Algorithms for Model Checking (2IW55)

Lecture 7

Boolean Equation Systems
Background material: Chapter 3 and 6 of
A. Mader, “Verification of Modal Properties using Boolean Equation Systems”, Ph.D. thesis, 1997

Tim Willemse
(timw@win.tue.nl)
http://www.win.tue.nl/~timw
HG 6.81

Outline

Boolean Equation Systems

Model Checking using BESs

Solving BESs

Exercise

Boolean Equation Systems

- Boolean Equation Systems are a versatile formal framework for verification.
- Boolean Equation Systems are systems of fixed point equations.

Given a set *Var* of propositional variables. A Boolean Expression is defined by:

$$f ::= X \mid \mathsf{true} \mid \mathsf{false} \mid f \wedge f \mid f \vee f$$

A Boolean Equation is an equation of the form $\sigma X = f$, where $X \in Var$, $\sigma \in \{\mu, \nu\}$ and f is a Boolean Expression. A Boolean Equation System is a sequence of Boolean Equations:

$$\mathcal{E} ::= \varepsilon \mid (\sigma X = f)\ \mathcal{E}$$

Note:

- Negation is not allowed, in order to ensure monotonicity.
- The order of equations is important. Intuitively, the topmost sign has priority.

Boolean Equation Systems

- A variable W that occurs in a Boolean Expression of a BES $\mathcal{E}$ is called bound, if there is an equation for W in $\mathcal{E}$, otherwise W is called free.
- If propositional variables are bound uniquely, the BES is well-formed; we only consider well-formed BESs.
- If $\mathcal{E}$ contains no free variables, $\mathcal{E}$ is closed, otherwise it is open.

Boolean Equation Systems

- A variable W that occurs in a Boolean Expression of a BES $\mathcal{E}$ is called bound, if there is an equation for W in $\mathcal{E}$, otherwise W is called free.
- If propositional variables are bound uniquely, the BES is well-formed; we only consider well-formed BESs.
- If $\mathcal{E}$ contains no free variables, $\mathcal{E}$ is closed, otherwise it is open.

Example

An example of a closed BES $\mathcal{E}$ with three propositional variables X, Y and Z:

$$(\mu X = (X \wedge Y) \vee Z)\ (\nu Y = X \wedge Y)\ (\mu Z = Z \wedge X)$$

An example of an open BES $\mathcal{F}$ with two propositional variables X and Y:

$$(\mu X = Y \vee Z)\ (\nu Y = X \wedge Y)$$

An example of a BES that is not well-formed:

$$(\mu X = X)\ (\nu X = X)$$

Boolean Equation Systems

Intuitive semantics:

- The solution of a BEAS is a valuation: $Val : Var \to \{\mathsf{false}, \mathsf{true}\}$.
- Let $[f](\eta)$ denote the value of boolean expression f under valuation η.
- For the solution η of a BES $\mathcal{E}$, we wish $\eta(X) = [f](\eta)$ for all equations $\sigma X = f$ in $\mathcal{E}$.
- Also, we want the smallest (for μ) or greatest (for ν) solution, where higher signs take priority.

Precise semantics: Given a BES $\mathcal{E}$, we define $[\mathcal{E}] : Val \to Val$ by recursion on $\mathcal{E}$.

$$\begin{cases} [\varepsilon](\eta) & := \eta \\ [(\mu X = f)\ \mathcal{E}](\eta) & := [\mathcal{E}](\eta[X := [f](\eta_\mu)]) \text{ where } \eta_\mu := [\mathcal{E}](\eta[X := \mathsf{false}]) \\ [(\nu X = f)\ \mathcal{E}](\eta) & := [\mathcal{E}](\eta[X := [f](\eta_\mu)]) \text{ where } \eta_\nu := [\mathcal{E}](\eta[X := \mathsf{true}]) \end{cases}$$

Outline

Boolean Equation Systems

Model Checking using BESs

Solving BESs

Exercise

Model Checking using BESs

Transformation of the μ-calculus model checking problem to BES

- Given is the following model checking problem:
  - a closed μ-calculus formula $\sigma X.\ f$ in Positive Normal Form and,
  - a Mixed Kripke Structure $M = \langle S, s_0, Act, R, L \rangle$.
- We define a BES $\mathcal{E}$ with the following property:

$$([\mathcal{E}](\eta))(X) = \textbf{true} \text{ iff } M, s \models \sigma X.\ f$$

  i.e. formula $\sigma X.\ f$ holds in state s if and only if the solution for X_s yields **true**.
- This BES is defined as follows:
  - For each subformula $\sigma X.g$ and for each state $s \in S$, we add the following equation:

$$\sigma X_s \ = \ RHS(s, g)$$

  - The order of the equations respects the subterm ordering in the original formula $\sigma X.\ f$.

