Algorithms for Model Checking (2IMF35)

Lecture 10

Parameterised Boolean Equation Systems (2)

Background material:

Model Checking Processes with Data,
J.F. Groote and T.A.C. Willemse (*Sc. Comp. Progr. 2005*)

Proof Graphs for Parameterised Boolean Equation Systems,
S. Cranen, B. Luttik and T.A.C. Willemse (*CONCUR 2013*)

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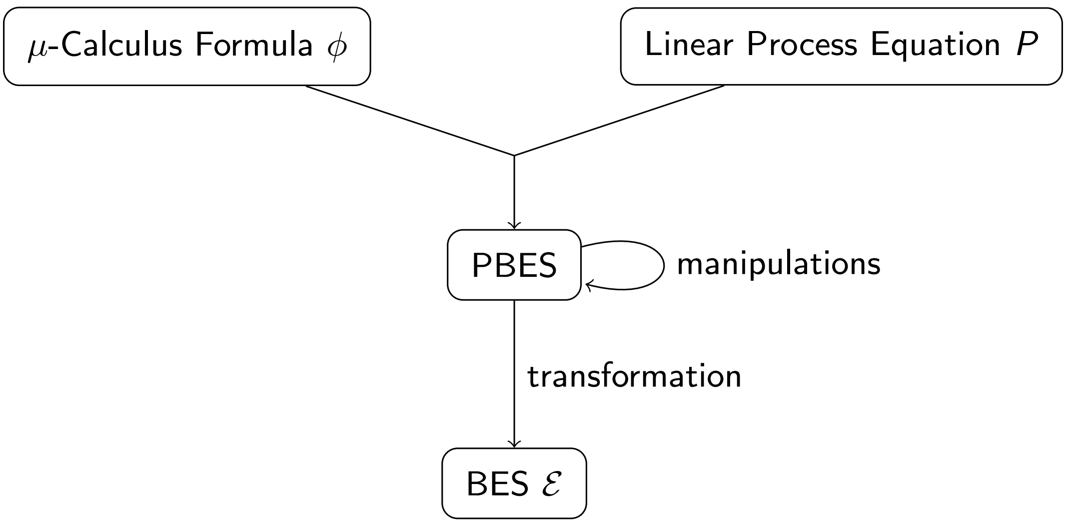
MF 6.073

Verification via PBESs

Verification Methodology:



Solving $\mathcal{E}$ answers $P \models \phi$

Parameterised Boolean Equation Systems

Problem Description

1. Given a process $X(e)$ described by an LPE X over Act
2. Given a first-order modal μ-calculus formula ϕ
3. Given environments η, ε
4. Check whether $X(e) \models \phi$ holds, where:

$$X(e) \models \phi \text{ iff } e \in [\![\phi]\!]\eta\varepsilon$$

- Decidable for finite data types
  - Compute LTS $[\![X(e)]\!]$
  - Evaluate ϕ on $[\![X(e)]\!]$ using standard model checking algorithms
- In general undecidable
- Transform problem to Parameterised Boolean Equation Systems (PBESs)

Parameterised Boolean Equation Systems

Grammar for predicate formulae

$$\phi, \psi ::= b \mid X(e) \mid \phi \wedge \psi \mid \phi \vee \psi \mid \forall d : D.\phi \mid \exists d : D.\phi$$

- b is a boolean expression . . . . $n + m \geq 5$
- $X \in \mathcal{P}$ is a sorted predicate variable (or *relation*) . . . . $X{:}2^D$
- e is an expression of sort D
- Interpreting ϕ requires two environments . . . . ε (for data) and $\eta{:}\mathcal{P} \rightarrow 2^D$

$$[\![b]\!]\eta\varepsilon = \begin{cases} \text{true} & \text{if } \varepsilon(b) \\ \text{false} & \text{else} \end{cases} \qquad [\![X(e)]\!]\eta\varepsilon = \begin{cases} \text{true} & \text{if } \varepsilon(e) \in \eta(X) \\ \text{false} & \text{else} \end{cases}$$

