit{false})$, ... until stabilization. The calculation of $\mu Y_2.p_2(\mathit{false}, Y_2)$ can involve up to $|M|$ steps.

Similarly, to compute $p_1^i(\mathit{false})$, we compute subformula $\mu Y_2.p_2(p_1^{i-1}(\mathit{false}), Y_2)$ by means of a subcomputation $p_2(p_1^{i-1}(\mathit{false}), \mathit{false})$, $p_2^2(p_1^{i-1}(\mathit{false}), \mathit{false})$, $p_2^3(p_1^{i-1}(\mathit{false}), \mathit{false})$, ... of up to $|M|$ steps. Thus $O(|M|^2)$ evaluations of $p_2^j(p_1^i(\mathit{false}), \mathit{false})$ are performed. In general, if we have a σ-formula nested k deep, the body of the innermost σ-subformula will be evaluated $O(|M|^k)$ times. Since the depth k of nesting σ-subformula can be $\geq \Omega(|p_0|)$, we get an algorithm of superexponential complexity, $(|M|\cdot|p_0|)^{c|p_0|}$, for some constant $c > 0$.

However, whenever the nested σ-subformulae are all of the same type (all μ's or all ν's) we can do better. In the above example, we must compute $p_1(\mathit{false})$, $p_1^2(\mathit{false})$, $p_1^3(\mathit{false})$, ... and for each $p_1^i(\mathit{false})$ we need to compute $\mu Y_2.p_2(p_1^{i-1}(\mathit{false}), \mathit{false})$. But, it is not necessary to restart the μY_2 subcomputation with the Y_2 variable $= \mathit{false}$ each time. Since $p_1^{i-1}(\mathit{false}) \subseteq p_1^i(\mathit{false})$ and $p_2(Y_1, Y_2)$ is monotonic in Y_1, we have that $\mu Y_2.p_2(p_1^{i-1}(\mathit{false}), Y_2) \subseteq \mu Y_2.p_2(p_1^i(\mathit{false}), Y_2)$. Thus we can start the computation for $\mu Y_2.p_2(p_1^i(\mathit{false}), Y_2)$ with the Y_2 variable equal to the previously computed $\mu Y_2.p_2(p_1^{i-1}(\mathit{false}), Y_2)$. Only about $O(|M|)$ calculations of $p_2^j(p_1^{i-1}(\mathit{false}), \mathit{false})$ are needed.

In general, instead of reinitializing the μ-variable Y each time the subformula $\mu Y.p(Y)$ is evaluated, it

suffices to reinitialize Y just when the ν-variable Z of the closest surrounding ν-formula νZ.q changes[4]. We can use the following rule, inserted at the beginning of the evaluation of μY.p(Y).

if the surrounding σ-formula of μY is νZ
then $Y = \emptyset$;
for each open subformula μX of μY such that there is no
subformula νW of μY containing μX, set $X = \emptyset$;[5]

A symmetric rule is used for reinitializing ν-variables. The improved algorithm is shown in Figure 3-2.

Theorem 3.2: The algorithm in Figure 3-2 is of time complexity $O[(|p_0|\cdot|M|)^{\mathcal{A}(p_0)+1}]$ where $\mathcal{A}(p_0)$ is the alternation depth of p_0 (defined above).

proof: It suffices to consider the case where p_0 does not contain any proper σ-subsentence (i.e., is a basic modality), since each subsentence is evaluated only once.

If p_0 is of alternation depth 0, so it contains no σ-subformulae, then it is easy to see that the algorithm takes time $O(|M|\cdot|p_0|)$. Let us next consider p_0 of alternation depth 1 so that it contains only σ-subformula of the same type, say, μ-subformula (A symmetric argument applies when it consists only of ν-subformulae). Define the μ-depth in p_0 of subformula μY_i, $D(\mu Y_i)$, to be the number of μ-subformulae surrounding μY_i. Suppose we have μY_k is a proper subformula of μY_{k-1} ... is a proper subformula of μY_1 is a proper subformula of no μ-subformula of p_0. Write #_evals(p_i) for the total number of times eval(p_i) is called when evaluating p_0. Now,

1. #_evals(p_1) $\leq |S| + 1$ since each time, except the last, p_1 is evaluated the size of Y_1 increases, and
2. #_evals(p_{i+1}) $\leq |S|$ + #_evals(p_i) since there can be up to $|S|$ calls to eval(p_{i+1}) which increase Y_{i+1} in addition to 1 call to eval(p_{i+1}) for each call to eval(p_i).

Thus, for any subformula q of p_0, #_evals(q) $\leq |p_0|\cdot|M|$. Now, the total cost to evaluate p_0 can be calculated as:

total cost $\leq \sum_{q \in SF(p_0)}$ (local cost to evaluate q once)·(total number of times q is evaluated)

Here, the local cost to evaluate q once is the time required for a single evaluation of q exclusive of the time spent (in recursive calls) to evaluate the immediate subformulae. Plainly, when q is not a σ-subformula, the local cost of q is at most $c_1\cdot|M|$, for some constant c_1. For example if $q = p_1 \wedge p_2$, it is essentially the time required to intersect the sets of states associated with p_1 and p_2; the time required to evaluate p_1 and p_2 is

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[4]Actually, we only need to reset Y when a surrounding ν-variable Z which is active in p changes. (See the remark near the end of subsection 2.3.)

[5]For future reference, call such an open formula μX of μY an *unguarded* open subformula.

not charged to the local cost of q, but rather to p_1 and p_2 respectively.

For $q = \sigma Y_i.p_i$, the local cost can include the charges for resetting S_i and any S_j corresponding to open unguarded σ-subformulae, and for storing (the old value of) S_i in S_i' and comparing S_i' with (the new value of) S_i (obtained after evaluating the body p_i). This can yield a local cost of $O(|q|\cdot|M| + |M|^2)$. However, the total cost estimate will not decrease if we calculate it on the basis an appropriately modified local cost.

