Algorithms for Model Checking (2IW55)

Lecture 2

Fairness & Basic Model Checking Algorithm for CTL and fair CTL
– based on strongly connected components –
Chapter 4.1, 4.2 + SIAM Journal of Computing 1(2), 1972

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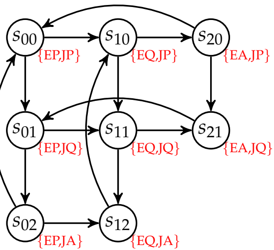
Outline

Temporal Logics: Fairness



- Atomic Propositions: EP, EQ, EA, JP, JQ, JA
- Intended meaning: John or Ella is either Playing, posing Questions, getting Answers
- To exclude that one child gets all attention, we want that both $\neg EQ$ as well as $\neg JQ$ hold infinitely often
- fairness constraints ensuring this:
  $F = \{\{s_{00}, s_{01}, s_{02}, s_{20}, s_{21}\}, \{s_{00}, s_{10}, s_{20}, s_{02}, s_{12}\}\}$

Temporal Logics: Fairness

Sometimes properties are violated by "unrealistic" paths only, for instance due to a scheduler. In this case, one may restrict to fair paths.

A Kripke Structure over AP with fairness constraints is a structure $M = \langle S, R, L, F \rangle$, where:

- $\langle S, R, L \rangle$ is an "ordinary" Kripke Structure as before
- $F \subseteq 2^S$ is a set of fairness constraints

A path is fair if it "hits" each fairness constraint infinitely often:

$$\mathsf{fair}(\pi) \ \textit{iff} \ \forall C \in F.\ \{i \mid \pi(i) \in C\} \ \text{is an infinite set}$$

Temporal Logics: Fairness

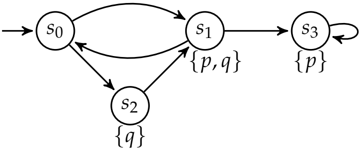
In CTL* with fairness semantics ($\models_F$), only fair paths will be considered.

Given a fixed Kripke Structure with fairness constraints $M = \langle S, R, L, F \rangle$, $s \models_F f$ means: formula f holds in state s in the fair CTL* semantics.

The definition of $\models_F$ coincides with $\models$ except for the following four clauses:

$s \models_F \mathsf{true}$	iff	there is some fair path starting in s
$s \models_F p$	iff	$p \in L(s)$ and there is some fair path starting in s
$s \models_F \mathsf{A}\, f$	iff	for all fair paths π starting in s, we have $\pi \models_F f$
$s \models_F \mathsf{E}\, f$	iff	for some fair path π starting in s, we have $\pi \models_F f$

Temporal Logics: Fairness



Note that $s_0 \models \mathsf{E\,F\,G}\ p$, but $s_0 \not\models \mathsf{A\,F\,G}\ p$

- First, consider as Fairness constraint: $F = \{\ \{s_3\}\ \}$
  - then all fair paths contain s_3 infinitely often
  - we have $s_0 \models_F \mathsf{A\,F\,G}\ p$
- Next, consider as Fairness constraint: $F = \{\ \{s_2\}\ \}$
  - then all fair paths contain s_2 infinitely often
  - in particular, fair paths cannot contain s_3
  - so $s_0 \not\models_F \mathsf{E\,F\,G}\ p$

Outline

Strongly Connected Components

Given a directed graph $G = \langle V, E \rangle$

- let $s \rightarrow_G^* t$ mean that there is a path from node s to t in G
- a strongly connected component (SCC) is a maximal subgraph S of G, such that for all $s, t \in S$, $s \rightarrow_G^* t$ and $t \rightarrow_G^* s$
- an SCC is non-trivial if it contains at least one edge

The SCCs of a graph (e.g. a Kripke Structure) can be computed in $\mathcal{O}(|V| + |E|)$ time with an algorithm based on depth-first search:

