Algorithms for Model Checking (2IMF35)

Lecture 8:
Small Progress Measures for Solving Parity Games

Background material:

M. Jurdziński, *Small Progress Measures for Solving Parity Games*

Tim Willemse
(timw@win.tue.nl)
http://www.win.tue.nl/∼timw
MF 6.073

Algorithms for Parity games

- McNaughton's/Zielonka's Recursive algorithm
- Today: Jurdziński's Small progress measures

Play values

Throughout this lecture: $G = (V, E, p, (V_\diamond, V_\Box))$ is an arbitrary parity game

Let $\pi = v_0 v_1 v_2 \dots$ be a play.

- dominating priority occurring infinitely often on π is odd . . . . π won by $\Box$
- dominating priority occurring infinitely often on π is even . . . . π won by $\diamond$

Observe: dominating priority on π is even iff every 'odd-dominated stretch' is finite

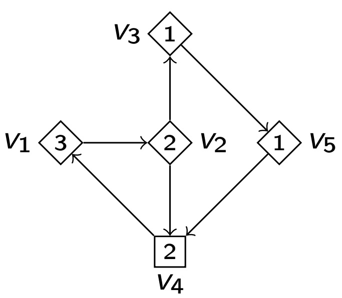
Definition (Stretch and k-dominated stretch)

- A stretch of play π is a subsequence $v_i v_{i+1} \dots v_{i+\ell}$
- A stretch $v_i v_{i+1} \dots v_{i+\ell}$ is k-dominated iff $p(v_{i+j}) \geq k$ for $0 \leq j \leq \ell$

Degree of a k-dominated stretch: the number of vertices with priority k in that stretch

Stretches and Degrees

Example



- $\pi = (v_2\ v_4\ v_1)^\omega$
  - 3-dominated stretch in π of degree 1: v_1
- $\pi = (v_2\ v_3\ v_5\ v_4\ v_1)^\omega$
  - 1-dominated stretch in π of degree 2: $v_3\ v_5\ v_4$ and $v_3\ v_5$ and $v_2\ v_3\ v_5$
  - 1-dominated stretch in π of degree 1: $v_2\ v_3$ and v_3
  - 3-dominated stretch in π of degree 1: v_1

Play values

- Let $d = 1 + \max\{p(v) \mid v \in V\}$
- $M \subseteq \mathbb{N}^d \cup \{\top\}$ is a set of measures with 0 on even positions (counting from 0)
- $\leq$ is lexicographic ordering on M with $m \leq \top$ for all m

Definition (Play values)

A play value is a function $\theta_\diamond : \mathit{Plays} \to M$ defined as:

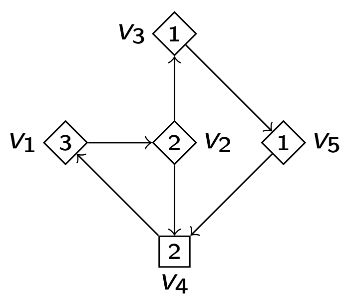
$$\theta_\diamond(\pi) = \begin{cases} (m_0, \dots, m_{d-1}) & \text{where, if } \pi \text{ is winning for } \diamond\text{, for all odd } i\text{,} \\ & m_i \text{ is the degree of the maximal } i\text{-dominated prefix of } \pi \\ \top & \text{if } \pi \text{ is won by } \square \end{cases}$$

Property

Player $\square$ wins v iff for all $\sigma : V_\diamond \to V$ there is some $\pi \in \mathit{Play}_\sigma(v)$, $\theta_\diamond(\pi) = \top$

Play values

Example



- $\theta_\diamond((v_2\ v_4\ v_1)^\omega) = (0, 0, 0, 0)$:
  - degree of 1-dominated and 3-dominated prefixes: 0
- $\theta_\diamond((v_1\ v_2\ v_4)^\omega) = (0, 0, 0, 1)$:
  - degree of 1-dominated prefix: 0; degree of 3-dominated prefix: 1
- $\theta_\diamond((v_2\ v_3\ v_5\ v_4\ v_1)^\omega) = \top$:
  - play is 1-dominated (i.e. won by $\square$)
- $\theta_\diamond(v_2\ v_3\ v_5\ v_4\ v_1\ (v_2\ v_4\ v_1)^\omega) = (0, 2, 0, 0)$
  - prefix $v_2\ v_3\ v_5\ v_4\ v_1$ is 1-dominated of degree 2
  - no 3-dominated stretch starting in v_2