Observe that it is, for example, allowable to modify the local costs of formulae q and q' by subtracting an amount a·b from q and adding amount a to q', provided that q' is evaluated at least b times for each time q is evaluated. Therefore, for, say, a μY_i formula such that the surrounding σ-formula is a ν-formula (or has no surrounding σ-formula), we charge to its (modified) local cost the cost to reinitialize S_i to $\emptyset$, which is at most $c_2\cdot|M|$ for some constant c_2. However, the cost to reinitialize S_j for μY_j, an open unguarded subformula of μY_i, is charged to the (modified) local cost of μY_j. Since μY_j will be evaluated at least once for each μY_i evaluation, this is allowable. Also, for an evaluation of $\mu Y_i.p_i$ we store S_i in and compare S_i with S_i' some number b times, where b is the number of times the body p_i is evaluated during this evaluation of $\mu Y_i.p_i$. The cost of a single such store and subsequent comparison is some amount $a \leq c_3\cdot|M|$, for some c_3. The total contribution of all storing and comparing of S_i to the local cost of an evaluation of $\mu Y_i.p_i$ is thus a·b. By our observation above, we can therefore modify the local costs of $\mu Y_i.p_i$ and p_i by subtracting a·b from $\mu Y_i.p_i$ and adding a to p_i. We can similarly modify the local costs for ν-subformulae.

Now the modified local cost of each subformula, whatever its main connective, is $\leq c\cdot|M|$ for some constant c. Thus the total cost of evaluating $p_0 \leq \sum_{q\in SF(p_0)}$ (modified local cost to evaluate q once)·(total number of times q is evaluated) $\leq |p_0|\cdot$(maximum of the modified local cost of any $q \in SF(p_0)$)·(maximum of the total number of times any $q \in SF(p_0)$ is evaluated) $\leq |p_0|\cdot(c\cdot|M|)\cdot(|p_0|\cdot|M|) \leq c\cdot(|p_0|\cdot|M|)^2$.

We can now argue by induction on the alternation depth, that a formula of alternation depth k can be evaluated in time $c\cdot(|p_0|\cdot|M|)^{k+1}$, where c the the constant mentioned in the previous paragraph. The basis step for $k \leq 1$ is as above. For the induction step, let p_0 be a sentence of alternation depth k+1. We assume that all σ-subformula in p_0 of alternation depth k+1 are μ-subformulae (The case where they are ν-subformulae is symmetric). Then p_0 can be expressed as a composition of the form $p'[\nu Y_1, ..., \nu Y_l]$ where the νY_i's are the top-level ν-subformulae, each of alternation depth $\leq$k, and p' is the formula obtained from p_0 by replacing each νY_i with a "fresh" variable Z_i; p' is of alternation depth 1.

In a manner similar to that used for the basis case, we can calculate the total cost for evaluating p_0 as follows: total cost $\leq \sum_{q\in SF(p')}$ (total number of times q is evaluated)·(modified local cost of evaluating q

once), where the modified local cost of evaluating Z_i is the cost of evaluating νY_i --- which is $c\cdot(|\mu Y_i|\cdot|M|)^{k+1}$ by induction hypothesis plus the additional cost due to the "modification" discussed above which is $c\cdot|M|$. Since p' is of alternating depth 1, we have, as above, that the maximal number of times say subformula q of p' is evaluated is $\leq |p'|\cdot|M|$. Hence the total cost

$$\leq (|p'|\cdot|M|)\cdot(\sum_{q\in SF(p')\setminus\{Z_1,\ldots Z_l\}} \text{(modified local cost of evaluating q once)} + \Sigma_{i=1}^{l} c\cdot(|\nu Y_i|\cdot|M|)^{k+1}))$$

$$\leq (|p_0|\cdot|M|)\cdot(|p'|\cdot(c\cdot|M|) + c\cdot((|p_0|-|p'|)\cdot|M|)^{k+1})$$

$$\leq (|p_0|\cdot|M|)\cdot c\cdot(|p_0|\cdot|M|)^{k+1}$$

$$\leq c\cdot(|p_0|\cdot|M|)^{k+2}$$

□

4. Expressiveness

Since $\mathcal{A}(p_0) \geq \Omega(|p_0|)$ our improved algorithm still has exponential worst case complexity $(|M|\cdot|p_0|)^{c|p_0|}$, for some constant $c > 0$. However, almost all useful modalities can be expressed in the Mu-Calculus with alternation depth 1 or 2. Indeed, the expressive power of the Mu-Calculus of limited alternation depth is underscored by the results below which establish that PDL-Δ (Propositional Dynamic Logic with Looping, cf [St82]) and FCTL[6] (Fair Computation Tree Logic, cf [EL85]) can be succinctly encoded into $L\mu_2$ while (ordinary) PDL (cf. [FL79]) and CTL (cf. [CE81]) can be succinctly encoded into $L\mu_1$. In practice, we therefore have a model checker for the useful portion of Mu-Calculus of small polynomial time complexity.

4.1. Mu-Calculus and Propositional Dynamic Logic

We denote by p^t the Mu-Calculus translation of PDL-Δ formula p, where the translation t is defined by the following rules (cf. [Pr79], [Ko83])

T1. $P^t = P$ for atomic propositional constants P
T2. $Y^t = Y$ for atomic propositional variables Y
T3. $(\neg p)^t = \neg p^t$
T4. $(<A>p)^t = <A>p^t$
T5. $(<a \cup b>p)^t = (<a>p)^t \vee (<b>p)^t$
T6. $(<p?>q)^t = p^t \wedge q^t$
T7. $(<a;b>p)^t = (<a><b>p)^t$
T8. $(<a^*>p)^t = \mu Y.(p^t \vee (<a>Y)^t)$, for "fresh" variable Y
T9. $(\Delta a)^t = \nu Y.(<a>Y)^t$, for "fresh" variable Y

---

[6] A brief description of PDL-Δ and FCTL is given in Appendix.