- Text book version (see Introduction to Algorothms, Corben *et al*)
- Tarjan's original algorithm (se SIAM Journal on Computing 1(2), 1972)

The second algorithm is most useful in model checking contexts

Strongly Connected Components

Idea behind Tarjan's SCC algorithm
Given is a directed graph $G = \langle V, E \rangle$

- compute spanning trees by depth-first search; number the nodes in the order they are visited
- the other, non-tree edges are either:
  - forward edges (can be ignored)
  - backward edges (to an ancestor)
  - cross edges (to another subtree)

  backward and cross edges lead to nodes with smaller numbers
- nodes are kept on a stack; the nodes of a discovered SCC will be popped immediately from this stack
- compute $root[v]$: the smallest node which is:
  - reachable from v by a sequence of tree-edges followed by at most one non-tree edge; and
  - if $root[v] = v$, the root of a new SCC is found, and the whole SCC is popped from the stack

Strongly Connected Components

Procedure FIND_SCC applies a repeated depth-first search on yet unprocessed nodes of the input graph $G = \langle V, E \rangle$
The depth-first search is delegated to the procedure DFS_SCC.

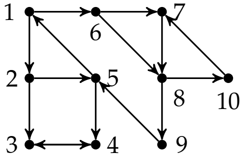
procedure FIND_SCC
    $i := 0$;
    empty the stack;
    leave all nodes unnumbered;
    for vertice $w \in V$ **do**
        if w is not yet numbered **then**
            DFS_SCC(w);
        end if
    end for
end procedure

Strongly Connected Components

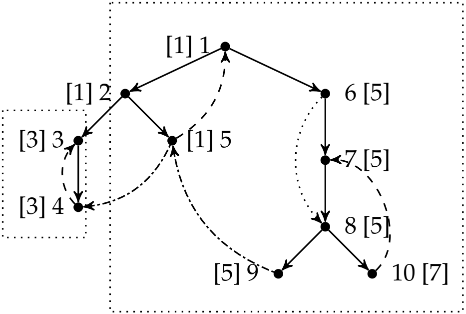
procedure DFS_SCC(v)
    $root[v] := number[v] := i := i + 1;$
    push v on the stack;
    for successor w of v **do**
        if w is not yet numbered **then** {tree edge}
            DFS_SCC(w);
            $root[v] := \min(root[v], root[w]);$
        else if $number[w] < number[v]$ and w on the stack **then** {cross/back edge}
            $root[v] := \min(root[v], number[w]);$
        end if
    end for
    if $root[v] = number[v]$ **then** {start new SCC}
        while top w of stack satisfies $number(w) \geq number(v)$ **do**
            pop w from stack;
        end while
    end if
end procedure

Strongly Connected Components

Example: SCC algorithm



A possible run of the SCC algorithm, with DFS node numbers, final root-values (in square brackets), tree edges (plain arrow), forward edges (dotted), back edges (dashed), cross edges (dash/dot). Two SCCs are found: number and root value are equal



Strongly Connected Components

We analyse the space and time requirements for running FIND_SCC on a graph $G = \langle V, E \rangle$:

- for every node:
  - DFS_SCC is called exactly once
  - all its outgoing edges are explored exactly once
- each node is pushed and popped from the stack exactly once
- checking whether a node is on the stack can be done in constant time, for instance by maintaining a Boolean array

Conclusion: Tarjan's algorithm for finding strongly connected components runs in time and space $\mathcal{O}(|V| + |E|)$

Outline

CTL Model Checking Algorithm

Recall that CTL has the following ten temporal operators:

- $\mathsf{A\,X}$ and $\mathsf{E\,X}$: for all/some next state
- $\mathsf{A\,F}$ and $\mathsf{E\,F}$: inevitably and potentially
- $\mathsf{A\,G}$ and $\mathsf{E\,G}$: invariantly and potentially always
- $\mathsf{A}\,[\;\mathsf{U}\;]$ and $\mathsf{E}\,[\;\mathsf{U}\;]$: for all/some paths, until
- $\mathsf{A}\,[\;\mathsf{R}\;]$ and $\mathsf{E}\,[\;\mathsf{R}\;]$: for all/some paths, releases

Besides atomic propositions (AP), the constant **true** and the Boolean connectives ($\neg, \vee$), the following temporal operators are sufficient: $\mathsf{E\,X}$, $\mathsf{E\,G}$, $\mathsf{E}\,[\;\mathsf{U}\;]$.