$$[\![\phi \wedge \psi]\!]\eta\varepsilon = [\![\phi]\!]\eta\varepsilon \text{ and } [\![\psi]\!]\eta\varepsilon \qquad [\![\phi \vee \psi]\!]\eta\varepsilon = [\![\phi]\!]\eta\varepsilon \text{ or } [\![\psi]\!]\eta\varepsilon$$

$$[\![\forall d{:}D.\phi]\!]\eta\varepsilon = \text{for all } v{\in}D: [\![\phi]\!]\eta(\varepsilon[d := v]) \qquad [\![\exists d{:}D.\phi]\!]\eta\varepsilon = \text{for some } v{\in}D: [\![\phi]\!]\eta(\varepsilon[d := v])$$

A parameterised Boolean equation is an equation of the form $\sigma X(d : D) = \phi$

- σ is a least fixed point sign μ or a greatest fixed point sign ν.
- ϕ is a predicate formula, X a predicate variable
- a parameterised Boolean equation system is a sequence of such equations

- bound (bnd), free, well-formedness, open, close, rank as in BESs
- As in BESs, the order of equations is important.
- Assume, for simplicity, that all equations range over sort D only

The solution of a PBES $\mathcal{E}$ is an environment: $\eta : \mathcal{P} \to 2^D$;

We define $[\![\mathcal{E}]\!]\eta\varepsilon$ by recursion on $\mathcal{E}$.

$$\begin{cases} [\![\epsilon]\!]\eta\varepsilon & := \eta \\ [\![(\mu X(d : D) = \phi)\ \mathcal{E}]\!]\eta\varepsilon & := [\![\mathcal{E}]\!]\eta[X := \mu\Phi^{X,d}_{\mathcal{E},\eta,\varepsilon}]\varepsilon \\ [\![(\nu X(d : D) = \phi)\ \mathcal{E}]\!]\eta\varepsilon & := [\![\mathcal{E}]\!]\eta[X := \nu\Phi^{X,d}_{\mathcal{E},\eta,\varepsilon}]\varepsilon \end{cases}$$

Note: $\nu\Phi^{X,d}_{\mathcal{E},\theta,\varepsilon}$ is the greatest fixpoint to the following monotone functional:

$$\Phi^{X,d}_{\mathcal{E},\eta,\varepsilon}(Z) := \{v \in D \mid [\![\phi]\!]([\![\mathcal{E}]\!]\eta[X := Z]\varepsilon)\varepsilon[d := v]\}$$

We write $X(v) = \mathit{true}$ iff $v \in [\![\mathcal{E}]\!]\eta\varepsilon(X)$

Let:

- $\eta : \mathcal{P} \to 2^D$ be a predicate environment and ε a data environment
- $\mathsf{sig}(\mathcal{E}) = \{(X, v) \mid X \in \mathsf{bnd}(\mathcal{E}), v \in D\}$ be the signatures

Definition (Signature Environments)

Assume $S \subseteq \mathsf{sig}(\mathcal{E})$.

- S_{true} is the environment defined as $S_{\mathsf{true}}(X) = \{v \mid (X, v) \in S\}$ for all X
- S_{false} is the environment defined as $S_{\mathsf{false}}(X) = \{v \mid (X, v) \notin S\}$ for all X

Definition (Dependency Graphs)

Let b be a Boolean.

A structure $\langle S, R, L \rangle$ is a b-dependency graph for closed $\mathcal{E}$ and data environment ε if:

- $S \subseteq \mathsf{sig}(\mathcal{E})$
- $L(X, v) = \mathsf{rank}(X)$
- $R \subseteq S \times S$ such that: if for $\sigma X(d : D) = \phi$ in $\mathcal{E}$, $(X, v) \in S$ then
  - If $b = \mathsf{true}$, we require: $[\![\phi]\!]((X, v)^{\bullet})_{\mathsf{true}}\varepsilon[d := v]$
  - If $b = \mathsf{false}$, we require: $\neg[\![\phi]\!]((X, v)^{\bullet})_{\mathsf{false}}\varepsilon[d := v]$

