# Algorithms for Model Checking (2IMF35)

## Lecture 4
## The $\mu$-Calculus
## (Chapter 7 in *Model Checking* by Clarke, Grumberg & Peled)

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# Outline



$\mu$-Calculus: syntax and semantics

Complexity

Emerson-Lei Algorithm

Embedding CTL-formulae

Conclusions

Exercise

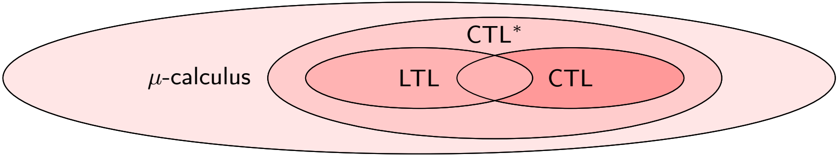
# $\mu$-Calculus: syntax and semantics



Recall: symbolic model checking for CTL was based on fixed points.

Idea of $\mu$-calculus: add fixed point operators as primitives to basic modal logic.

- $\mu$-calculus is very expressive (subsumes CTL, LTL, CTL$^*$).
- $\mu$-calculus is very pure ("assembly language" for modal logic, cf: $\lambda$-calculus for functional programming).
- drawback: lack of intuition.
- fragments of the $\mu$-calculus are the basis for practical model checkers, such as $\mu$CRL, mCRL2, CADP, Concurrency Workbench.



## Kripke Structures and Labelled Transition Systems

Mix of Kripke Systems and Labelled Transition Systems: $M = \langle S, Act, R, L \rangle$ over a set $AP$ of atomic propositions:

- $S$ is a set of states
- $Act$ is a set of action labels
- $R$ is a labelled transition relation: $R \subseteq S \times Act \times S$
- $L$ is a labelling: $L \in S \to 2^{AP}$

Notation: $s \xrightarrow{a} t$ denotes $(s, a, t) \in R$

Special cases:

- Kripke Structures: $Act$ is a singleton (only one transition relation)
- LTS (process algebra): $AP$ is empty (only propositions true and false)

# $\mu$-Calculus: syntax and semantics



Let the following sets be given:

- *AP* (atomic propositions),
- *Act* (action labels) and
- *Var* (formal variables).

The syntax of $\mu$-calculus formulae $f, g$ is defined by the following grammar:

$$f, g ::= \text{true} \mid p \mid X \mid \neg f \mid f \wedge g \mid [a]f \mid \nu X.f$$

Note:

- $p \in AP, X \in Var, a \in Act$.
- $[a]f$ means "for all direct $a$-successors, $f$ holds" (compare to CTL: A X $f$).

**Some notation and terminology:**

- An occurrence of $X$ is bound by a surrounding fixed point symbol $\nu X$. Unbound occurrences of $X$ are called free.
- A formula is closed if it has no free variables, otherwise it is called open
- An environment $e$ interprets the free formal variables $X$ as a set of states
  - Mixed Kripke Structure $M = \langle S, Act, R, L \rangle$
  - $e : Var \to 2^S$
  - $e[X := V]$ is an environment like $e$, but $X$ is set to $V$:

$$e[X := V](Y) := \begin{cases} V & \text{if } Y = X \\ e(Y) & \text{otherwise} \end{cases}$$

- The semantics of a formula $f$ is a set of states of a Mixed Kripke Structure

# $\mu$-Calculus: syntax and semantics

Fix a system: $M = \langle S, Act, R, L \rangle$

- $[\![f]\!]_e$ denotes the set of states where $f$ holds given context $e : Var \to 2^S$:

$$
\begin{array}{lcl}
[\![\mathrm{true}]\!]_e & = & S \\
[\![p]\!]_e & = & \{s \mid p \in L(s)\} \\
[\![X]\!]_e & = & e(X) \\[1ex]
[\![\neg f]\!]_e & = & S \setminus [\![f]\!]_e \\
[\![f \wedge g]\!]_e & = & [\![f]\!]_e \cap [\![g]\!]_e \\[1ex]
[\![[a]f]\!]_e & = & \{s \mid \forall t.\ s \xrightarrow{a} t \ \Rightarrow\ t \in [\![f]\!]_e\} \\[1ex]
[\![\nu X.f]\!]_e & = & \nu(Z \mapsto [\![f]\!]_{e[X:=Z]})
\end{array}
$$

# $\mu$-Calculus: syntax and semantics



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- $[\![\nu X.f]\!]_e$ requires monotonicity of $[\![f]\!]_{e[X:=Z]}$.
- Syntactic Monotonicity Criterion: monotonicity is guaranteed if, in $\nu X.f$, formal variable $X$ occurs under an even number of negations ($\neg$) in $f$.