Hence: only algorithms for computing formulae of the above form are needed.

CTL Model Checking Algorithm

Main loop of model checking CTL: check formula f on a Kripke Structure $\langle S, R, L\rangle$.

By recursion on f, algorithm $\textsc{mc\_ctl}(f)$ computes $label(s)$ for all states $s \in S$, where $label(s)$ shall contain those subformulae of f that hold in s.

Algorithm $\textsc{mc\_ctl}(f)$ employs a case distinction on the structure of f:

$f = p$	add p to $label(s)$ for those states s with $p \in L(s)$
$f = g_0 \vee g_1$	$\textsc{mc\_ctl}(g_0)$; $\textsc{mc\_ctl}(g_1)$; add f to all states labelled with g_0 or g_1
$f = \neg g$	$\textsc{mc\_ctl}(g)$; add f to all states not labelled with g
$f = \mathsf{E}\,\mathsf{X}\,g$	$\textsc{mc\_ctl}(g)$; add f to all states with an R-successor labelled by g
$f = \mathsf{E}\,[g_0\;\mathsf{U}\;g_1]$	$\textsc{mc\_ctl}(g_0)$; $\textsc{mc\_ctl}(g_1)$; $\textsc{check\_eu}(g_0, g_1)$
$f = \mathsf{E}\,\mathsf{G}\,g$	$\textsc{mc\_ctl}(g)$; $\textsc{check\_eg}(g)$

Upon termination, $s \models f$ if and only if $f \in label(s)$

CTL Model Checking Algorithm

procedure CHECK_EU(f,g)
    $T := \{s \mid g \in label(s)\}$;
    for all $s \in T$ **do** $label(s) := label(s) \cup \{\mathsf{E}\,[f\ \mathsf{U}\ g]\}$;
    end for
    while $T \neq \emptyset$ **do**
        choose $s \in T$;
        $T := T \setminus \{s\}$;
        for all t satisfying $t\ R\ s$ **do**
            if $\mathsf{E}\,[f\ \mathsf{U}\ g] \notin label(t)$ and $f \in label(t)$ **then**
                $label(t) := label(t) \cup \mathsf{E}\,[f\ \mathsf{U}\ g]$;
                $T := T \cup \{t\}$;
            end if
        end for
    end while
end procedure

Observations:

- label all states where g holds
- search backwards over states where f holds

CTL Model Checking Algorithm

```
procedure CHECK_EG(f)
    S' := {s | f ∈ label(s)};
    SCC := {C | C is a nontrivial SCC of S'};
    T := ⋃_{C∈SCC} {s | s ∈ C};
    for all s ∈ T do label(s) := label(s) ∪ {E G f};
    end for
    while T ≠ ∅ do
        choose s ∈ T;
        T := T \ {s};
        for all t satisfying t ∈ S' and t R s do
            if E G f ∉ label(t) then
                label(t) := label(t) ∪ {E G f};
                T := T ∪ {t};
            end if
        end for
    end while
end procedure
```

Observations:

- restrict attention to subgraph where f holds
- an infinite path in a finite graph eventually reaches a non-trivial SCC

CTL Model Checking Algorithm

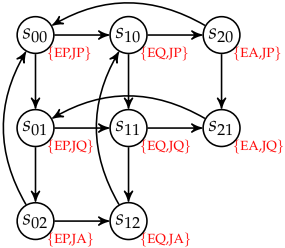
We analyse the time complexity for the standard **CTL** model checking algorithm of formula f (with $|f|$ the number of subformulae) on Kripke Structure $M = \langle S, R, L \rangle$.

