PBES Exercises, March 25, 2015

Consider the LPE description of a lossy channel system, where actions r, s and l represent *receiving*, *sending* and *losing*, respectively, and the action τ represents some internal behaviour of the system.

$$\begin{array}{rcl} P(b{:}Bool, n{:}Nat) & = & \sum_{m:Nat} \neg b \longrightarrow r(m) \cdot P(\mathsf{true}, m) \\ & + & b \longrightarrow s(n) \cdot P(\mathsf{false}, n) \\ & + & b \longrightarrow l \cdot P(\mathsf{true}, n) \end{array}$$

Let ϕ be the first-order modal μ-calculus formula given below:

$$\nu X.\ \mu Y.\ (([\neg l]X \wedge (\nu Z.\ \exists j{:}Nat.\langle r(j) \vee l\rangle Z)) \vee [\neg l]Y)$$

1. Compute the PBES that is the result of the transformation $\mathbf{E}(\phi)$ applied to P.

   Solution: Applying the translation scheme gives rise to three fixpoint equations: two greatest fixpoints and one least fixpoint. After clean-up, the equation system is as follows:

$$\Big(\nu X(b{:}Bool, n{:}Nat) = Y(b, n)\Big)$$

$$\begin{array}{l} \Big(\mu Y(b{:}Bool, n{:}Nat) = \\ \quad ((\neg b \Rightarrow \forall m{:}Nat.X(\mathit{true}, m)) \wedge (b \Rightarrow X(\mathit{false}, n)) \wedge Z(b, n))) \\ \vee \quad ((\neg b \Rightarrow \forall m{:}Nat.Y(\mathit{true}, m)) \wedge (b \Rightarrow Y(\mathit{false}, n)))\Big) \end{array}$$

$$\begin{array}{l} \Big(\nu Z(b{:}Bool, n{:}Nat) = \\ \quad \exists j{:}Nat.(\exists m{:}Nat.\neg b \wedge m = j \wedge Z(\mathit{true}, m)) \vee (b \wedge Z(\mathit{true}, n))\Big) \end{array}$$

   Observe that, using the one-point rule, the last equation can be rewritten to:

$$\Big(\nu Z(b{:}Bool, n{:}Nat) = \exists j{:}Nat.(\neg b \wedge Z(\mathit{true}, j)) \vee (b \wedge Z(\mathit{true}, n))\Big)$$

   We will use the latter one in our calculations.

2. Solve the resulting PBES using symbolic approximation. Show all steps in all your computations.

   Solution: we start by approximating the last equation, i.e., the equation for Z. Since we are dealing with a greatest fixpoint, we have to start with the function $Z_0(b, n) =$ true. The successive approximants are represented by Z_i below:

$$\begin{array}{rl} Z_0(b, n) & = \mathit{true} \\ Z_1(b, n) & = (\exists j{:}Nat.\neg b \wedge Z_0(\mathit{true}, j)) \vee (b \wedge \neg Z_0(\mathit{true}, n)) \\ & = \neg b \vee b \\ & = \mathit{true} \end{array}$$

So, we know the solution to Z, viz., $Z(b,n) = \textsf{true}$. We now substitute this solution upwards in the equation for Y. The modification this induces in the equation for Y is then as follows:

$$\begin{array}{ll} \Big(& \mu Y(b{:}Bool, n{:}Nat) = \\ & \quad ((\neg b \Rightarrow \forall m{:}Nat.X(\textsf{true}, m)) \wedge (b \Rightarrow X(\textsf{false}, n))) \\ \vee & ((\neg b \Rightarrow \forall m{:}Nat.Y(\textsf{true}, m)) \wedge (b \Rightarrow Y(\textsf{false}, n)))\Big) \end{array}$$