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# $\mu$-Calculus: syntax and semantics



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- Syntactic Monotonicity Criterion: monotonicity is guaranteed if, in $\nu X.f$, formal variable $X$ occurs under an even number of negations ($\neg$) in $f$.

The semantics immediately gives rise to a naive algorithm for model checking $\mu$-calculus (compute *gfp* by iteration).



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# $\mu$-Calculus: Positive Normal Form



- Extend the grammar with the following shorthands with semantics:

| | |
|---|---|
| $\text{false} := \neg\text{true}$<br>$f \vee g := \neg((\neg f) \wedge (\neg g))$ | $[\![\text{false}]\!]_e = \emptyset$<br>$[\![f \vee g]\!]_e = [\![f]\!]_e \cup [\![g]\!]_e$ |
| $\langle a \rangle f := \neg([a](\neg f))$ | $[\![\langle a \rangle f]\!]_e = \{s \mid \exists t. s \xrightarrow{a} t \wedge t \in [\![f]\!]_e\}$ |
| $\mu X.f := \neg(\nu X.\neg f[X := \neg X])$ | $[\![\mu X.f]\!]_e = \mu(Z \mapsto [\![f]\!]_{e[X:=Z]})$ |

- A $\mu$-calculus formula is in positive normal form if negations occur only in front of propositions.
- Transform a formula into positive normal form by driving negations inward.
- Syntactic monotonicity prevents single negations in front of formal variables.



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# Outline

**Complexity of naive $\mu$-Calculus algorithm**

- We check formula $f$ with at most $k$ nested fixed points on the Kripke Structure $M = \langle S, R, Act, L\rangle$.
- In $\nu X_1.\ \langle a\rangle\big(\mu X_2.\ (X_1 \wedge h) \vee \langle a\rangle X_2\big)$:
  - The outermost (greatest) fixed point can decrease at most $|S|$ times (recall that $S$ is finite)
  - In total, the innermost fixed point of formula $f$ is evaluated at most $|S|^2$ times.
- In general: the innermost fixed point of formula $f$ is evaluated at most $|S|^k$ times.
- Each iteration requires up to $|M| \times |f|$ steps.
- Total time complexity of naive algorithm: $\mathcal{O}((|S| + |R|) \times |f| \times |S|^k)$.

A more careful analysis will yield a more optimal treatment for nested fixed points of the same type.


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- Let $Act = \{a\}$:
  - E G $f$ ............................................................ $\nu X.f \wedge \langle a \rangle X$
  - E [$f$ U $g$] ........................................................ $\mu X.g \vee (f \wedge \langle a \rangle X)$
  - Every $p$ is inevitably followed by a $q$: $\nu X_1. \Big( (p \Rightarrow (\mu X_2.\ q \vee [a]X_2)) \wedge [a]X_1 \Big)$
- Special case: $X_1$ does not occur within the scope of $\mu X_2$.
- The last formula can therefore be evaluated "inside-out":

$$\begin{array}{rcl} X_2^0 & = & \text{false} \\ X_2^1 & = & q \vee [a]X_2^0 \\ X_2^2 & = & q \vee [a]X_2^1 \\ \ldots\ X_2^\omega & = & q \vee [a]X_2^\omega \end{array} \quad \Longrightarrow \quad \begin{array}{rcl} X_1^0 & = & \text{true} \\ X_1^1 & = & (p \Rightarrow X_2^\omega) \wedge [a]X_1^0 \\ X_1^2 & = & (p \Rightarrow X_2^\omega) \wedge [a]X_1^1 \\ \ldots\ X_1^\omega & = & (p \Rightarrow X_2^\omega) \wedge [a]X_1^\omega \end{array}$$

## A more difficult case

- On some path, $h$ holds infinitely often: $\nu X_1.\ \langle a\rangle\big(\mu X_2.\ (X_1 \wedge h) \vee \langle a\rangle X_2\big)$
- Problem: the inner fixed point depends crucially on $X_1$.