- There are at most $|f|$ calls to MC_CTL
- Backward reachability and detecting strongly connected components can be done in time linear to the Kripke Structure: $\mathcal{O}(|S| + |R|)$
- Hence, each recursive call takes at most $\mathcal{O}(|S| + |R|)$ time

So, the complexity of this **CTL** model checking algorithm is $\mathcal{O}(|f| \cdot (|S| + |R|))$, which is linear in both the formula and the state space.

Outline

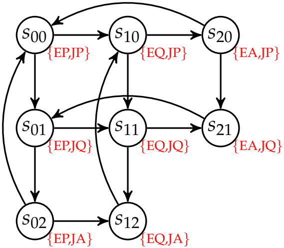
Example: demanding children



- Atomic Propositions: EP, EQ, EA, JP, JQ, JA
- Intended meaning: John or Ella is either Playing, posing Questions, getting Answers

Requirement: Whenever John asks a question, he eventually gets an answer
Formula: $\mathsf{A}\,\mathsf{G}\,(JQ \rightarrow \mathsf{A}\,\mathsf{F}\,JA)$

Example: demanding children

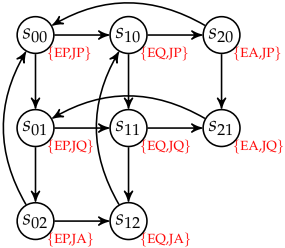


- Atomic Propositions: EP, EQ, EA, JP, JQ, JA
- Intended meaning: John or Ella is either Playing, posing Questions, getting Answers

- *Step 1: express using basic operators*

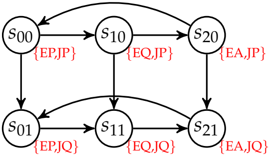
$$\begin{aligned} & \mathsf{A\,G}\,(JQ \rightarrow \mathsf{A\,F}\,JA) \\ \equiv\; & \\ & \neg\mathsf{E}\,[\mathsf{true}\ \mathsf{U}\ \neg(\neg JQ \vee \neg\mathsf{E\,G}\,\neg JA)] \end{aligned}$$

Example: demanding children



- *Step 2: treat* $\mathsf{E}\,\mathsf{G}\,\neg JA$
  - Restrict to the subgraph where $\neg JA$ holds
  - Find non-trivial SCCs
  - Backward reachability

Example: demanding children



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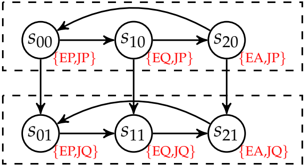

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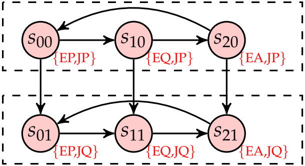
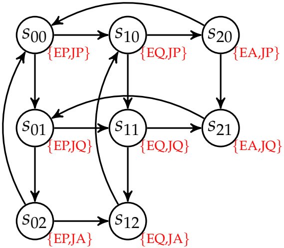


- *Step 2: treat* $\mathsf{E}\,\mathsf{G}\,\neg JA$
    - Restrict to the subgraph where $\neg JA$ holds
    - Find non-trivial SCCs
    - Backward reachability

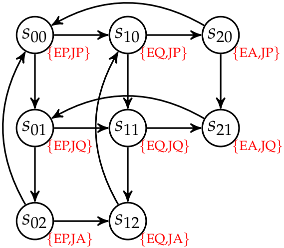
No new states are found. So, $\mathsf{E}\,\mathsf{G}\,\neg JA$ holds in the states $\{s_{00}, s_{10}, s_{20}, s_{01}, s_{11}, s_{21}\}$;