Play values

Key idea behind Small Progress Measures:

compute some $\varrho : V \to M$ such that for all $v \in V$:

- there is some strategy σ for $\Diamond$ such that for each $\pi \in Play_\sigma(v)$: $\theta_\Diamond(\pi) \leq \varrho(v)$
- there is some strategy σ for $\Box$ such that for each $\pi \in Play_\sigma(v)$: $\theta_\Diamond(\pi) \geq \varrho(v)$

Observe:

- M is infinite .... we need to have upper bounds on $\varrho(v)$
- If the degree of a k-dominated stretch exceeds $|V_k|$ .... stretch revisits a vertex
- Revisiting vertices .... cycles

Play values

Bird's-eye view of small progress measures

- *Solitaire games and reachable cycles*
  - Cycles can be used to decide the winner.
- Assign a certain *measure* to each vertex that approximates play values
  - solitaire games: parity progress measures
  - two-player games: game parity progress measures
- Efficiently compute measure
  - fixed point approximation

Definition (Solitaire game)

G is an $\Box$-solitaire game if only $\Box$ makes (nontrivial) choices;

i.e. for all vertices $v \in V_\Diamond$ we have $|\{w \in V \mid (v,w) \in E\}| \leq 1$

Strategy σ for player $\Diamond$ in G induces an $\Box$-solitaire game G_σ:

$G_\sigma = (V, E_\sigma, p, (V_\Diamond, V_\Box))$, where

$$E_\sigma = \{(v,w) \in E \mid v \in V_\Diamond \Rightarrow w = \sigma(v)\}\}$$

Property

If σ is closed on some $W \subseteq V$, then $G_\sigma \cap W$ is also an $\Box$-solitaire game

Note: $G_\sigma \cap W := G_\sigma \setminus (V \setminus W)$

- An even cycle is a cycle in which the lowest priority is even
- Let σ be an $\Diamond$-strategy closed on W

Property

σ is winning for player $\Diamond$ from all $v \in W$ iff all cycles in $G_\sigma \cap W$ are even

We will annotate vertices with information ('measures') about plays such that:

- when, along a play we encounter priority i, we will ignore information about less significant priorities (i.e., $> i$)
- the information we record about priorities k outweighs information about $\ell > k$
  - vertex with 'bad' priority: measure of such vertex is larger than its neighbours
  - vertex with 'good' priority: measure of such vertex may be smaller than its neighbours

Represent information as follows:

- Tuples to record information about priorities
- Order tuples lexicographically (same as measures in play values)

Let $\alpha \in \mathbb{N}^d$ be a d-tuple of natural numbers

- We number its components from 0 to $d-1$, i.e. $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{d-1})$
- $<, \leq, =, \neq, \geq, >$ on tuples denote lexicographic ordering
- $(n_0, n_1, \dots, n_k) \equiv_i (m_0, m_1, \dots, m_\ell)$ iff $(n_0, n_1, \dots, n_i) \equiv (m_0, m_1, \dots, m_i)$, for $\equiv \in \{<, \leq, =, \neq, \geq, >\}$
- When $i > k$ or $i > \ell$, the tuples will be suffixed with 0s

Progress Measures

Example (d-tuples)

- $(0,1,0,1) =_0 (0,2,0,1) \equiv (0) = (0) \equiv$ true
- $(0,1,0,1) <_1 (0,2,0,1) \equiv (0,1) < (0,2) \equiv$ true
- $(0,1,0,1) \geq_3 (0,2,0,1) \equiv (0,1,0,1) \geq (0,2,0,1) \equiv$ false

Parity progress measures and Solitaire games

Let $d = 1 + \max\{p(v) \mid v \in V\}$.