$$
\begin{array}{llll}
X_1^0 & = & \text{true} & \\
 & & & X_2^{00} = \text{false} \\
 & & & X_2^{01} = (X_1^0 \wedge h) \vee \langle a\rangle X_2^{00} \\
 & & & X_2^{02} = (X_1^0 \wedge h) \vee \langle a\rangle X_2^{01} \\
 & & \ldots & X_2^{0\omega} = (X_1^0 \wedge h) \vee \langle a\rangle X_2^{0\omega} \\
X_1^1 & = & \langle a\rangle X_2^{0\omega} & \\
 & & & X_2^{10} = \text{false} \\
 & & & X_2^{11} = (X_1^1 \wedge h) \vee \langle a\rangle X_2^{10} \\
 & & \ldots & X_2^{1\omega} = (X_1^1 \wedge h) \vee \langle a\rangle X_2^{1\omega} \\
X_1^2 & = & \langle a\rangle X_2^{1\omega} & \\
\ldots\ X_1^\omega & = & \langle a\rangle X_2^{\omega\omega} &
\end{array}
$$

# Complexity



The complexity of a $\mu$-calculus formula depends on the fixed points (*analogue:* the complexity of first-order formulae depends on the universal/existential quantifiers and their alternations)

- Basic idea: find a syntactic complexity measure that approaches the semantic complexity
- Nesting Depth:
  maximum number of nested fixed points in a positive normal form

$$
\begin{array}{rcll}
ND(f) & := & 0 & \text{for } f \in \{p, \neg p, X\} \\
ND(ⓐ f) & := & ND(f) & \text{for } ⓐ \in \{[a], \langle a \rangle\} \\
ND(f \Box g) & := & max(ND(f), ND(g)) & \text{for } \Box \in \{\wedge, \vee\} \\
ND({}^{\mu}_{\nu}\, X.f) & := & 1 + ND(f) & \text{for } {}^{\mu}_{\nu} \in \{\mu, \nu\}
\end{array}
$$

- Example: $ND\Big((\mu X_1.\ \nu X_2.\ X_1 \vee X_2) \wedge (\mu X_3.\ \mu X_4.\ (X_3 \wedge \mu X_5.\ p \vee X_5))\Big)$

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| $ND(f \Box g)$ | $:=$ | $max(ND(f), ND(g))$ | for $\Box \in \{\wedge, \vee\}$ |
| $ND(^{\mu}_{\nu} X.f)$ | $:=$ | $1 + ND(f)$ | for $^{\mu}_{\nu} \in \{\mu, \nu\}$ |

- Example: $ND\Big((\mu X_1.\ \nu X_2.\ X_1 \vee X_2) \wedge (\mu X_3.\ \mu X_4.\ (X_3 \wedge \mu X_5.\ p \vee X_5))\Big) = 3$
- $X_3, X_4$ and $X_5$ have no alternation between fixed point signs



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# Complexity

- Capture alternation
- Alternation Depth: number of alternating fixed points of a formula in positive normal form.

$$
\begin{array}{rcll}
AD(f) & := & 0 & \text{for } f \in \{p, \neg p, X\} \\
AD(ⓐ f) & := & AD(f) & \text{for } ⓐ \in \{[a], \langle a \rangle\} \\
AD(f \Box g) & := & max(AD(f), AD(g)) & \text{for } \Box \in \{\wedge, \vee\} \\
AD(\mu X.f) & := & 1 + max\{AD(g) \mid g \text{ is a } \nu\text{-subformula of } f\} & \\
AD(\nu X.f) & := & 1 + max\{AD(g) \mid g \text{ is a } \mu\text{-subformula of } f\} &
\end{array}
$$

- Examples:

$$AD\Big((\mu X_1.\ \nu X_2.\ X_1 \vee X_2) \wedge (\mu X_3.\mu X_4.\ (X_3 \wedge \mu X_5.p \vee X_5))\Big)$$

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- Examples:

$$AD\Big((\mu X_1.\ \nu X_2.\ X_1 \vee X_2) \wedge (\mu X_3.\mu X_4.\ (X_3 \wedge \mu X_5.p \vee X_5))\Big) = 2$$

$$AD\Big((\mu X_1.\ \nu X_2.\ X_1 \vee X_2) \wedge (\mu X_3.\nu X_4.\ (X_3 \wedge \mu X_5.p \vee X_5))\Big) = 3$$

- $X_5$ does not depend on $X_3$ and $X_4$



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# Complexity

- Dependent Alternation Depth (dAD): number of alternating fixed points, such that the innermost fixed point depends on the outermost.
- The definition of $dAD$ is identical to $AD$, except for

$$
\begin{array}{lcl}
dAD(\mu X.f) & := & max(dAD(f), \\
 & & \quad 1 + max\{dAD(g) \mid \\
 & & \qquad g \text{ is a } \nu\text{-subformula of } f \text{ and } X \text{ occurs in } g\} \\
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- Examples:

$$dAD\Big((\mu X_1.\ \nu X_2.\ X_1 \vee X_2) \wedge (\mu X_3.\mu X_4.\ (X_3 \wedge \mu X_5.p \vee X_5))\Big)$$