Example: demanding children



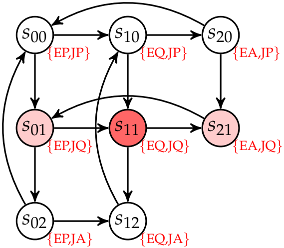
- *Step 3: treat* $\neg \mathsf{E}\,\mathsf{G}\,\neg JA$
  - $\mathsf{E}\,\mathsf{G}\,\neg JA$ holds in $\{s_{00}, s_{10}, s_{20}, s_{01}, s_{11}, s_{21}\}$, so $\neg \mathsf{E}\,\mathsf{G}\,\neg JA$ holds in $\{s_{02}, s_{12}\}$
- *Step 4: treat* $\neg JQ$
  - $\neg JQ$ holds in $\{s_{00}, s_{10}, s_{20}, s_{02}, s_{12}\}$
- *Step 5: treat* $\neg JQ \vee \neg \mathsf{E}\,\mathsf{G}\,\neg JA$
  - $\neg JQ \vee \neg \mathsf{E}\,\mathsf{G}\,\neg JA$ holds in $\{s_{00}, s_{10}, s_{20}, s_{02}, s_{12}\} \cup \{s_{02}, s_{12}\} = \{s_{00}, s_{10}, s_{20}, s_{02}, s_{12}\}$

Example: demanding children



- *Step 6: treat* $\neg(\neg JQ \vee \neg \mathsf{E}\,\mathsf{G}\,\neg JA)$
  - $\neg JQ \vee \neg \mathsf{E}\,\mathsf{G}\,\neg JA$ holds in $\{s_{00}, s_{10}, s_{20}, s_{02}, s_{12}\}$, so $\neg(\neg JQ \vee \neg \mathsf{E}\,\mathsf{G}\,\neg JA)$ holds in $\{s_{01}, s_{11}, s_{12}\}$
- *Step 7: compute* $\mathsf{E}\,[\textit{true}\ \mathsf{U}\ \neg(\neg JQ \vee \neg \mathsf{E}\,\mathsf{G}\,\neg JA)]$
  - Start in $\{s_{01}, s_{11}, s_{12}\}$
  - Perform a backward reachability analysis over states for which **true** holds

Example: demanding children



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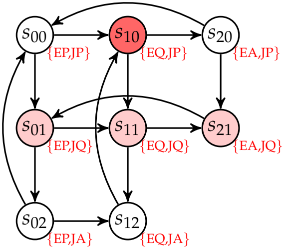

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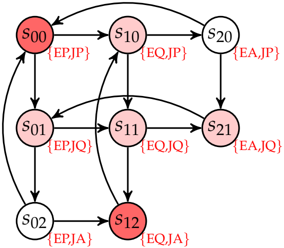

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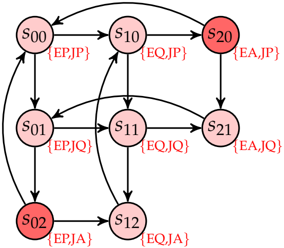

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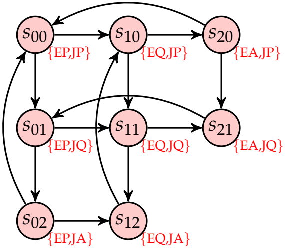

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  - Perform a backward reachability analysis over states for which **true** holds

Example: demanding children

Conclusion:

- So, $\mathsf{E}\,[\mathsf{true}\;\mathsf{U}\;\neg(\neg JQ \vee \neg\mathsf{E}\,\mathsf{G}\,\neg JA)]$ holds in all states
- Hence, its negation $\mathsf{A}\,\mathsf{G}\,(JQ \rightarrow \mathsf{A}\,\mathsf{F}\,JA)$ holds in no state
- The requirement does not hold for the full Kripke Structure
- Why? Because in this case, there is a path in which only Ella progresses while John is not being served.
- Next, we look at the Kripke Structure with Fairness Constraints