Where

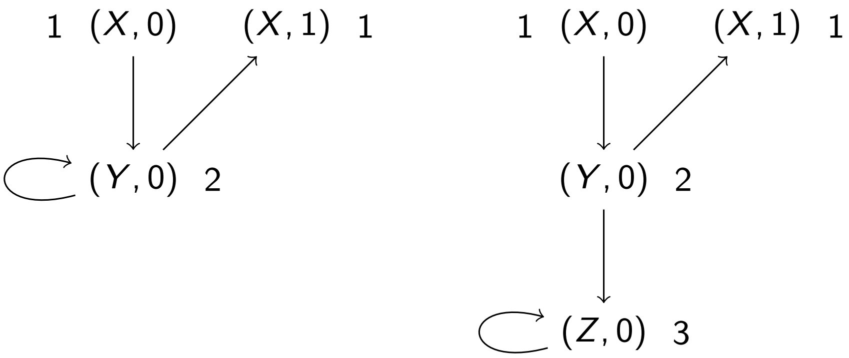
- $(X, v)^{\bullet} = \{(Z, w) \in S \mid (X, v) \; R \; (Z, w)\}$

Example

Consider the following equation system $\mathcal{E}$.

$$\mu X(b:Bit) = Y(b) \vee b = 1$$
$$\nu Y(b:Bit) = Y(b) \vee (X(1) \wedge Z(b))$$
$$\mu Z(b:Bit) = Z(b)$$

Below are two true-dependency graphs $\langle S, R, L\rangle$ with $(X,0) \in S$.
Note that $\mathrm{sig}(\mathcal{E}) = \{(U,0),(U,1) \mid U = X, Y, Z\}$.



Example (Continued)

To see that the graph on the left satisfies the true-dependency graph property:

for $(X,0)$: $[\![Y(b) \vee b = 1]\!]\{(Y,0)\}_{\mathsf{true}}\delta[b := 0] = \mathsf{true}$
for $(X,1)$: $[\![Y(b) \vee b = 1]\!]\emptyset_{\mathsf{true}}\delta[b := 1] = \mathsf{true}$
for $(Y,0)$: $[\![Y(b) \vee X(1)]\!]\{(Y,0),(X,1)\}_{\mathsf{true}}]\delta[b := 0] = \mathsf{true}$

- Any infinite path goes through states with label 2, hence, it satisfies the true-proof graph property.
- Note that in this true-dependency graph, $(Y,0) \to (X,1)$ can be left out, because the right hand side of the equation for Y is disjunctive and the left disjunct is true.

Definition (Proof Graphs)

A true-dependency graph $\langle S, R, L\rangle$ is a proof graph iff for all $s \in S$ and all infinite paths $\pi \in \text{path}(s)$:

$$\min\{r \mid \text{label } r \text{ occurs infinitely often on } \pi\} \text{ is even}$$

Theorem

For all closed PBESs $\mathcal{E}$ and all η, ε:

$$v \in [\![\mathcal{E}]\!]\eta\varepsilon(X) \textit{ iff there is a proof graph } \langle S, R, L\rangle \textit{ such that } (X, v) \in S$$

Dually:

Definition (Refutation Graphs)

A false-dependency graph $\langle S, R, L\rangle$ is a refutation graph iff for all $s \in S$ and all infinite paths $\pi \in \text{path}(s)$:

$$\min\{r \mid \text{label } r \text{ occurs infinitely often on } \pi\} \text{ is odd}$$

Theorem

For all closed PBESs $\mathcal{E}$ and all η, ε:

$$v \notin [\![\mathcal{E}]\!]\eta\varepsilon(X) \textit{ iff there is a refutation graph } \langle S, R, L\rangle \textit{ such that } (X, v) \in S$$