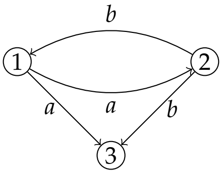
Model Checking using BESs

The Right-Hand Side of an equation is defined inductively by the structure of the μ-calculus formula:

$$
\begin{array}{lcl}
RHS(s,p) & = & L(s) \\
RHS(s,X) & = & X_s \\
 & & \\
RHS(s,f \wedge g) & = & RHS(s,f) \wedge RHS(s,g) \\
RHS(s,f \vee g) & = & RHS(s,f) \vee RHS(s,g) \\
 & & \\
RHS(s,[a]f) & = & \bigwedge_{t \in S} \{RHS(t,f) \mid s \xrightarrow{a} t\} \\
RHS(s,\langle a \rangle f) & = & \bigvee_{t \in S} \{RHS(t,f) \mid s \xrightarrow{a} t\} \\
 & & \\
RHS(s,\mu X.\, f) & = & X_s \\
RHS(s,\nu X.\, f) & = & X_s
\end{array}
$$

Model Checking using BESs

Example



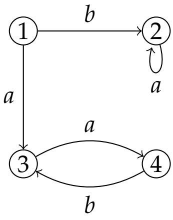
- $RHS(1, [a]X) = RHS(2, X) \wedge RHS(3, X) = X_2 \wedge X_3$.
- $RHS(2, \langle b\rangle Y) = RHS(1, Y) \vee RHS(2, Y) = Y_1 \vee Y_2$.
- $RHS(3, \langle b\rangle Y) = \mathsf{false}$ (empty disjunction!)
- $$\begin{array}{ll} & RHS(1, [a]\langle b\rangle \mu Z.\ Z) \\ = & RHS(2, \langle b\rangle \mu Z.\ Z) \wedge RHS(3, \langle b\rangle \mu Z.\ Z) \wedge \\ = & (RHS(1, \mu Z.Z) \vee RHS(3, \mu Z.Z)) \wedge \mathsf{false} \\ = & (Z_1 \vee Z_3) \wedge \mathsf{false} \end{array}$$
- Translation of $\mu X.\langle b\rangle \mathsf{true} \vee \langle a\rangle X$ to BES:

$$(\mu X_1 = X_3 \vee X_2)\ (\mu X_2 = \mathsf{true})\ (\mu X_3 = \mathsf{false})$$

Model Checking using BESs

Example

μ-calculus formula: $\nu X.\big([a]X \wedge \nu Y.\mu Z.(\langle b\rangle Y \vee \langle a\rangle Z)\big)$
Translates to the following BES:



$$
\begin{array}{lcl}
\nu X_1 & = & X_3 \wedge Y_1 \\
\nu X_2 & = & X_2 \wedge Y_2 \\
\nu X_3 & = & X_4 \wedge Y_3 \\
\nu X_4 & = & \mathsf{true} \wedge Y_4 \\
\nu Y_1 & = & Z_1 \\
\nu Y_2 & = & Z_2 \\
\nu Y_3 & = & Z_3 \\
\nu Y_4 & = & Z_4 \\
\mu Z_1 & = & Y_2 \vee Z_3 \\
\mu Z_2 & = & \mathsf{false} \vee Z_2 \\
\mu Z_3 & = & \mathsf{false} \vee Z_4 \\
\mu Z_4 & = & Y_3 \vee \mathsf{false}
\end{array}
$$

Outline

Solving BESs

- We reduced the model checking problem $M, s \models f$ to the solution of a BES with $\mathcal{O}(|M| \times |f|)$ equations.
- We now want a fast procedure to solve such BESs.
- An extremely tedious way to solve a BES is to unfold its semantics.
- A very appealing solution is to solve it by Gauß Elimination.

Solving BESs

Gauß Elimination uses the following 4 basic operations to solve a BES:

- local solution: eliminate X in its defining equation:

$$\begin{array}{lll} \mathcal{E}_0\ (\mu X = f)\ \mathcal{E}_1 & \text{becomes} & \mathcal{E}_0\ (\mu X = f[X := \mathsf{false}])\ \mathcal{E}_1 \\ \mathcal{E}_0\ (\nu X = f)\ \mathcal{E}_1 & \text{becomes} & \mathcal{E}_0\ (\nu X = f[X := \mathsf{true}])\ \mathcal{E}_1 \end{array}$$