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- Examples:

$$dAD\Big((\mu X_1.\ \nu X_2.\ X_1 \vee X_2) \wedge (\mu X_3.\mu X_4.\ (X_3 \wedge \mu X_5.p \vee X_5))\Big) = 2$$

$$dAD\Big((\mu X_1.\ \nu X_2.\ X_1 \vee X_2) \wedge (\mu X_3.\nu X_4.\ (X_3 \wedge \mu X_5.p \vee X_5))\Big) = 2$$

# Outline

# Emerson-Lei Algorithm



- Given a finite set $S$ and a monotonic $\tau : 2^S \to 2^S$ in the partial order $(2^S, \subseteq)$.
- We used to compute the least fixed point from $\emptyset$:

$$\emptyset \subseteq \tau(\emptyset) \subseteq \tau^2(\emptyset) \subseteq \ldots \subseteq \tau^i(\emptyset) = \tau^{i+1}(\emptyset)$$

  then $\mu X.\tau(X) = \tau^i(\emptyset)$
- Actually, instead of $\emptyset$, we can start in any set known to be smaller than the fixed point:

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  then $\mu X.\tau(X) = \tau^i(\emptyset)$
- Actually, instead of $\emptyset$, we can start in any set known to be smaller than the fixed point:
  - Assume $W \subseteq \mu X.\tau(X)$, so we have:

$$\emptyset \subseteq W \subseteq \tau^i(\emptyset)$$

  - By monotonicity and the definition of fixed points:

$$\tau^i(\emptyset) \subseteq \tau^i(W) \subseteq \tau^{2i}(\emptyset) = \tau^i(\emptyset)$$

  - So if $W \subseteq \mu X.\tau(X)$ we compute the least fixed point as:

$$W, \tau(W), \tau^2(W), \ldots, \tau^j(W) = \tau^{j+1}(W)$$

    This converges at some $j \leq i$ (may be $j < i$)

# Emerson-Lei Algorithm



- The observations on the previous slide can speed up computations of nested fixed points.
- Consider two nested $\mu$-fixed points: $\mu X_1.f(X_1, \mu X_2.\ g(X_1, X_2))$
- Start approximation of $X_1$ and $X_2$ with $X_1^0 = X_2^0 = \text{false}$:

$$\begin{array}{llcll} X_1^0 & = \text{false} & & & \\ & & & X_2^{00} & = \text{false} \\ & & & X_2^{01} & = g(X_1^0, X_2^{00}) \\ & & \ldots & X_2^{0\omega} & = g(X_1^0, X_2^{0\omega}) \\ X_1^1 & = f(X_1^0, X_2^{0\omega}) & & & \end{array}$$

- Clearly, $X_1^0 \subseteq X_1^1$, so also $X_2^{0\omega} = \mu X_2.g(X_1^0, X_2) \subseteq \mu X_2.g(X_1^1, X_2) = X_2^{1\omega}$. So, approximating $X_2$ can start at $X_2^{0\omega}$ instead of at false:

$$\begin{array}{llcll} & & & X_2^{10} & = X_2^{0\omega} \\ & & \ldots & X_2^{1\omega} & = g(X_1^1, X_2^{1\omega}) \\ X_1^2 & = f(X_1^1, X_2^{1\omega}) & & & \end{array}$$

# Emerson-Lei Algorithm

Given:

- Mixed Kripke Structure: $M = \langle S, R, Act, L \rangle$
- A $\mu$-Calculus formula $f$ and an environment $e$

Returns: $[\![f]\!]_e$, the set of states in $S$ where $f$ holds.

Idea:

- The function eval($f$) proceeds by recursion on $f$, using iteration for the fixed points.
- The value of the current approximation for variable $X_i$ is stored in array $A[i]$, in order to reuse it in later iterations.
- Reset $A[i]$ only if:
  - a higher $X_j$ of different sign changed, and
  - $^{\mu}_{\nu} X_i.f$ contains free variables.