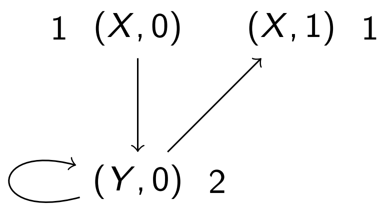
Parameterised Boolean Equation Systems

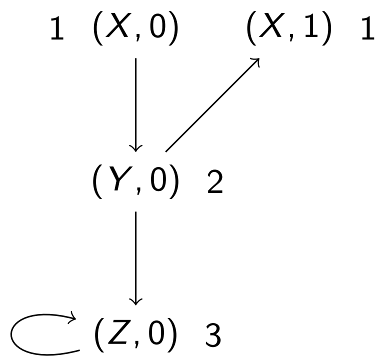
Example

Consider the following equation system $\mathcal{E}$.

$$\begin{aligned}\mu X(b : Bit) &= Y(b) \vee b = 1\\ \nu Y(b : Bit) &= Y(b) \vee (X(1) \wedge Z(b))\\ \mu Z(b : Bit) &= Z(b)\end{aligned}$$

Below are two true-dependency graphs $\langle S, R, L \rangle$ with $(X, 0) \in S$. The left one is a true-proof graph; the right one is not.





Verification via PBESs

First-order Modal μ-Calculus model checking problem

- Given is a First-order Modal μ-Calculus formula $\sigma Z.\ \phi$
- Given a system described by an LPE $X(e)$

Compute whether $X(e) \models \sigma Z.\ \phi$

- Transform the model checking problem to solving a PBES $\mathcal{E}$
- The transformation is similar to the transformation to BES.
- Idea: for each fixed point subformula $\sigma' X.\ \psi$ of $\sigma Z.\phi$, add an equation

$$\sigma' \tilde{X}(d : D, \cdots) = RHS(\psi)$$

- The order of the equations respects the subterm ordering in $\sigma Z.\phi$
- Transformation is such that $X(e) \models \sigma Z.\ \phi$ iff $e \in [\![\mathcal{E}]\!]\eta\varepsilon(\tilde{Z})$

- Identify a list of data variables bound outside the scope of a fixed point formula
- Given a formula ψ and some formal variable Z

Identify Bound Data Variables

$$\begin{aligned}
Par(Z, b, l) &= Par(Z, X, l) &&= [] \\
Par(Z, \phi \wedge \psi, l) &= Par(Z, \phi \vee \psi, l) &&= Par(Z, \phi, l) +\!\!+ Par(Z, \psi, l) \\
Par(Z, \forall d{:}D.\phi, l) &= Par(Z, \exists d{:}D.\phi, l) &&= Par(Z, \phi, [d{:}D] +\!\!+ l) \\
Par(Z, [\alpha]\phi, l) &= Par(Z, \langle\alpha\rangle\phi, l) &&= Par(Z, \phi, l)
\end{aligned}$$

$$Par(Z, \sigma X.\phi, l) = \begin{cases} l & \text{if } Z = X \\ Par(Z, \phi, l) & \text{otherwise} \end{cases}$$

Example

The one-place buffer system described by process B:

$$\begin{aligned}
B(b : Bool, n : Nat) \quad = &\quad \sum_{m:Nat} b \longrightarrow r(m) \cdot B(\text{false}, m) \\
+ &\quad \neg b \longrightarrow s(n) \cdot B(\text{true}, n)
\end{aligned}$$

- Property ψ: if the input stream is constant, so is the output stream:

$$\forall k : Nat.\ (\nu X.(\forall l : Nat.\ [r(l)](l = k \Rightarrow X) \wedge [s(l)](l = k \wedge X)))$$

- Transform ψ to a formula Ψ that starts with a dummy fixed point:

$$\nu A.\ \forall k : Nat.\ (\nu X.(\forall l : Nat.\ [r(l)](l = k \Rightarrow X) \wedge [s(l)](l = k \wedge X)))$$