- Define $V_i = \{v \in V \mid p(v) = i\}$,
- Denote $n_i = \mid V_i \mid$, the number of vertices with priority i,

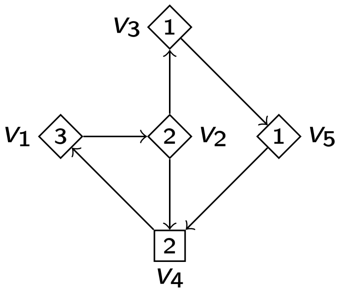
Define $\mathbb{M} \subseteq \mathbb{N}^d$ with:

- 0 on even positions
- Natural numbers $\leq n_i$ on odd positions i

Progress Measures

Example

Determine maximum value of $\mathbb{M}$ for the following parity game:



- Maximum value of $\mathbb{M}$ is $(0, 2, 0, 1)$
- $\mathbb{M} = \{0\} \times \{0, 1, 2\} \times \{0\} \times \{0, 1\}$

Parity progress measures and Solitaire games

Definition (Parity progress measure)

Let G be an $\Box$-solitaire game. Mapping $\varrho : V \to \mathbb{M}$ is a parity progress measure for G if for all $(v, w) \in E$:

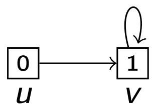
- $\varrho(v) \geq_{p(v)} \varrho(w)$ if $p(v)$ is even
- $\varrho(v) >_{p(v)} \varrho(w)$ if $p(v)$ is odd

For all strategies σ for player $\Diamond$, closed on W:

- Recall: σ is winning for player $\Diamond$ from W iff all cycles in $G_\sigma \cap W$ are even
- All cycles in $G_\sigma \cap W$ are even iff there exists a parity progress measure ϱ for $G_\sigma \cap W$

Progress Measures

Problem: parity progress measures only exist for even-dominated cycles.



Second clause requires $\varrho(v) >_1 \varrho(v)$

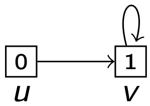
Parity progress measures and Solitaire games

Solitaire games with odd-dominated cycles.

- Define $\mathbb{M}^\top = \mathbb{M} \cup \{\top\}$
- Extend ordering:
  - for all $m \in \mathbb{M}$, define $m < \top$, $m <_i \top$, $m \neq \top$ and $m \neq_i \top$
  - $\top \leq_i \top$ for all i
- The set of mappings $([V \to \mathbb{M}^\top], \sqsubseteq)$ is a complete lattice
  - $\varphi, \varrho : V \to \mathbb{M}^\top$.
  - Define $\varphi \sqsubseteq \varrho$ if $\varphi(v) \leq \varrho(v)$ for all $v \in V$
  - write $\varphi \sqsubset \varrho$ if $\varphi \sqsubseteq \varrho$ and $\varphi \neq \varrho$.
- Replace co-domain of parity progress measures with $\mathbb{M}^\top$.

Progress Measures

Example



- Observe: $\varrho(u) = \varrho(v) = \top$
- Measure can identify both even and odd reachable cycles in a solitaire game.

Game Parity Progress Measures and Solitaire games

Definition (Prog)

If $\varrho{:}V \to \mathbb{M}^\top$ and $(v, w) \in E$, then $Prog(\varrho, v, w)$ is the least $m \in \mathbb{M}^\top$, such that

- if $p(v)$ is even, then $m \geq_{p(v)} \varrho(w)$
- if $p(v)$ is odd, then either $m >_{p(v)} \varrho(w)$, or both $m = \varrho(w) = \top$

Define $Inc_v(\varrho)$ for $v \in V$ as $\varrho[v := \varrho(v) \max \max\{Prog(\varrho, v, w) \mid (v, w) \in E\}]$

- For every $v \in V$, Inc_v is $\sqsubseteq$-monotone
- The least ϱ satisfying $Inc_v(\varrho) \sqsubseteq \varrho$ for all v is a parity progress measure
- can be computed using fixpoint approximation

For an $\Box$-solitaire game and least parity progress measure ϱ:

- $W_\Diamond = \{v \in V \mid \exists \varrho : \ \varrho(v) \neq \top\}$
- $W_\Box = V \setminus W_\Diamond$.