Since p is duplicated in translating $<a \cup b>p$, $|p^t|$ the length of p^t considered as a string of symbols in the usual way, can be exponential in $|p|$. However, if we have an intelligent representation scheme which consolidates common subformulae (so that, e.g., the parse "graph" of $<a>p \vee <b>p$ would have the form of Figure 4-1), then $\|p^t\|$, the size of the representation of p^t (e.g., the size of its parse graph) is linear in $|p|$. In the sequel, we will assume that the "size" of a formula is measured by the size its parse graph instead of number of symbols. We thus have

Proposition 4.1: For every PDL-Δ formula $f(X_1, ..., X_n)$ having free variables $X_1, ..., X_n$, $f^t(X_1, ..., X_n)$ is an Lμ formula also having free variables $X_1, ..., X_n$, which is equivalent to f; i.e., $\forall$ structure $M = (S, \mathcal{R}, L)$, $\forall$ state s of M, and for every valuation $\mathcal{V} = (V_1, ..., V_n)$, $M, s \models f(\mathcal{V})$ iff $M, s \models f^t(\mathcal{V})$. Moreover, $\|f^t\| = O(|f|)$.

Proof: The proof is given in the Appendix. □

Theorem 4.2: For any PDL sentence p, p^t is an equivalent Mu-Calculus sentence of alternation depth ≤ 1. Thus, PDL is succinctly translatable into $L\mu_1$ in linear time.

Proof: Correctness and succinctness of the translation follow from Proposition 4.1, and the observation that translation rules T1-9 can plainly be implemented in linear time.

To see that p^t is of alternation depth ≤ 1, note that the translation rules T1-9 above define a mapping from PDL sentences to Mu-Calculus sentences. Since we only introduce least fixpoint operators (μ) during the translation and negations are only applied to subsentences, it is clear that p^t, the Mu-Calculus counterpart of the the PDL sentence p, is of alternation depth at most 1 according to definition 2.2. □

Since the $L\mu_1$ formula $\mu X.[A]<A>X$ is not expressible in PDL (cf. [Ko83]), we have the following:

Corollary 4.3: $L\mu_1$ is strictly more expressive than PDL. □

Let p be a PDL-Δ formula and q be a subsentence of p. We observe that the translation t has the following property: $[p(q/Q)]^t(Q^t/q^t) = [p(q/Q)]^t(Q/q^t) = p^t$, where Q is a fresh atomic variable not occurring in p and p(x/y) denotes the substitution of y for x in p.

Theorem 4.4: For any PDL-Δ sentence p, p^t is a Mu-Calculus sentence of alternation depth ≤ 2. Thus, PDL-Δ is succinctly translatable into $L\mu_2$ in linear time.

Proof: The correctness and succinctness of the translation follow immediately from Proposition 4.1 and it is clear that the translation can be implemented in linear time complexity. To see that the alternation depth is ≤ 2, we argue as follows: Let p be an arbitrary PDL-Δ sentence, we will prove by induction on $\#\Delta(p)$, the number of Δ operators in p, that $\mathcal{A}(p^t) \leq 2$.

Basis: If $\#\Delta(p) = 0$ then $\mathcal{A}(p^t) \leq 1$ by Theorem 4.2. If $\#\Delta(p) = 1$, let Δa be the unique Δ-sentence of p, then $(\Delta a)^t = \nu Y.(<a>Y)^t$. Let $q_1, ..., q_n$ be all the top-level subsentences of $<a>Y$. Let $<a'>Y =$

$<a>Y[q_1/Q_1, ..., q_n/Q_n]$. By the above observation, we have $(<a>Y)^t = (<a'>Y)^t(Q_1/q_1^t, ..., Q_n/q_n^t)$. By definition 2.2, $\mathcal{A}((<a>Y)^t) = \max(\mathcal{A}((<a'>Y)^t), \mathcal{A}(q_1^t), ..., \mathcal{A}(q_n^t))$. Since, q_i is a sentence of PDL $\mathcal{A}(q_i^t) \leq 1$ by Theorem 4.2. Note that the only negations in $<a>Y$ are applied to subsentences (because Δa is a sentence). Since any negation of a subsentence is a subsentence and we replace all top-level proper subsentences of a by atomic propositional constants to get a', a' is negation free. Therefore, $(<a'>Y)^t$ contains only μ operators (if it contains any σ-operators) and is in PNF. Applying definition 2.2, $\mathcal{A}((<a'>Y)^t) \leq 1$. Hence, $\mathcal{A}((<a>Y)^t) = \max(\mathcal{A}((<a'>Y)^t), \mathcal{A}(q_1^t), ..., \mathcal{A}(q_n^t)) \leq \max(1, 1, ..., 1) = 1$. By definition 2.2, $\mathcal{A}((\Delta a)^t) \leq 1 + \mathcal{A}((<a>Y)^t) \leq 2$. Now, we can evaluate $\mathcal{A}(p^t)$. $\mathcal{A}(p^t) \leq \max(2, \mathcal{A}((p')^t)$, where p' is obtained from p by replacing Δa with a fresh atomic propositional constant. Since p' is an ordinary PDL formula, $\mathcal{A}((p')^t) \leq 1$ by Theorem 4.2. Therefore $\mathcal{A}(p^t) \leq 2$.

Induction step: Assume $\#\Delta(p) = k + 1 \geq 2$. Let Δb be a Δ-sentence of p with $\#\Delta(\Delta b) \leq k$. By induction hypothesis, $\mathcal{A}((\Delta b)^t) \leq 2$. By rule AD1 of Definition 2.2, $\mathcal{A}(p^t) \leq \max[2, \mathcal{A}(p'^t)]$, where p' is the formula p with Δb replaced by a "fresh" atomic proposition. Again by induction hypothesis, $\mathcal{A}(p'^t) \leq 2$. Therefore, $\mathcal{A}(p^t) \leq 2$. □

In [Ni84] it has been shown that $\nu X.(<A>X \wedge <B>X)$ is not expressible in PDL-Δ, we thus have the following corollary:

Corollary 4.5: $L\mu_2$ is more expressive than PDL-Δ. □

Theorem 4.6: There exists an algorithm for model checking in PDL-Δ which runs in time $O[(|M|\cdot|p_0|)^3]$ for input structure M and sentence p_0.

Proof: If we modify the model checking algorithm given in the previous section so that it does not evaluate common subformulae twice, it is then straightforward to see that the modified model checking algorithm runs in time $O[(||p_0||\cdot|M|)^{\mathcal{A}(p_0)+1}]$ instead of $O[(|p_0|\cdot|M|)^{\mathcal{A}(p_0)+1}]$. The theorem thus follows from Theorem 4.4 and the modified algorithm. □

4.2. Mu-Calculus and FCTL.

To compare the Mu-Calculus with endogenous Temporal Logics such as CTL and FCTL we only have to consider $L\mu$ over one program letter. In the sequel, we therefore assume that $\Pi_0 = \{A\}$.