- Substitute definitions backwards:

$$\begin{array}{ll} & \mathcal{E}_0\ (\sigma_1 X = X \vee Y)\ \mathcal{E}_1\ (\sigma_2 Y = Y \wedge X)\ \mathcal{E}_2 \\ \text{becomes:} & \mathcal{E}_0\ (\sigma_1 X = X \vee (Y \wedge X))\ \mathcal{E}_1\ (\sigma_2 Y = Y \wedge X)\ \mathcal{E}_2 \end{array}$$

- Substitute closed equations forward:

$$\begin{array}{ll} & \mathcal{E}_0\ (\sigma_1 X = \mathsf{true})\ \mathcal{E}_1\ (\sigma_2 Y = Y \wedge X)\mathcal{E}_2 \\ \text{becomes:} & \mathcal{E}_0\ (\sigma_1 X = \mathsf{true})\ \mathcal{E}_1\ (\sigma_2 Y = Y \wedge \mathsf{true})\ \mathcal{E}_2 \end{array}$$

- Boolean simplication: At least the following:

$$b \wedge \mathsf{true} \to b \qquad b \vee \mathsf{true} \to \mathsf{true} \qquad b \wedge \mathsf{false} \to \mathsf{false} \qquad b \vee \mathsf{false} \to b$$

Solving BESs

Example

	$(\mu X = X \vee Y)\ (\nu Y = X \vee (Y \wedge Z))\ (\mu Z = Y \wedge Z)$
local $\rightarrow$	$(\mu X = \mathsf{false} \vee Y)\ (\nu Y = X \vee (\mathsf{true} \wedge Z))\ (\mu Z = Y \wedge \mathsf{false})$
simplifications $\rightarrow$	$(\mu X = Y)\ (\nu Y = X \vee Z))\ (\mu Z = \mathsf{false})$
substitution backwards $\rightarrow$	$(\mu X = Y)\ (\nu Y = X \vee \mathsf{false})\ (\mu Z = \mathsf{false})$
simplifications $\rightarrow$	$(\mu X = Y)\ (\nu Y = X)\ (\mu Z = \mathsf{false})$
substitution backwards $\rightarrow$	$(\mu X = X)\ (\nu Y = X)\ (\mu Z = \mathsf{false})$
local $\rightarrow$	$(\mu X = \mathsf{false})\ (\nu Y = X)\ (\mu Z = \mathsf{false})$
substitution forwards $\rightarrow$	$(\mu X = \mathsf{false})\ (\nu Y = \mathsf{false})\ (\mu Z = \mathsf{false})$

Solving BESs

Gauß Elimination is a decision procedure for computing the solution to a BES.

Input: a BES $(\sigma_1 X_1 = f_1)\ \ldots\ (\sigma_n X_n = f_n)$. Returns: the solution for X_1.

for $i = n$ **downto** 1 **do**
  if $\sigma_i = \mu$ **then** $f_i := f_i[X_i := \mathsf{false}]$
  else $f_i := f_i[X_i := \mathsf{true}]$
  end if
  for $j = 1$ **to** $i - 1$ **do** $f_j := f_j[X_i := f_i]$
  end for
end for

Note:

- Invariant of the outer loop: f_i contains only variables x_j with $j < i$.
- Upon termination, $\sigma_1 X_1 = f_1$ is closed and evaluates to true or false.
- One could forward-substitute the solution for X_1 and repeat the procedure to solve X_2, *etcetera*.

Solving BESs

Complexity of Gauß Elimination.

- Note that in $\mathcal{O}(n^2)$ substitutions, we obtain the final answer for X_1.
- However, f_1 can have $\mathcal{O}(2^n)$ different copies of e_n as subterms, so intermediate expressions could become exponentially big.
- Practical efficiency increases a lot if one keeps all intermediate terms simplified all the time.
- Gauß Elimination can be sped up if a forward dependency analysis is conducted (so-called local model checking).
- Precise efficiency depends heavily on the set of simplification rules.
- Precise complexity of Gauß Elimination is yet unknown.
- Interesting: the complexity seems to be independent of the alternation depth of the μ-Calculus formula.

Outline

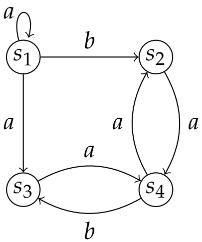
Boolean Equation Systems

Model Checking using BESs

Solving BESs

Exercise

Exercise



Consider the following μ-Calculus formula f:

$$\nu X.\big([a]X \wedge \nu Y.\mu Z.(\langle b\rangle Y \vee \langle a\rangle Z)\big)$$

- Use the Emerson-Lei algorithm for computing whether $M, s_1 \models f$.
- Translate the model checking question $M \models f$ to a BES; indicate how $M, s \models \phi$ corresponds to the variables in the BES.
- Solve the BES by Gauß Elimination.