Example

Let $\mathbb{M} = \{0\} \times \{0,1,2\} \times \{0\} \times \{0,1\}$

- Suppose $p(v) = 0$, $\varrho(w) = (0,2,0,0)$.
  Then $Prog(\varrho, v, w) = (0,0,0,0)$
- Suppose $p(v) = 1$, $\varrho(w) = (0,2,0,0)$.
  Then $Prog(\varrho, v, w) = \top$
- Suppose $p(v) = 3$, $\varrho(w) = (0,2,0,0)$.
  Then $Prog(\varrho, v, w) = (0,2,0,1)$

Definition (Game parity progress measure)

Mapping $\varrho : V \to \mathbb{M}^\top$ is a game parity progress measure if for all $v \in V$:

- if $v \in V_\diamond$, then $\exists_{(v,w)\in E}\ \varrho(v) \geq_{p(v)} Prog(\varrho, v, w)$
- if $v \in V_\Box$, then $\forall_{(v,w)\in E}\ \varrho(v) \geq_{p(v)} Prog(\varrho, v, w)$

If ϱ is the least game parity progress measure for G, then:

$$\varrho(v) \neq \top$$
$$\Leftrightarrow$$
player $\diamond$ can prevent reaching $\Box$-dominated cycles

For the least game parity progress measure ϱ we have:

- there is some $\diamond$ strategy σ such that for each $\pi \in Play_\sigma(v)$: $\theta_\diamond(\pi) \leq \varrho(v)$
- there is some $\Box$ strategy σ such that for each $\pi \in Play_\sigma(v)$: $\theta_\diamond(\pi) \geq \varrho(v)$

Recall: the set of mappings $([V \to \mathbb{M}^{\top}], \sqsubseteq)$ is a complete lattice

Define $Lift_v(\varrho)$ for $v \in V$ as follows:

$$\begin{cases} \varrho[v := \varrho(v) \ \max\ \min\{Prog(\varrho, v, w) \mid (v,w) \in E\}] & \text{if } v \in V_\Diamond \\ \varrho[v := \varrho(v) \ \max\ \max\{Prog(\varrho, v, w) \mid (v,w) \in E\}] & \text{if } v \in V_\Box \end{cases}$$

Observe:

- For every $v \in V$, $Lift_v$ is $\sqsubseteq$-monotone.
- A mapping $\varrho{:}V \to \mathbb{M}^{\top}$ is a game parity progress measure if and only if $Lift_v(\varrho) \sqsubseteq \varrho$ for all $v \in V$.
- Least game parity progress measure computable by fixpoint approximation (algorithm Lfp of Lecture 2)

Algorithm $SPM(G)$

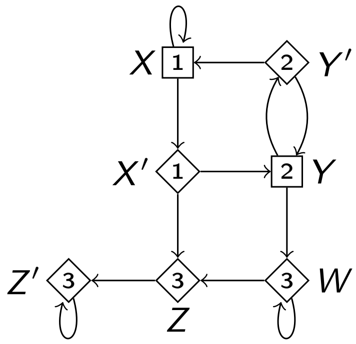
$\varrho\colon V \to \mathbb{M}^{\top} \leftarrow \lambda v \in V.(0, \ldots, 0)$
while $\varrho \sqsubset Lift_v(\varrho)$ for some $v \in V$ **do**
  $\varrho \leftarrow Lift_v(\varrho)$
end while

Post condition:

- ϱ is least game parity progress measure
- $\{v \in V \mid \varrho(v) \neq \top\}$ is winning set for player $\Diamond$
- $\{v \in V \mid \varrho(v) = \top\}$ is winning set for player $\Box$

Small progress measures (example)

Consider parity game G:



Maximum value of $\mathbb{M}$ is $(0, 2, 0, 3)$

Small progress measures (example) (1)

Initially: $\varrho \leftarrow \lambda v \in V.(0, 0, 0, 0)$, so

v	$\varrho(v)$
X	$(0, 0, 0, 0)$
X'	$(0, 0, 0, 0)$
Y	$(0, 0, 0, 0)$
Y'	$(0, 0, 0, 0)$
Z	$(0, 0, 0, 0)$
Z'	$(0, 0, 0, 0)$
W	$(0, 0, 0, 0)$