Theorem 4.7. There is a succinct, efficient (i.e., linear size and computable in linear time) translation of CTL into $L\mu_1$ over 1 program letter. Moreover, $L\mu_1$ over 1 program letter is strictly more expressive than CTL.

Proof sketch: The existence of the desired translation follows from the following fixpoint characterizations of the basic CTL modalities: $EXP = <A>P$, $EFP = \mu Y.P \vee <A>Y$, $EGP = \nu Y.P \wedge <A>Y$, and $E[P \mathcal{U} Q] = \mu Y.[Q \vee (P \wedge <A>Y)]$. To see that CTL is strictly less expressive than $L\mu_1$

over 1 program letter, observe that the formula $\nu Y.[P \wedge <A><A>Y]$ which asserts that p occurs every other time is not expressible in CTL (cf. [Wo81]). □

It can be shown that CTL cannot express correctness properties such as $E\overset{\infty}{F}P$ (along some path P occurs infinitely often) which are important in dealing with fairness (cf. [EH83].) Thus, in [EL85] we extended CTL to Fair CTL (FCTL). An FCTL specification (p_0, Φ_0) consists of a functional assertion p_0, which is a state formula, and an underlying fairness assumption Φ_0, which is a pure path formula. The functional assertion p_0 is expressed in essentially CTL syntax with basic modalities of the form either A_Φ ("for all *fair* paths") or E_Φ ("for some *fair* path") followed by one of the linear time operators F, G, X, or U. We subscript the path quantifiers with the symbol Φ to emphasize that they range over paths meeting the fairness constraint Φ_0, and to syntactically distinguish FCTL from CTL. A fairness constraint Φ_0 is a boolean combination of the infinitary linear time operators $\overset{\infty}{F}p$ ("infinitely often p") and $\overset{\infty}{G}p$ ("almost always p"), applied to propositional arguments (cf. [EC80], [Ha86]). We can then view a subformula such as $A_\Phi FP$ of functional assertion p_0 as an abbreviation for the CTL* formula $A[\Phi_0 \Rightarrow Fp]$. Similarly, $E_\Phi GP$ abbreviates $E[\Phi_0 \wedge GP]$.

One key result of [EL85] was that almost all practical types of fairness notions can be succinctly expressed using the canonical form $\Phi = \bigwedge_{i=1}^{n}(\overset{\infty}{F}p_i \vee \overset{\infty}{G}q_i)$. We will show, in this section, how the fairness canonical form can be succinctly expressed in $L\mu_2$. In the following discussion we take the liberty of slightly mixing CTL and $L\mu$ operators[7]. Also, we let $\Phi' = \bigwedge_{i=1}^{n}(\overset{\infty}{F}p_i \vee Gq_i)$, and $\tau(Y) = \bigwedge_{i=1}^{n}[EXE(Y\ \mathcal{U}\ (p_i \wedge Y)) \vee (q_i \wedge EXY)]$. Then we have the following succinct characterization of $E\Phi$ in $L\mu_2$:

Proposition 4.8: $E\Phi = EF\nu Y.\tau(Y)$.

Proof: We first establish the following lemmas:

Lemma 4.9: $\nu Y.\tau(Y) \subseteq E\Phi$.

Proof: We will argue that if $Y = \tau(Y)$ then $Y \subseteq E\Phi$, from which it will follow that $\nu Y.\tau(Y) \subseteq E\Phi$. Suppose $s_0 \in Y$. Then for each $j \in [1{:}n]$, $s_0 \models EXE(Y\ \mathcal{U}\ (p_j \wedge Y)) \vee (q_j \wedge EXY)$. Consider a fixed $j_0 \in [1{:}n]$. We have two cases:

Case (a): $s_0 \models EXE(Y\ \mathcal{U}\ (p_{j_0} \wedge Y))$, in which case there exists a finite sequence $(s_0{=}t_0, t_1, \ldots, t_k{=}s_1)$ of states, each satisfying Y, which starts at s_0 and ends at state s_1 such that $s_1 \models p_{j_0} \wedge Y$. Denote this sequence by $s_0 \underline{\underline{\ }}^{j_0,a}\!\!\rightarrow s_1$.

Case (b): $s_0 \models (q_{j_0} \wedge EXY) \wedge \neg EXE(Y\ \mathcal{U}\ (p_{j_0} \wedge Y))$, in which case s_0 has a successor s_1 such that $s_1 \models Y$. Denote this sequence (s_0, s_1) by $s_0 \underline{\underline{\ }}^{j_0,b}\!\!\rightarrow s_1$.

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[7] By Theorem 4.7 we can view CTL formulae as abbreviations for $L\mu_1$ formulae if we wish

In either case, there is a finite sequence from s_0 to s_1, which we can denote by $s_0 \rightarrow s_1$, where for simplicity we suppress which case applies.

We will construct an infinite path x of the form $s_0 \rightarrow s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow \ldots$ by splicing together segments of the form $s_i \rightarrow s_{i+1}$ in such a way as to ensure that $x \models \Phi$. The ground rules are (1) at time i, try to handle p_k, q_k where $k = 1 + i \ mod \ n$, and (2) always favor Case (a) to Case (b), if possible. So initially the path is just s_0. Inductively, the path is of the form $s_0 \rightarrow s_1 \rightarrow \ldots \rightarrow s_i$. Let $k = 1 + i \ mod \ n$. If $s_i \models EXE[Y \ \mathcal{U} \ (p_k \wedge Y)]$ (case (a)) then take $s_i \rightarrow s_{i+1}$ to be $s_i \xrightarrow{k,a} s_{i+1}$; otherwise (case (b)) take $s_i \rightarrow s_{i+1}$ to be $s_i \xrightarrow{k,b} s_{i+1}$. Let x be the infinite path $s_0 \rightarrow s_1 \rightarrow s_2 \rightarrow \ldots$ obtained by repeating this process for each $i \in \mathbf{N}$. Observe that, since $s_0 \models Y$ and Y holds of each state of each $s_i \rightarrow s_{i+1}$, (1) Y holds of every state along x.