We next approximate the equation for Y; since Y is a least fixpoint equation, we start our approximation $Y_0(b,n) = \textsf{false}$:

$$\begin{array}{ll}
Y_0(b,n) & = \textsf{false} \\
\\
Y_1(b,n) & = ((\neg b \Rightarrow \forall m{:}Nat.X(\textsf{true}, m)) \wedge (b \Rightarrow X(\textsf{false}, n))) \\
 & \qquad \vee((\neg b \Rightarrow \forall m{:}Nat.\textsf{false}) \wedge (\neg b)) \\
 & = ((\neg b \Rightarrow \forall m{:}Nat.X(\textsf{true}, m)) \wedge (b \Rightarrow X(\textsf{false}, n))) \\
\\
Y_2(b,n) & = ((\neg b \Rightarrow \forall m{:}Nat.X(\textsf{true}, m)) \wedge (b \Rightarrow X(\textsf{false}, n))) \\
 & \qquad \vee((\neg b \Rightarrow \forall m{:}Nat.Y_1(\textsf{true}, m)) \wedge (b \Rightarrow Y_1(\textsf{false}, n))) \\
 & = ((\neg b \Rightarrow \forall m{:}Nat.X(\textsf{true}, m)) \wedge (b \Rightarrow X(\textsf{false}, n))) \\
 & \qquad \vee((\neg b \Rightarrow \forall m{:}Nat.X(\textsf{false}, m)) \wedge (b \Rightarrow \forall m{:}Nat.X(\textsf{true}, m))) \\
\\
Y_3(b,n) & = ((\neg b \Rightarrow \forall m{:}Nat.X(\textsf{true}, m)) \wedge (b \Rightarrow X(\textsf{false}, n))) \\
 & \qquad \vee((\neg b \Rightarrow \forall m{:}Nat.Y_2(\textsf{true}, m)) \wedge (b \Rightarrow Y_2(\textsf{false}, n))) \\
 & = ((\neg b \Rightarrow \forall m{:}Nat.X(\textsf{true}, m)) \wedge (b \Rightarrow X(\textsf{false}, n))) \\
 & \qquad \vee((\neg b \Rightarrow ((\forall m{:}Nat.X(\textsf{false}, m)) \vee (\forall m{:}Nat.X(\textsf{true}, m)))) \\
 & \qquad\qquad \wedge(b \Rightarrow ((\forall m{:}Nat.X(\textsf{true}, m)) \vee (\forall m{:}Nat.X(\textsf{false}, m)))) \\
 & = ((\neg b \Rightarrow \forall m{:}Nat.X(\textsf{true}, m)) \wedge (b \Rightarrow X(\textsf{false}, n))) \\
 & \qquad \vee(\forall m{:}Nat.X(\textsf{false}, m)) \vee (\forall m{:}Nat.X(\textsf{true}, m)) \\
 & = (\forall m{:}Nat.X(\textsf{false}, m)) \vee (\forall m{:}Nat.X(\textsf{true}, m)) \vee (b \wedge X(\textsf{false}, n)) \\
\\
Y_4(b,n) & = ((\neg b \Rightarrow \forall m{:}Nat.X(\textsf{true}, m)) \wedge (b \Rightarrow X(\textsf{false}, n))) \\
 & \qquad \vee((\neg b \Rightarrow \forall m{:}Nat.Y_3(\textsf{true}, m)) \wedge (b \Rightarrow Y_3(\textsf{false}, n))) \\
 & = ((\neg b \Rightarrow \forall m{:}Nat.X(\textsf{true}, m)) \wedge (b \Rightarrow X(\textsf{false}, n))) \\
 & \qquad \vee((\neg b \Rightarrow (X(\textsf{false}, n) \vee (\forall m{:}Nat.X(\textsf{true}, m)))) \\
 & \qquad\qquad \wedge(b \Rightarrow ((\forall m{:}Nat.X(\textsf{true}, m)) \vee (\forall m{:}Nat.X(\textsf{false}, m)))) \\
 & = X(\textsf{false}, n) \vee \forall m{:}Nat.X(\textsf{true}, m) \\
\\
Y_5(b,n) & = Y_4(b,n)
\end{array}$$

So, the solution to Y is $Y(b,n) = X(\textsf{false}, n) \vee \forall m{:}Nat.X(\textsf{true}, m)$. We now subsitute this solution upwards in the equation for X. The modification this induces in the *equation for X is then as follows:*

$$\Big(\mu X(b{:}Bool, n{:}Nat) = X(\textsf{false}, n) \lor \forall m{:}Nat.X(\textsf{true}, m)\Big)$$

We next approximate the equation for X; since X is a greatest fixpoint equation, we start our approximation $X_0(b,n) =$ false:

$$X_0(b,n) \quad = \textsf{true}$$

$$\begin{aligned} X_1(b,n) &= X_0(\textsf{false}, n) \lor \forall m{:}Nat.X_0(\textsf{true}, m) \\ &= \textsf{true} \lor \forall m{:}Nat.\textsf{true} \\ &= \textsf{true} \end{aligned}$$

Since the approximation terminates immediately, the solution to the equation for X is $X(b,n) =$ true.