- We have: $Par(A, \Psi, []) = []$ and $Par(X, \Psi, []) = [k : Nat]$

- Let $\psi := \sigma Z.\ \phi$
- Given LPE $X(d{:}D) = \sum_{i \leq n} \sum_{e_i : D_i} c_i(d, e_i) \longrightarrow a_i(f_i(d, e_i)) \cdot X(g_i(d, e_i))$

Create Equation System Outline

$$\mathbf{E}(b) = \epsilon \qquad \mathbf{E}(Z) = \epsilon$$

$$\mathbf{E}(\phi \wedge \psi) = \mathbf{E}(\phi)\ \mathbf{E}(\psi) \qquad \mathbf{E}(\phi \vee \psi) = \mathbf{E}(\phi)\ \mathbf{E}(\psi)$$

$$\mathbf{E}(\forall d'{:}D'.\phi) = \mathbf{E}(\phi) \qquad \mathbf{E}(\exists d'{:}D'.\phi) = \mathbf{E}(\phi)$$

$$\mathbf{E}([\alpha]\phi) = \mathbf{E}(\phi) \qquad \mathbf{E}(\langle\alpha\rangle\phi) = \mathbf{E}(\phi)$$

$$\mathbf{E}(\sigma Z.\phi) = \left(\sigma \tilde{Z}(d{:}D, Par(Z, \psi, [])) = \mathbf{RHS}(\phi)\right)\ \mathbf{E}(\phi)$$

Example

Applying operator **E** on formula Ψ given the buffer process B:

$$
\begin{array}{rl}
 & \mathbf{E}(\Psi) \\
= & \mathbf{E}(\underline{\nu A.}\ \Psi_1) \\
= & (\nu \tilde{A}(b : Bool, n : Nat) = \mathbf{RHS}(\Psi_1))\ \mathbf{E}(\Psi_1) \\
= & (\nu \tilde{A}(b : Bool, n : Nat) = \mathbf{RHS}(\Psi_1))\ \mathbf{E}(\forall k : Nat.\ \Psi_2) \\
= & (\nu \tilde{A}(b : Bool, n : Nat) = \mathbf{RHS}(\Psi_1))\ \mathbf{E}(\forall k : Nat.\ \Psi_2) \\
= & (\nu \tilde{A}(b : Bool, n : Nat) = \mathbf{RHS}(\Psi_1))\ \mathbf{E}(\nu X.\Psi_3) \\
= & (\nu \tilde{A}(b : Bool, n : Nat) = \mathbf{RHS}(\Psi_1)) \\
 & (\nu \tilde{X}(b : Bool, n : Nat, k : Nat) = \mathbf{RHS}(\Psi_3))\ \mathbf{E}(\Psi_3) \\
= & \ldots \\
 & (\nu \tilde{A}(b : Bool, n : Nat) = \mathbf{RHS}(\Psi_1)) \\
 & (\nu \tilde{X}(b : Bool, n : Nat, k : Nat) = \mathbf{RHS}(\Psi_3))
\end{array}
$$

So, **E**(Ψ) yields two equations.

- Let $\psi := \sigma Y.\ \phi$
- Given LPE $X(d{:}D) = \sum_{i \leq n} \sum_{e_i:D_i} c_i(d, e_i) \longrightarrow a_i(f_i(d, e_i)) \cdot X(g_i(d, e_i))$

RHS:

$$\mathbf{RHS}(b) = b \qquad \mathbf{RHS}(Z) = \tilde{Z}(d, Par(Z, \psi, []))$$

$$\mathbf{RHS}(\phi \wedge \psi) = \mathbf{RHS}(\phi) \ \wedge \ \mathbf{RHS}(\psi) \qquad \mathbf{RHS}(\phi \vee \psi) = \mathbf{RHS}(\phi) \ \vee \ \mathbf{RHS}(\psi)$$

$$\mathbf{RHS}(\forall d'{:}D'.\phi) = \forall d'{:}D'.\ \mathbf{RHS}(\phi) \qquad \mathbf{RHS}(\exists d'{:}D'.\phi) = \exists d'{:}D'.\ \mathbf{RHS}(\phi)$$