We claim that if $s_i \models \neg EXE[Y \ \mathcal{U} \ (p_k \wedge Y)]$ then q_k holds of every state along the infinite suffix $x' = s_i \rightarrow s_{i+1} \rightarrow s_{i+2} \ldots$ of x. To see this, assume $s_i \models \neg EXE[Y \ \mathcal{U} \ (p_k \wedge Y)]$ and suppose the claim is false. Let t be the first state along x' after s_i such that q_k is false at t. Since Y holds at every state along x, $t \models Y$. Since $Y = \tau(Y)$, $t \models EXE[Y \ \mathcal{U} \ (p_k \wedge Y)]$. However, every state between s_i and t satisfies Y. This implies that $s_i \models EXE[Y \ \mathcal{U} \ (p_k \wedge Y)]$ contradicting our assumption that $s_i \models \neg EXE[Y \ \mathcal{U} \ (p_k \wedge Y)]$. Thus, each state along x' satisfies q_k, thereby establishing the Lemma.

It is now easy to verify that the path x thus constructed satisfies Φ: For each $j \in [1:n]$, either Case (a) applied at every time i for which $1 + i \ mod \ n = j$ -- so that $x \models \overset{\infty}{F} p_k$ -- or eventually there is a time i, such that $1 + i \ mod \ n = j$ and Case (b) applies -- so that, by the claim, $x \models \overset{\infty}{G} q_k$. Hence, $Y \subseteq E\Phi$ for any fixpoint Y of τ. This completes our proof. □

Lemma 4.10: $E\Phi' \subseteq \nu Y.\tau(Y)$.

Proof: It suffices to show that $E\Phi' \subseteq \tau(E\Phi')$ since $\nu Y.\tau(Y) = \cup\{Z: Z \subseteq \tau(Z)\}$. Suppose $s_0 \models E\Phi'$. Then there exists a path $x = (s_0, s_1, s_2, \ldots)$ such that $x \models \Phi'$. Moreover, for each state s_j on x, $s_j \models E\Phi'$. We will show that $s_0 \models \tau(E\Phi')$.

For each $i \in [1:n]$, if $x \models Gq_i$ then $s_0 \models q_i \wedge EXE\Phi'$. Otherwise, $x \models \overset{\infty}{F} p_i$. In this case, $s_0 \models EXE[E\Phi' \ \mathcal{U} \ (p_i \wedge E\Phi')]$. Thus, $s_0 \models EXE[E\Phi' \ \mathcal{U} \ (p_i \wedge E\Phi')] \vee (q_i \wedge EXE\Phi')$ for each $i \in [1:n]$. I.e., $s_0 \models \bigwedge_{i=1}^{n} [EXE[E\Phi' \ \mathcal{U} \ (p_i \wedge E\Phi')] \vee (q_i \wedge EXE\Phi')] = \tau(E\Phi')$. Therefore, $E\Phi' \subseteq \nu Y.\tau(Y)$. □

Lemma 4.11: $EFE\Phi = E\Phi$.

Proof: Obvious. Left to the reader. □

Lemma 4.12: $EFE\Phi' = E\Phi$.

Proof: $\subseteq$: Since $E\Phi' \subseteq E\Phi$, $EFE\Phi' \subseteq EFE\Phi$ by monotonicity of EF operator. Hence, $EFE\Phi' \subseteq E\Phi$ by Lemma 4.11.

$\supseteq$: If $s_0 \models E\Phi$, then there exists a fullpath $x = (s_0, s_1, s_2, ...)$ such that $x \models \Phi$. It is quite straightforward to verify that there exists a state s_j on x such that $s_j \models E\Phi'$. Thus $s_0 \models EFE\Phi'$. Therefore, $EFE\Phi' \supseteq E\Phi$. □

To complete the proof of Proposition 4.8, we use the following "squeezing" argument to show that $EF\nu Y.\pi(Y) = E\Phi$. Since $E\Phi' \subseteq \nu Y.\pi(Y)$ (Lemma 4.10) $\subseteq E\Phi$ (Lemma 4.9), we have $E\Phi = EFE\Phi'$ (Lemma 4.12) $\subseteq EF\nu Y.\pi(Y)$ (by monotonicity of EF) $\subseteq EFE\Phi = E\Phi$ (Lemma 4.11).

Thus $E\Phi \subseteq EF\nu Y.\pi(Y) \subseteq E\Phi$, and the theorem is proved. □

Remark: We feel that the existence of a *succinct* fixpoint characterization in $L\mu_2$ of $E\Phi$ is a rather nonobvious point. Now, in the case where Φ contains only a single conjunct, then $E\Phi$ pretty obviously has the fixpoint characterization $\nu Y.EGEF((q_1 \wedge Y) \vee EG(p_1 \wedge Y))$. But this fixpoint characterization does not appear to generalize to $n > 1$ conjuncts. An alternative approach would be to multiply out the n conjuncts, but then the resulting formula is of length about 2^n and the fixpoint characterization will not be succinct. We also point out that our succinct fixpoint characterization is correct for arbitrary sets of states p_i, q_i, and not just in the (easier) case when, e.g., $p_i = executed_i$, and $q_i = \neg enabled_i$ (cf. [EC80], [deR81].)

Theorem 4.13: There is a succinct, efficient (i.e., linear size and computable in linear time) translation of FCTL into $L\mu_2$ over 1 program letter. Moreover, $L\mu_2$ over 1 program letter is strictly more expressive than FCTL.

Proof Sketch: Since all temporal operators in FCTL can be defined in terms of the basic modalities $E_\Phi Xq$, $E_\Phi Gq$ and $E_\Phi[p\ \mathcal{U}\ q]$, it suffices to show how to encode these formulae in $L\mu_2$. For $E_\Phi Xq$ and $E_\Phi[p\ \mathcal{U}\ q]$, it is quite straightforward to get the fixpoint characterizations because $E_\Phi Xq \equiv EX(q \wedge E\Phi)$, $E_\Phi[p\ \mathcal{U}\ q] \equiv E[p\ \mathcal{U}\ (q \wedge E\Phi)]$, and we have the fixpoint characterization of $E\Phi$ in $L\mu_2$ by Proposition 4.8.