Small progress measures (example) (2)

Step 2: $\varrho \leftarrow \mathit{Lift}_X(\varrho) = \varrho[X := \max\{\mathit{Prog}(\varrho, X, X'), \mathit{Prog}(\varrho, X, X)\}] = \varrho[X := \max\{(0,1,0,0),(0,1,0,0)\}] = \varrho[X := (0,1,0,0)]$

v	$\varrho(v)$
X	(0, 1, 0, 0)
X'	(0, 0, 0, 0)
Y	(0, 0, 0, 0)
Y'	(0, 0, 0, 0)
Z	(0, 0, 0, 0)
Z'	(0, 0, 0, 0)
W	(0, 0, 0, 0)

Small progress measures (example) (3)

Step 3: $\varrho \leftarrow \mathit{Lift}_X(\varrho) = \varrho[X := \max\{\mathit{Prog}(\varrho, X, X'), \mathit{Prog}(\varrho, X, X)\}] = \varrho[X := \max\{(0,1,0,0),(0,2,0,0)\}] = \varrho[X := (0,2,0,0)]$

v	$\varrho(v)$
X	(0, 2, 0, 0)
X'	(0, 0, 0, 0)
Y	(0, 0, 0, 0)
Y'	(0, 0, 0, 0)
Z	(0, 0, 0, 0)
Z'	(0, 0, 0, 0)
W	(0, 0, 0, 0)

Small progress measures (example) (4)

Step 4: $\varrho \leftarrow \mathit{Lift}_X(\varrho) = \varrho[X := \max\{\mathit{Prog}(\varrho, X, X'), \mathit{Prog}(\varrho, X, X)\}] = \varrho[X := \max\{(0,1,0,0), \top\}] = \varrho[X := \top]$

v	$\varrho(v)$
X	$\top$
X'	$(0,0,0,0)$
Y	$(0,0,0,0)$
Y'	$(0,0,0,0)$
Z	$(0,0,0,0)$
Z'	$(0,0,0,0)$
W	$(0,0,0,0)$

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Small progress measures (example) (5)

Step 5: $\mathit{Lift}_{Y'}(\varrho) = \varrho[Y' := \min\{\mathit{Prog}(\varrho, Y', X), \mathit{Prog}(\varrho, Y', Y)\}] = \varrho[Y' := \min\{\top, (0,0,0,0)\}] = \varrho[Y' := (0,0,0,0)]$
$\mathit{Lift}_Y(\varrho) = \varrho[Y := \max\{\mathit{Prog}(\varrho, Y, W), \mathit{Prog}(\varrho, Y, Y')\}] = \varrho[Y := \max\{(0,0,0,0), (0,0,0,0)\}] = \varrho[Y := (0,0,0,0)]$
$\varrho \leftarrow \mathit{Lift}_{X'}(\varrho) = \varrho[X' := \min\{\mathit{Prog}(\varrho, X', Y), \mathit{Prog}(\varrho, X', Z)\}] = \varrho[X' := \min\{(0,1,0,0), (0,1,0,0)\}] = \varrho[X' := (0,1,0,0)]$

v	$\varrho(v)$
X	$\top$
X'	$(0,1,0,0)$
Y	$(0,0,0,0)$
Y'	$(0,0,0,0)$
Z	$(0,0,0,0)$
Z'	$(0,0,0,0)$
W	$(0,0,0,0)$

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Small progress measures (example) (6)

Step 6: $\varrho \leftarrow \mathit{Lift}_{Z'}(\varrho) = \varrho[Z' := \min\{\mathit{Prog}(\varrho, Z', Z')\}] = \varrho[Z' := \min\{(0,0,0,1)\}] = \varrho[Z' := (0,0,0,1)]$

v	$\varrho(v)$
X	$\top$
X'	$(0,1,0,0)$
Y	$(0,0,0,0)$
Y'	$(0,0,0,0)$
Z	$(0,0,0,0)$
Z'	$(0,0,0,1)$
W	$(0,0,0,0)$

Small progress measures (example) (7)