$$\mathbf{RHS}(\sigma Z.\phi) = \tilde{Z}(d, Par(Z, \psi, []))$$

$$\mathbf{RHS}(\langle\alpha\rangle\phi) = \bigvee_{i \leq n} \exists e_i{:}D_i.\ \Big(c_i(d, e_i) \wedge a_i(f_i(d, e_i)) \text{ in } \alpha \wedge ((\mathbf{RHS}(\phi))[d := g_i(d, e_i)])\Big)$$

$$\mathbf{RHS}([\alpha]\phi) = \bigwedge_{i \leq n} \forall e_i{:}D_i.\ \Big((c_i(d, e_i) \wedge a_i(f_i(d, e_i)) \text{ in } \alpha) \Rightarrow ((\mathbf{RHS}(\phi))[d := g_i(d, e_i)])\Big)$$

Example (Verification of the Buffer process B, continued)

- Consider subformula $(\forall l : Nat.\ [r(l)](l = k \Rightarrow X) \wedge [s(l)](l = k \wedge X))$ of Ψ

$$\begin{aligned} & \mathbf{RHS}(\forall l : Nat.\ [r(l)](l = k \Rightarrow X) \wedge [s(l)](l = k \wedge X)) \\ = \ & \forall l : Nat.\ \mathbf{RHS}([r(l)](l = k \Rightarrow X) \wedge [s(l)](l = k \wedge X)) \\ = \ & \forall l : Nat.\ (\mathbf{RHS}([r(l)](l = k \Rightarrow X)) \wedge \mathbf{RHS}([s(l)](l = k \wedge X))) \end{aligned}$$

- Computing $\mathbf{RHS}([r(l)](l = k \Rightarrow X))$ requires process B.

$$\begin{aligned} & \mathbf{RHS}(([r(l)](l = k \Rightarrow X))) \\ = \ & (\forall m : Nat.\ (b \wedge r(m) \text{ in } r(l)) \Rightarrow \mathbf{RHS}(l = k \Rightarrow X)[b := \text{false}, n := m]) \\ \wedge \ & ((\neg b \wedge s(n) \text{ in } r(l)) \Rightarrow \mathbf{RHS}(l = k \Rightarrow X)[b := \text{true}, n := n]) \\ = \ & (\forall m : Nat.\ (b \wedge r(m) \text{ in } r(l)) \Rightarrow (l = k \Rightarrow \tilde{X}(\text{false}, m, k))) \\ \wedge \ & ((\neg b \wedge s(n) \text{ in } r(l)) \Rightarrow (l = k \Rightarrow \tilde{X}(\text{true}, n, k))) \end{aligned}$$

Matching parameterised actions with action formulae:

$$
\begin{array}{ll}
a(e) \text{ in true} & = \text{true} \\
a(e) \text{ in } a'(e') & = (a = a' \wedge e = e') \\
a(e) \text{ in } \neg\alpha & = \neg(a(e) \text{ in } \alpha) \\
a(e) \text{ in } (\alpha \wedge \beta) & = (a(e) \text{ in } \alpha) \ \wedge (a(e) \text{ in } \beta) \\
a(e) \text{ in } (\alpha \vee \beta) & = (a(e) \text{ in } \alpha) \ \vee (a(e) \text{ in } \beta)
\end{array}
$$

Observations:

- in yields a predicate formula
- in does not introduce predicate variables

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Example

- The expression $r(m)$ in $r(l)$ yields $r = r \wedge m = l$, which simplifies to $m = l$
- The expression $s(n)$ in $r(l)$ yields $s = r \wedge n = l$, which simplifies to false

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Verification via PBESs

Example (Verification of the Buffer process, continued)

Buffer system and constant stream revisited

$$\begin{aligned} B(b:Bool, n:Nat) &= \sum_{m:Nat} b \longrightarrow r(m) \cdot B(\text{false}, m) \\ &+ \neg b \longrightarrow s(n) \cdot B(\text{true}, n) \end{aligned}$$