It remains to show how $E_\Phi Gq$ is expressed in $L\mu_2$. It is not hard to verify that $E_\Phi Gq \equiv E[q\ U\ E(Gq \wedge \Phi')]$. Using the same squeezing technique used to established Proposition 4.8 above, we can show that the corresponding $L\mu_2$ formula for $E[q\ U\ E(Gq \wedge \Phi')]$ is $E[q\ \mathcal{U}\ \nu Y.\{\bigwedge_{i=1}^{n}[EXE(Y\ U\ (p_i \wedge Y)) \vee (q_i \wedge EXY)] \wedge q\}]$.

Finally, it can be shown that the $L\mu_2$ (in fact, $L\mu_1$) formula $\nu Y.p \wedge AXAXY$ is not expressible in CTL*, and hence not in FCTL. □

Note that since the canonical form FCTL is as expressive as the full FCTL (but less succinct), we conclude that $L\mu_2$ is more expressive than FCTL.[8]

[8]The reader who is familiar with GFCTL will note that Theorem 4.13 also holds for GFCTL as well as FCTL (cf. [EL85]).

5. Conclusion

In this paper we have considered the problem of model checking in the propositional Mu-calculus $L\mu$. It provides a least fixpoint operator (μ) and greatest fixpoint operator (ν) which make it possible to give fixpoint characterizations of temporal modalities. For the full logic $L\mu$ our model checking algorithm has exponential time worst case complexity. However, we show that for each k, for the restricted fragment $L\mu_k$ where the depth of alternating nestings of μ's and ν's is at most k, our algorithm runs in polynomial time with the degree of the polynomial = k+1. It turns out that these restricted fragments of the Mu-Calculus have considerable expressive power. Indeed, we show that PDL and CTL can be succinctly translated into (and are strictly less expressive than) $L\mu_1$ while PDL-Δ and FCTL can be succinctly translated into (and are strictly less expressive than) $L\mu_2$. (CTL* can be translated into $L\mu_2$ also, but our translation is not succinct.) In practice, we therefore have an efficient model checking algorithm for a language $L\mu_2$ which subsumes a number of commonly used program logics. An additional advantage is that $L\mu_2$ provides a uniform framework for handling fairness constraints as in [EL85], extended temporal operators similar to those in [Wo83], and (branching) past-tense modalities (through the use of converse relations.)

Appendix.

A.1: PDL-Δ and FCTL.

For the convenience of the reader, we briefly describe the syntax and semantics of PDL-Δ and FCTL in this appendix.

For PDL-Δ, we are given a set Ψ_0 whose elements are called atomic propositions and a set Π_0 whose elements are (atomic) program letters. The set of programs, Π, and the set of formulae, Ψ, of PDL-Δ are then defined inductively as follows:

(P1) $\Pi_0 \subseteq \Pi$
(P2) if a, b $\in \Pi$ then a;b, a$\cup$b, a* $\in \Pi$
(P3) if p $\in \Psi$ then p? $\in \Pi$
(F1) $\Psi_0 \subseteq \Psi$
(F2) if p $\in \Psi$ then $\neg$p $\in \Psi$
(F3) if a el Π and p $\in \Psi$ then $<$a$>$p, Δa $\in \Psi$

Sentences of PDL-Δ are interpreted with respect to a structure M = (S, $\mathcal{R}$, L) as defined in section 2.3. Intuitively, Π is a set of regular languages over Π_0 and tests. If p is a formula, the test p? is a program equivalent to 'if p then skip else abort'. Thus $<$p?$>$q is true at a state s iff p $\wedge$ q is true at s. If a $\in \Pi$, the formula $<$a$>$p is true at a state s when there is a a path labelled by some word matching the regular expression a leading from s to a state satisfying p, and the formula Δa is true at a state s when there is an infinite chain of edges labelled with an infinite word from the ω-regular language a^ω.

Basic modalities of FCTL are formed by fair path quantifiers A_Φ (along every fair computation path satisfying the underlying fairness constraint Φ_0) and E_Φ (along some fair path) followed by one of the usual linear time temporal operators X (nexttime), F (sometime), G (always), or $\mathcal{U}$ (until). Formally, formulae of FCTL are formed as follows:

1. Any atomic proposition P is a formula.
2. If p, q are formulae then so are $\neg$p, and (p $\wedge$ q).
3. If p, q are formulae then so are A_ΦXp, E_ΦXp, A_Φ[p U q], and E_Φ[p U q].

A propositional formula is one formed by rules 1, 2 above. A fairness constraint is then formed by the following rules:

4. If p, q are propositional formulae then $\overset{\infty}{F}$p (infinitely often p), $\overset{\infty}{G}$p (almost always p) are *fairness constraints.*
5. If p, q are fairness constraints then so are $\neg$p, and (p $\wedge$ q).

The other connectives can then be defined as abbreviations in the usual way: A_ΦFq abbreviates A_Φ[*true* U q], E_ΦFq abbreviates E_Φ[*true* U q], A_ΦGq abbreviates $\neg E_\Phi F \neg$q, E_ΦGq abbreviates $\neg A_\Phi F \neg$q. Again, formulae of FCTL is interpreted in a structure M = (S, $\mathcal{R}$, L) as defined in section 2.3 except that $\mathcal{R}$ is simply a binary relation over S. For example, the formula E_ΦFp is true at a state s when there is a path starting at s which meets the underlying fairness constraint Φ_0 and p is true at some state of the path. The formulae A_Φ[p $\mathcal{U}$ q] is true at a state s iff along every fair path starting at s, p holds until q becomes true. The reader is referred to [EL85] for more details.

We define the language LX to be the set of formulae formed by the rules of Lμ, FCTL, and PDL-Δ. Since FCTL and PDL-Δ are just sublanguages of Lμ, LX is not more expressive than Lμ. However, it is clear that by allowing mixing FCTL, PDL-Δ, and Lμ modalities the task of specification will be much more easier. Moreover, it is straightforward to modify our model checking algorithm to accept formulae in LX.