Outline

CTL Model Checking with Fairness

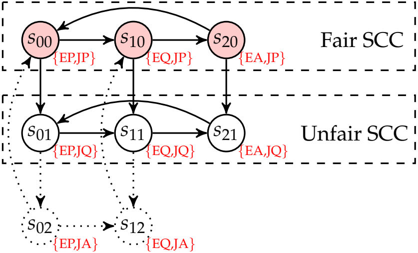
Recall: Kripke Structure $M = \langle S, R, L, F \rangle$ with fairness constraints $F \subseteq 2^S$.

- A path is fair if it "hits" each fairness constraint infinitely often
- A fair SCC is an SCC that contains an element from each constraint $C \in F$

Main idea of fair model checking for CTL:

- Special treatment for $s \models_F \mathsf{E\,G}\, f$: CHECK_FAIR_EG
  - Restrict attention to $S' \subseteq S$ where f holds
  - Find a path to a fair non-trivial SCC in S'
- Label states where $\mathsf{E\,G}$ true fairly holds with a new proposition symbol *fair*
- Treat the other operators using the original "unfair" procedures:
  - $s \models_F p$ .... $s \models p \wedge fair$
  - $s \models_F \mathsf{E\,X}\, f$ .... $s \models \mathsf{E\,X}\,(f \wedge fair)$
  - $s \models_F \mathsf{E}\,[f\,\mathsf{U}\,g]$ .... $s \models \mathsf{E}\,[f\,\mathsf{U}\,(g \wedge fair)]$

CTL Model Checking with Fairness



- Assume fairness constraints $\neg EQ$ and $\neg JQ$.
- Remark: full graph is one big fair SCC, so E G true holds everywhere

- E G $\neg JA$:
  - Restrict to subgraph with $\neg JA$
  - Find fair non-trivial SCCs
  - Do backward reachability
- Hence: $JQ \wedge$ E G $\neg JA$ holds fairly in NO state
- Hence E F $(JQ \wedge$ E G $\neg JA)$ holds nowhere fairly
- Hence, its negation, the requirement A G $(JQ \rightarrow$ A F $JA)$ fairly holds everywhere!

Outline

Summary

CTL model checking:

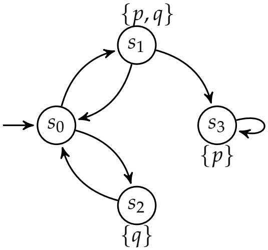
- SCC algorithm is used
- Tarjan's SCC algorithm runs one depth-first search, computing SCCs on-the-fly. Time complexity is linear
- CTL model checking can be done in time linear in the size of the formula as well as in the Kripke Structure
- Extension with Fairness Constraints is straightforward and is useful in practice
- Why not treat fairness in formulae?

$$\mathsf{A}\,[(\mathsf{G}\,\mathsf{F}\,C_1 \wedge \mathsf{G}\,\mathsf{F}\,C_2) \rightarrow \mathit{Requirement}]$$

  - fairness cannot be expressed in CTL
  - for LTL all known algorithms are exponential in the size of the formula

Outline

Exercise



CTL formulae: p, $\mathsf{E}\ [q\ \mathsf{R}\ p]$, $\mathsf{A}\ \mathsf{G}\ \mathsf{E}\ \mathsf{F}\ p$, $\mathsf{A}\ ((\mathsf{G}\ p) \vee (\mathsf{F}\ q))$

- Determine for each formula in which states of the above Kripke Structure it holds; use both the semantics and use the appropriate algorithms
- Extend the Kripke structure with the Fairness constraints $F = \{\ \{s_1\}, \{s_2\}\ \}$. In which states do the above formulae *fairly* hold?
- Similarly for the Fairness constraint $F = \{\ \{s_3\}\ \}$