Step 7: $\varrho \leftarrow \mathit{Lift}_{Z'}(\varrho) = \varrho[Z' := \min\{\mathit{Prog}(\varrho, Z', Z')\}] = \varrho[Z' := \min\{(0,0,0,2)\}] = \varrho[Z' := (0,0,0,2)]$

v	$\varrho(v)$
X	$\top$
X'	$(0,1,0,0)$
Y	$(0,0,0,0)$
Y'	$(0,0,0,0)$
Z	$(0,0,0,0)$
Z'	$(0,0,0,2)$
W	$(0,0,0,0)$

Small progress measures (example) (8)

Step 8: $\varrho \leftarrow \mathit{Lift}_{Z'}(\varrho) = \varrho[Z' := \min\{\mathit{Prog}(\varrho, Z', Z')\}] = \varrho[Z' := \min\{(0,0,0,3)\}] = \varrho[Z' := (0,0,0,3)]$

v	$\varrho(v)$
X	$\top$
X'	$(0,1,0,0)$
Y	$(0,0,0,0)$
Y'	$(0,0,0,0)$
Z	$(0,0,0,0)$
Z'	$(0,0,0,3)$
W	$(0,0,0,0)$

Small progress measures (example) (9)

Step 9: $\varrho \leftarrow \mathit{Lift}(\varrho, Z') = \varrho[Z' := \min\{\mathit{Prog}(\varrho, Z', Z')\}] = \varrho[Z' := \min\{(0,1,0,0)\}] = \varrho[Z' := (0,1,0,0)]$

v	$\varrho(v)$
X	$\top$
X'	$(0,1,0,0)$
Y	$(0,0,0,0)$
Y'	$(0,0,0,0)$
Z	$(0,0,0,0)$
Z'	$(0,1,0,0)$
W	$(0,0,0,0)$

Small progress measures (example) (10)

Step 10: $\varrho \leftarrow \mathit{Lift}_{Z'}(\varrho) = \varrho[Z' := \min\{\mathit{Prog}(\varrho, Z', Z')\}] = \varrho[Z' := \min\{(0,1,0,1)\}] = \varrho[Z' := (0,1,0,1)]$

v	$\varrho(v)$
X	$\top$
X'	$(0,1,0,0)$
Y	$(0,0,0,0)$
Y'	$(0,0,0,0)$
Z	$(0,0,0,0)$
Z'	$(0,1,0,1)$
W	$(0,0,0,0)$

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Small progress measures (example) (11)

Step 11*: Repeat lifting Z' even more often
$\varrho \leftarrow \mathit{Lift}_{Z'}(\varrho) = \varrho[Z' := \min\{\mathit{Prog}(\varrho, Z', Z')\}] = \varrho[Z' := \min\{\top\}] = \varrho[Z' := \top]$

v	$\varrho(v)$
X	$\top$
X'	$(0,1,0,0)$
Y	$(0,0,0,0)$
Y'	$(0,0,0,0)$
Z	$(0,0,0,0)$
Z'	$\top$
W	$(0,0,0,0)$

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Small progress measures (example) (12)

Step 12: $\varrho \leftarrow \mathit{Lift}_Z(\varrho) = \varrho[Z := \min\{\mathit{Prog}(\varrho, Z, Z')\}] = \varrho[Z := \min\{\top\}] = \varrho[Z := \top]$

v	$\varrho(v)$
X	$\top$
X'	$(0, 1, 0, 0)$
Y	$(0, 0, 0, 0)$
Y'	$(0, 0, 0, 0)$
Z	$\top$
Z'	$\top$
W	$(0, 0, 0, 0)$

Small progress measures (example) (13)

Step 13:
$\varrho \leftarrow \mathit{Lift}_W(\varrho) = \varrho[W := \min\{\mathit{Prog}(\varrho, W, Z), \mathit{Prog}(\varrho, W, W')\}] = \varrho[W := \min\{\top, (0, 0, 0, 1)\}] = \varrho[W := (0, 0, 0, 1)]$

v	$\varrho(v)$
X	$\top$
X'	$(0, 1, 0, 0)$
Y	$(0, 0, 0, 0)$
Y'	$(0, 0, 0, 0)$
Z	$\top$
Z'	$\top$
W	$(0, 0, 0, 1)$