A.2: Proof of Theorem 4.1.

The proof is by induction on the lexicographic order of (#Δ(f), |f|, ld(f)), where #Δ(f) is the number of Δ-operators in f, and ld(f) is the length of the leading diamond of f (i.e., ld(f) = |a| if f = <a>p, otherwise ld(f) = 0).

In the following, we will use p instead of p($\mathcal{V}$) if the valuation $\mathcal{V}$ is obvious.

1. If f=P (an atomic propositional constant), clearly M, s $\models$ P iff M, s $\models$ P^t.
2. If f=X_i (an atomic propositional variable), M, s $\models$ $X_i(\mathcal{V})$ iff M, s $\models$ $X_i^t(\mathcal{V})$.
3. f=$\neg$p: By induction hypothesis, M, s $\models$ p iff M, s $\models$ p^t. Therefore, M, s $\models \neg$p iff M, s $\models \neg p^t$ iff M, s $\models (\neg p)^t$.

4. f=$<A>p$: By induction hypothesis, M, s $\models$ $<A>p$ iff $\exists(s, t)\in R(A)[M, t \models p]$ iff $\exists(s, t)\in R(A)[M, t \models p^t]$ iff M, s $\models$ $<A>p^t$ iff M, s $(<A>p^t)$.
5. f=$<a \cup b>p$: By induction hypothesis, M, s $\models$ $<a>p$ iff M, s $\models (<a>p)^t$, and M, s $\models$ $<b>p$ iff M, s $\models (<b>b)^t$. Therefore, M, s $\models$ $<a \cup b>p$ iff M, s $\models$ $<a>p$ or M, s $\models$ $<b>p$ iff M, s $\models$ $(<a>p)^t$ or M, s $\models (<b>b)^t$ iff M, s $\models (<a \cup b>p)^t$.
6. f=$<p?>q$: By induction hypothesis, M, s $\models$ p iff M, s $\models p^t$, and M, s $\models$ q iff M, s $\models q^t$. Therefore, M, s $\models$ $<p?>q$ iff M, s $\models p \wedge q$ iff M, s $\models$ p and M, s $\models$ q iff M, s $\models p^t$ and M, s $\models q^t$ iff M, s $\models (p \wedge q)^t$ iff M, s $\models (<p?>q)^t$.
7. f=$<a;b>p$: M, s $\models$ $<a;b>p$ iff M, s $\models$ $<a><b>p$. Note that $\#\Delta(<a;b>p) = \#\Delta(<a><b>p)$, $|<a;b>p| = |<a><b>p|$, and $ld(<a><b>p) < ld(<a;b>p)$. Therefore, M, s $\models$ $<a><b>p$ iff M, s $\models (<a><b>p)^t$ by induction hypothesis.
8. f=$<a^*>p$: Since $<a^*>p$ is the least fixpoint of $p \vee <a>Y$, M, s $\models$ $<a^*>p$ iff M, s $\models \mu Y.[p \vee (<a>Y)]$. By induction hypothesis, M, s $\models$ p iff M, s $\models p^t$ and M, s $\models$ $<a>Y$ iff M, s $\models$ $(<a>Y)^t$. Therefore, M, s $\models$ $<a^*>p$ iff M, s $\models \mu Y.[p^t \vee (<a>Y)^t]$
9. f=Δa: Since Δa is the greatest fixpoint of $<a>Y$, M, s $\models \Delta a$ iff M, s $\models \nu Y.<a>Y$. By induction hypothesis, M, s $\models$ $<a>Y$ iff M, s $\models$ $(<a>Y)^t$. Therefore, M, s $\models \Delta a$ iff M, s $\models \nu Y.(<a>Y)^t$.

□

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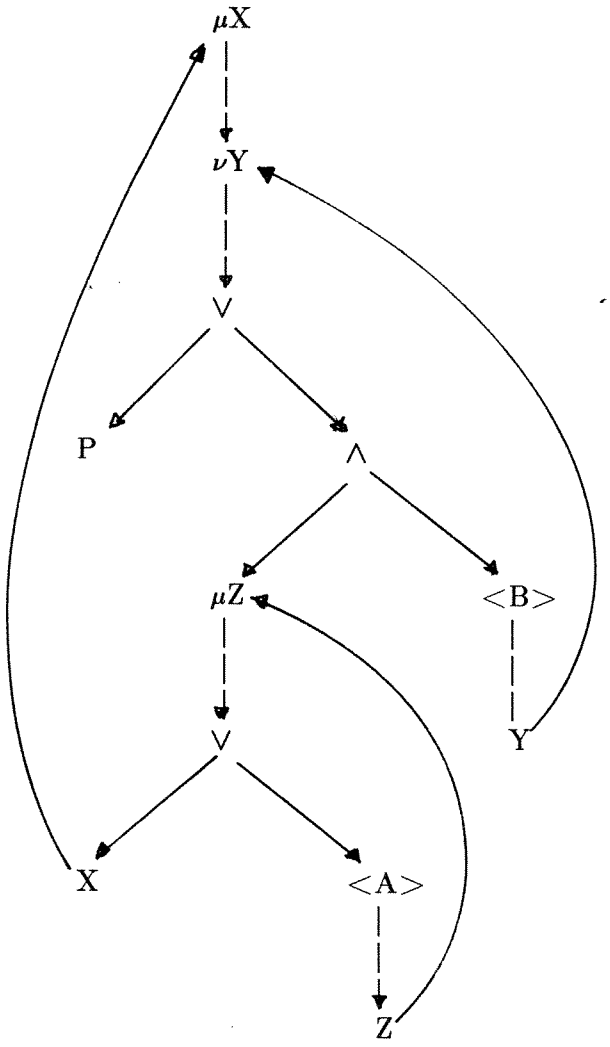
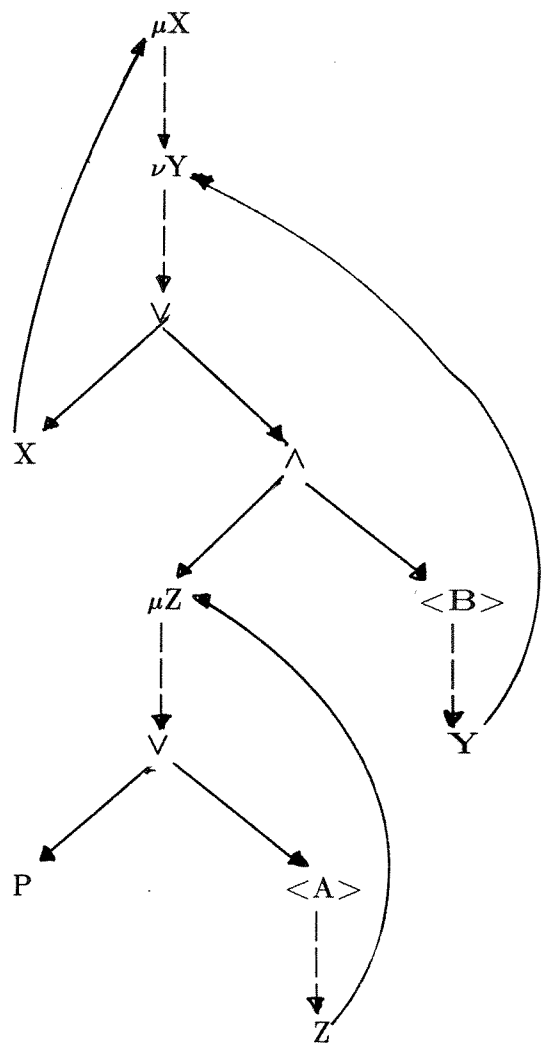