Property Ψ: $\nu A.\ \forall k:Nat.\ (\nu X.(\forall l:Nat.\ [r(l)](l=k \Rightarrow X) \land [s(l)](l=k \land X)))$

Result after translation to PBES $\mathcal{E}$ (note: cleanup using ordinary first-order logic):

$$(\nu \tilde{A}(b:Bool, n:Nat) = \forall k:Nat.\ \tilde{X}(b,n,k))$$

$$\begin{aligned} &(\nu \tilde{X}(b:Bool, n:Nat, k:Nat) = \\ &\quad \forall l:Nat.\ ((\forall m:Nat.\ (b \land m = l) \Rightarrow (l = k \Rightarrow \tilde{X}(\text{false}, m, k))) \\ &\land((\neg b \land n = l) \Rightarrow (l = k \land \tilde{X}(\text{true}, n, k))))) \end{aligned}$$

For all $b:Bool$ and $n:Nat$, we have: $B(b,n) \models \Psi$ iff $(b,n) \in ([\![\mathcal{E}]\!]\theta\varepsilon)(\tilde{A}) = \text{true}$

Solving PBESs

How to solve PBESs

$$e \stackrel{?}{\in} X_i \text{ in } \mathcal{E} := \ (\sigma_1 X_1(d_1:D_1) = \phi_1) \cdots (\sigma_n X_n(d_n:D_n) = \phi_n)$$

Known techniques for solving/simplifying $\mathcal{E}$:

- Gauß Elimination on PBES + symbolic approximation of equations
- Instantiation to BES and subsequently solve the BES
- Using patterns
- Using under/over approximation
- Invariants

Definition (Logical Equivalence)

Let ϕ, ψ be two predicates. Then ψ is logically equivalent to ϕ, denoted $\phi \leftrightarrow \psi$ iff

$$\forall \varepsilon, \eta : \ [\![\phi]\!]\eta\varepsilon = [\![\psi]\!]\eta\varepsilon$$

- If $\phi \leftrightarrow \psi$, then equation $\nu X(d : D) = \phi$ has the same solution as $\nu X(d : D) = \psi$ (likewise for μ)
- Useful simplifications:
  - false $\wedge\ \phi \leftrightarrow$ false
  - true $\vee\ \phi \leftrightarrow$ true
  - if $d \notin \mathsf{FV}(\phi)$, then $(\exists d : D.\ \phi) \leftrightarrow (\forall d : D.\ \phi) \leftrightarrow \phi$
  - One-point rule: $(\exists d : D. d = e \wedge \phi(d)) \leftrightarrow \phi(e)$
  - One-point rule: $(\forall d : D. d = e \Rightarrow \phi(d)) \leftrightarrow \phi(e)$
- Apply logical simplifications before applying PBES manipulations/solving techniques.

Gauß elimination on PBESs + Symbolic Approximation:

$$e \overset{?}{\in} X_i \text{ in } \mathcal{E} := \ (\sigma_1 X_1(d_1 : D_1) = \phi_1) \cdots (\sigma_n X_n(d_n : D_n) = \phi_n)$$

- Local solution: eliminate X in its defining equation:

$$\mathcal{E}_0\ (\sigma X(d{:}D) = \phi)\ \mathcal{E}_1 \text{ becomes } \mathcal{E}_0\ (\sigma X(d{:}D) = X^\omega)\ \mathcal{E}_1$$

  - X^ω can be found by symbolic approximation:
  - $X^0 =$ false if $\sigma = \mu$, else $X^0 =$ true
  - $X^{n+1} = \phi[X := X^n]$
  - X^ω may require transfinite approximation; else $X^\omega = X^n$ for $X^n \leftrightarrow X^{n+1}$
- Substitute definition backwards:

$$\mathcal{E}_0\ (\sigma_1 X_1(d_1{:}D_1) = \phi_1)\ \mathcal{E}_1\ (\sigma_2 X_2(d_2{:}D_2) = \phi_2)\ \mathcal{E}_2$$
becomes:
$$\mathcal{E}_0\ (\sigma_1 X_1(d_1{:}D_1) = \phi_1[X_2 := \phi_2])\ \mathcal{E}_1\ (\sigma_2 X_2(d_2{:}D_2) = \phi_2)\ \mathcal{E}_2$$