As a result, we know that $P(b,n) \models \phi$ for every b, n.

3. Solve the resulting PBES using instantiation. Hint: first eliminate redundant parameters of the given PBES, and use logic to rewrite the right-hand side of the PBES. Show all steps in all your computations.

   Solution: we start by detecting the significant parameters in the equations for X, Y and Z. These yield the parameter b only. The marked influence graph will simply yield two disconnected subgraphs, one for the vertices X_b, Y_b and Z_b, and another one for the vertices X_n, Y_n and Z_n. Since from none of the vertices of X_n, Y_n or Z_n we can reach X_b, Y_b or Z_b, all parameters n are redundant. This leads to the following simplified equation system:

$$\Big(\nu X(b{:}Bool) = Y(b)\Big)$$

$$\begin{aligned} \Big(\mu Y(b{:}Bool) = & \\ & ((\neg b \Rightarrow \forall m{:}Nat.X(\textsf{true})) \land (b \Rightarrow X(\textsf{false})) \land Z(b))) \\ \lor \quad & ((\neg b \Rightarrow \forall m{:}Nat.Y(\textsf{true})) \land (b \Rightarrow Y(\textsf{false})))\Big) \end{aligned}$$

$$\Big(\nu Z(b{:}Bool) = \exists j{:}Nat.(\neg b \land Z(\textsf{true})) \lor (b \land Z(\textsf{true}))\Big)$$

   Observe that we can even further simplify this equation system by i) removing redundant quantifiers, ii) simplifying the equation for Z, leading to the following simplified *system of equations:*

$$
\begin{array}{l}
\left(\nu X(b{:}Bool) = Y(b)\right) \\
\left(\mu Y(b{:}Bool) = \right. \\
\quad ((\neg b \Rightarrow X(\mathsf{true})) \wedge (b \Rightarrow X(\mathsf{false})) \wedge Z(b))) \\
\vee \quad \left. ((\neg b \Rightarrow Y(\mathsf{true})) \wedge (b \Rightarrow Y(\mathsf{false})))\right) \\
\left(\nu Z(b{:}Bool) = Z(\mathsf{true})\right)
\end{array}
$$

Instantiating the resulting equation system using any possible value will yield the following 6 Boolean equations:

$$
\begin{array}{l}
\left(\nu X_{\mathsf{false}} = Y_{\mathsf{false}}\right) \\
\left(\nu X_{\mathsf{true}} = Y_{\mathsf{true}}\right) \\
\left(\mu Y_{\mathsf{false}} = (X_{\mathsf{true}} \wedge Z_{\mathsf{false}}) \vee Y_{\mathsf{true}}\right) \\
\left(\mu Y_{\mathsf{true}} = (X_{\mathsf{false}} \wedge Z_{\mathsf{true}}) \vee Y_{\mathsf{false}}\right) \\
\left(\nu Z_{\mathsf{false}} = Z_{\mathsf{true}}\right) \\
\left(\nu Z_{\mathsf{true}} = Z_{\mathsf{true}}\right)
\end{array}
$$

Gauß Elimination will yield:

$$
\begin{array}{l}
\left(\nu X_{\mathsf{false}} = \mathsf{true}\right) \\
\left(\nu X_{\mathsf{true}} = \mathsf{true}\right) \\
\left(\mu Y_{\mathsf{false}} = \mathsf{true}\right) \\
\left(\mu Y_{\mathsf{true}} = \mathsf{true}\right) \\
\left(\nu Z_{\mathsf{false}} = \mathsf{true}\right) \\
\left(\nu Z_{\mathsf{true}} = \mathsf{true}\right)
\end{array}
$$