In $\mu X.\nu Y.(P \vee (\mu Z.(X \vee <A>Z) \wedge <B>Y))$:
X is active in both νY and μZ
Y is active in both μX and μZ
Z is active in both μX and νY
(a)



In $\mu X.\nu Y.(X \vee (\mu Z.(P \vee <A>Z) \wedge <B>Y))$:
X is active in νY but not μZ
Y is active in μX but not μ.
Z is not active in either μX or νY
(b)

Figure 2-1: Active Variables in Mu-Calculus

Algorithm 3-1: model checking for a μ-calculus sentence
Input: given a structure M=(S, $\mathcal{R}$, L), and a sentence p_0 which contains variables Y_1, Y_2, ... Y_n
Output: determine whether M is a model for p_0

Step 1. Convert p_0 to its equivalent PNF p'_0.
Step 2. S' = eval(p'_0); /* Compute the set of states at which p'_0 holds */
Step 3. if S'$\neq\emptyset$ then M is a model for p_0 else M is not a model for p_0

```
Recursive function eval(f); var S', S";
      /* return the set of states which satisfy f */
begin
      case f of the form
          P (atomic prop.): S'={s∈S: P∈L(s)};
          Y_i (σ-variable): S'=S_i;
          ¬p: S'=S\eval(p);
          p∧q: S'=eval(p) ∩ eval(q);
          p∨q: S'=eval(p) ∪ eval(q);
          <A>p: S"=eval(p); S'={s∈S: ∃t∈S"[(s,t)∈R(A)]};
          [A]p: S"=eval(p); S'={s∈S: ∀t∈S[(s,t)∈R(A) ⇒ t∈S"]};
          μY_i.p_i:
             begin
               S_i = ∅;
               repeat S' = S_i; S_i = eval(p_i) until S'=S_i;
             end;
          νY_i.p_i:
             begin
               S_i = S;
               repeat S' = S_i; S_i = eval(p_i) until S'=S_i;
             end;
      end;
      return(S');
end.
```

Figure 3-1: Model Checker for Mu-Calculus

(a) Parse Tree (b) Parse Graph

Figure 4-1: Representation of <a>p ∨ <b>p

Algorithm 3-2: model checking for a μ-calculus sentence
Input: given a structure M=(S, $\mathcal{R}$, L), and a sentence p_0 which contains variables Y_1, Y_2, ... Y_n
Output: determine whether M is a model for p_0

Step 1. Convert p_0 to its equivalent PNF p'_0.
Step 2. for i=1 to n do if Y_i is a μ-variable then $S_i=\emptyset$ else $S_i=S$;
/* set appropriate initial conditions for iteration */
for each subsentence q of p'_0 do already-eval(q)=*false*;
/* already-eval() is a global boolean array used to prevent redundant evaluation of a subsentence. When a subsentence q of p'_0 is evaluated, its value is kept in the global array value(). Whenever the value of q is needed again we simply return value(q). */
Step 3. S' = eval(p'_0); /* Compute the set of states at which p'_0 holds */
Step 4. if S'$\neq\emptyset$ then M is a model for p_0 else M is not a model for p_0

```
Recursive function eval(f); var S', S";
      /* return the set of states which satisfy f */
begin
      if already-eval(f) then return(value(f));
      case f of the form
          P (atomic prop.): S'={s∈S: P∈L(s)};
          Y_i (σ-variable): S'=S_i;
          ¬p: S'=S\eval(p);
          p∧q: S'=eval(p) ∩ eval(q);
          p∨q: S'=eval(p) ∪ eval(q);
          <A>p: S"=eval(p); S'={s∈S: ∃t∈S"[(s,t)∈R(A)]};
          [A]p: S"=eval(p); S'={s∈S: ∀t∈S[(s,t)∈R(A) ⇒ t∈S"]};
          μY_i.p_i(..., Y_i, ...):
            begin
              if the surrounding σ-formula of μY_i is νZ
                 then S_i = ∅;
                      for each open subformula μY_j of μY_i such that there is no
                      subformula νW of μY_i containing μY_j, set S_j = ∅;
              repeat S' = S_i; S_i = eval(p_i) until S'=S_i;
            end;
          νY_i.p_i(..., Y_i, ...):
            begin
              if the surrounding σ-formula of νY_i is μZ
                 then S_i = S;
                      for each open subformula νY_j of νY_i such that there is no
                      subformula μW of νY_i containing νY_j, set S_j = S;
              repeat S' = S_i; S_i = eval(p_i) until S'=S_i;
            end;
      end;
      if f is a sentence then begin already-eval(f) = true; value(f) = S' end;
      return(S');
end.
```

Figure 3-2: Improved Model Checker for Mu-Calculus