Small progress measures (example) (14)

Step 14*: Repeat lifting of W often
$\varrho \leftarrow \mathit{Lift}_W(\varrho) = \varrho[W := \min\{\mathit{Prog}(\varrho, W, Z), \mathit{Prog}(\varrho, W, W')\}] = \varrho[W := \min\{\top, \top\}] = \varrho[W := \top]$

v	$\varrho(v)$
X	$\top$
X'	$(0, 1, 0, 0)$
Y	$(0, 0, 0, 0)$
Y'	$(0, 0, 0, 0)$
Z	$\top$
Z'	$\top$
W	$\top$

Small progress measures (example) (15)

Step 15: $\varrho \leftarrow \mathit{Lift}_Y(\varrho, Y) = \varrho[Y := \max\{\mathit{Prog}(\varrho, Y, W), \mathit{Prog}(\varrho, Y, Y')\}] = \varrho[Y := \max\{\top, (0, 0, 0, 0)\}] = \varrho[Y := \top]$

v	$\varrho(v)$
X	$\top$
X'	$(0, 1, 0, 0)$
Y	$\top$
Y'	$(0, 0, 0, 0)$
Z	$\top$
Z'	$\top$
W	$\top$

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Small progress measures (example) (16)

Step 16: $\varrho \leftarrow \textit{Lift}_{X'}(\varrho) = \varrho[X' := \min\{\textit{Prog}(\varrho, X', Z), \textit{Prog}(\varrho, X', Y)\}] = \varrho[X' := \min\{\top, \top\}] = \varrho[X' := \top]$

v	$\varrho(v)$
X	$\top$
X'	$\top$
Y	$\top$
Y'	$(0, 0, 0, 0)$
Z	$\top$
Z'	$\top$
W	$\top$

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Small progress measures (example) (17)

Step 17: $\varrho \leftarrow \textit{Lift}_{Y'}(\varrho) = \varrho[Y' := \min\{\textit{Prog}(\varrho, Y', X), \textit{Prog}(\varrho, Y', Y)\}] = \varrho[Y' := \min\{\top, \top\}] = \varrho[Y' := \top]$

v	$\varrho(v)$
X	$\top$
X'	$\top$
Y	$\top$
Y'	$\top$
Z	$\top$
Z'	$\top$
W	$\top$

ϱ is least game parity progress measure, and $\{v \in V \mid \varrho(v) \neq \top\} = \emptyset$ is winning set for player $\Diamond$. Hence player $\Box$ wins from all vertices

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Let $\varrho: V \to \mathbb{M}^{\top}$ be the least game parity progress measure.

- Define strategy $\sigma: V_{\diamond} \to V$ for player $\diamond$, by setting $\sigma(v)$ to be a successor w of $v \in V_{\diamond}$ that minimises $\varrho(w)$
- σ is a winning strategy for player $\diamond$ from $\{v \in V \mid \varrho(v) \neq \top\}$
- Strategy for $\square$ cannot be inferred directly
  - ...but can be computed on-the-fly . . . . Gazda&Willemse'14

Set $n = |V|$, $m = |E|$, $d = \max\{p(v) \mid v \in V\}$.

Worst-case running time complexity:

$$\mathcal{O}(d \cdot m \cdot (\frac{n}{\lfloor d/2 \rfloor})^{\lfloor d/2 \rfloor})$$

Lowerbound on worst-case:

$$\Omega((\lceil n/d \rceil)^{\lceil d/2 \rceil})$$

Summary Part II

- Model checking L_μ = solving Boolean equation systems
  - Gauß Elimination for solving BES .... $\mathcal{O}(2^{|\mathcal{E}|})$
- Solving BES = solving Parity games
  - Recursive .... $\mathcal{O}(m \cdot n^d)$
  - Small progress measures .... $\mathcal{O}(d \cdot m \cdot (\frac{n}{\lfloor d/2 \rfloor})^{\lfloor d/2 \rfloor})$
  - bigstep (combination of the two above) .... $\approx \mathcal{O}(n^{d/3})$
  - new quasi-polynomial algorithm (2016) .... $\approx \mathcal{O}(n^{6+\log d})$