Initialisation:

**for all** variables $X_i$ **do**
  **if** $X_i$ is bound by a $\mu$ **then** $A[i] :=$ false;
  **else if** $X_i$ is bound by a $\nu$ **then** $A[i] :=$ true;
  **else** $A[i] := e(X_i)$
  **end if**
**end for**

# Emerson-Lei algorithm



**function** eval($f$)
    **if** $f = X_i$ **then return** $A[i]$
    **else if** $f = g_1 \vee g_2$ **then return** eval($g_1$) $\cup$ eval($g_2$)
    **else if** ... **then** ...
    **else if** $f = \mu X_i.g(X_i)$ **then**
        **if** the surrounding binder of $f$ is a $\nu$ **then**
            **for all** open subformulae of $f$ of the form $\mu X_k.g$ **do** $A[k] := \mathit{false}$
            **end for**
        **end if**
        **repeat**
            $X_{old} := A[i]$; {continue from previous value}
            $A[i] :=$ eval($g$);
        **until** $A[i] = X_{old}$
        **return** $A[i]$
    **end if**
**end function**

# Emerson-Lei algorithm



Given a formula $\nu X_1.\nu X_2.\mu X_3.\mu X_4.(X_1 \vee X_2 \vee (\mu X_5.X_5 \wedge p))$

- When computing $\nu X_2$, $\mu X_4$ and $\mu X_5$: no reset is needed because the surrounding binder has the same sign.
- When computing $X_3$:
  - Reset $X_3, X_4$: their subformula contains $X_1$ and $X_2$ as free variables
  - Do not reset $X_5$: the subformula $(\mu X_5.X_5 \wedge p)$ is closed

Modifications with respect to the book (p. 105):

- We identified $e$ and $A[i]$ (they play the same role)
- The restriction to reset open formulae only makes the algorithm more efficient. This is essential for CTL (see later).
- The book has a slightly different algorithm (correctness unclear to me): we presented the original Emerson and Lei algorithm (1986).



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# Emerson-Lei algorithm



Complexity analysis

- Let formula $f$ be given, with dependent alternation depth $dAD(f) = d$.
- Let the Kripke Structure be $\langle S, Act, R, L \rangle$.
- Take a block of fixed points of the same type:
  - its length is at most $|f|$.
  - the value of each fixed point in it can grow/shrink at most $|S|$ times.
- In total, the innermost block will have no more than $(|f| \cdot |S|)^d$ iterations of the repeat-loop.
- Each iteration requires time at most $\mathcal{O}(|f| \cdot (|S| + |R|))$.
- Hence: the overall complexity of the Emerson-Lei algorithm is $\mathcal{O}(|f| \cdot (|S| + |R|) \cdot (|f| \cdot |S|)^d)$



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# Outline

# Embedding CTL-formulae



Again, assume $Act = \{a\}$. Given the fixed point characterisation of CTL, there is a straightforward translation of CTL to the $\mu$-calculus:

- $Tr(p) = p$
- $Tr(\neg f) = \neg Tr(f)$
- $Tr(f \wedge g) = Tr(f) \wedge Tr(g)$
- $Tr(\mathsf{E}\ \mathsf{X}\ f) = \langle a \rangle\ Tr(f)$
- $Tr(\mathsf{E}\ \mathsf{G}\ f) = \nu Y.(Tr(f) \wedge \langle a \rangle\ Y)$
- $Tr(\mathsf{E}\ [f\ \mathsf{U}\ g]) = \mu Y.(Tr(g) \vee (Tr(f) \wedge \langle a \rangle\ Y))$

Note:

- $Tr(f)$ is syntactically monotone
- $Tr(f)$ is a closed $\mu$-calculus formula
- $dAD(Tr(f)) \leq 1$, which is called the alternation free fragment of the $\mu$-calculus
- $AD(Tr(f))$ is not bounded!



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# Conclusions



- the $\mu$-calculus incorporates least and greatest fixed points directly in the logic.
- the naive algorithm is exponential in the nesting depth of fixed points.
- a careful analysis leads to an algorithm which is exponential in the (dependent) alternation depth only,
- Hence: alternation free $\mu$-calculus is linear in the Kripke Structure and polynomial in the formula.
- CTL translates into the alternation free fragment of the $\mu$-calculus.
- for the latter we essentially needed the dependent alternation depth.
- fairness constraints typically lead to one extra alternation ($dAD(f) = 2$)

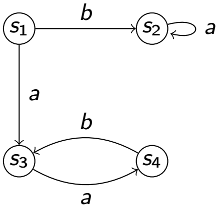
# Outline

# Exercise



Consider the following $\mu$-calculus formula $\phi$ and LTS $\mathcal{L}$:

$$\phi := \nu X.\Big([a]X \wedge \nu Y.\mu Z.(\langle b\rangle Y \vee \langle a\rangle Z)\Big)$$




- Compute the set of states where $\phi$ holds with the naive algorithm (give all intermediate approximations).
- Compute the set of states where $\phi$ holds with the Emerson-Lei's algorithm (give all intermediate approximations).
- Explain in natural language the meaning of formula $\phi$.