- Substitute solved equations (i.e. not containing predicate variables) forward:

$$\mathcal{E}_0\ (\sigma_1 X_1(d_1{:}D_1) = \phi_1)\ \mathcal{E}_1\ (\sigma_2 X_2(d_2{:}D_2) = \phi_2)\ \mathcal{E}_2$$
becomes:
$$\mathcal{E}_0\ (\sigma_1 X_1(d_1{:}D_1) = \phi_1)\ \mathcal{E}_1\ (\sigma_2 X_2(d_2{:}D_2) = \phi_2[X_1 := \phi_1])\ \mathcal{E}_2$$

Example

PBES: $(\nu X(n : Nat) = n \leq 2 \land Y(n))\ (\mu Y(n : Nat) = \text{odd}(n) \lor X(n+1))$

1. Eliminate Y from $(\mu Y(n : Nat) = \text{odd}(n) \lor X(n+1))$ .... done
2. Substitute definition of Y backwards:

$$\begin{array}{ll} & (\nu X(n : Nat) = n \leq 2 \land Y(n)) \\ \text{becomes} & (\nu X(n : Nat) = n \leq 2 \land (\text{odd}(n) \lor X(n+1))) \end{array}$$

3. Eliminate X from $(\nu X(n : Nat) = n \leq 2 \land (\text{odd}(n) \lor X(n+1)))$:

$$\begin{array}{ll} X^0 & \equiv \text{true} \\ X^1 & \equiv n \leq 2 \land (\text{odd}(n) \lor \text{true}) \ \leftrightarrow n \leq 2 \\ X^2 & \equiv n \leq 2 \land (\text{odd}(n) \lor n+1 \leq 2) \ \leftrightarrow n \leq 2 \land (\text{odd}(n) \lor n \leq 1) \ \leftrightarrow n \leq 1 \\ X^3 & \equiv n \leq 2 \land (\text{odd}(n) \lor n+1 \leq 1) \ \leftrightarrow n \leq 2 \land (\text{odd}(n) \lor n = 0) \ \leftrightarrow n \leq 1 \end{array}$$

So, solution to X is $n \leq 1$ (i.e., X semantically consists of the set $\{0, 1\}$)

Gauß Elimination terminates; symbolic approximation may not terminate

- Due to infinite data types, a transfinite approximation may be needed
- Evaluating predicates may be impossible: $\exists k, l, m : Nat. x^k + y^l = z^m$
- Theorem proving technology may be added in symbolic approximation

Exercise

Consider the lossy channel system described by the following LPE:

$$\begin{array}{rcl} C(b:Bool, m:M) & = & \sum_{k:M} b \longrightarrow r(k) \cdot C(\text{false}, k) \\ & + & \neg b \longrightarrow s(m) \cdot C(\text{true}, m) \\ & + & \neg b \longrightarrow l \cdot C(\text{true}, m) \end{array}$$

Action r stands for reading, s stands for sending and l stands for losing a message.

1. $\nu X.([\text{true}]X \wedge (\mu Y.[l]Y \wedge \forall m{:}M.[r(m)]Y \wedge \langle\text{true}\rangle\text{true}))$
2. $\nu X.\mu Y.\nu Z.(\forall m{:}M.[s(m)]X) \wedge ((\forall m{:}M.\ [s(m)]\text{false}) \vee ([l]Y \wedge \forall m{:}M.[r(m)]Y)) \wedge [l]Z \wedge \forall m{:}M.[r(m)]Z$

Questions:

- Explain the first formula in natural language
- Translate both formulae to PBESs given process C
- Use Gauß Elimination to solve the PBES
- For which initial states of C do the properties hold?