Infinity in Mathematics & Computer Science

Lecture at *QUANTA 2007*

15 November 2007 / CMS, Lucknow, India

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Motivations for Working on Scientific Problems

- Direct application in real life
- Foundation for working on other problems
- Aid to acquisition of knowledge and development of skills
- Fun and enjoyment

Termination of Computations: A Recent Success

- Start with finite sequence over $\{a, b, c\}$

  $bbaa$

- Repeatedly replace subsequences:

$$\begin{array}{rcl} aa & \longrightarrow & bc \\ bb & \longrightarrow & ac \\ cc & \longrightarrow & ab \end{array}$$

Example:

$$bb\underline{aa} \rightarrow b\underline{bb}c \rightarrow ba\underline{cc} \rightarrow b\underline{aa}b \rightarrow \underline{bb}cb \rightarrow a\underline{cc}b \rightarrow aa\underline{bb} \rightarrow a\underline{aa}c \rightarrow ab\underline{cc} \rightarrow abab$$

Does this terminate for every start sequence?

Termination of Computations: Famous Open Problems

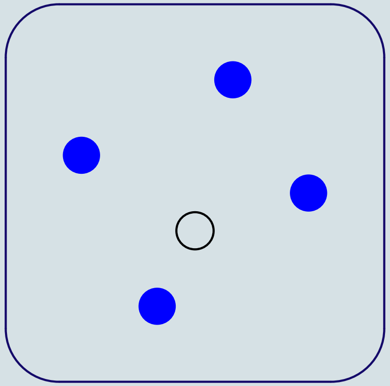
- If N even $\rightarrow$ $N/2$, if N odd $\rightarrow$ $3N+1$ (Collatz)

  $3 \rightarrow 10 \rightarrow 5 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1 \rightarrow 4 \rightarrow \cdots$

- While N is not a palindrome, add it to its reverse

  $152 \rightarrow 152 + 251 = 403 \rightarrow 403 + 304 = 707$ $\qquad$ $196 \rightarrow ?$

Marble Game 1

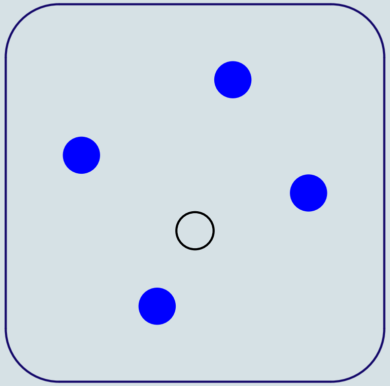


Each time take a marble:

- Remove it

Does this terminate? After how many steps?

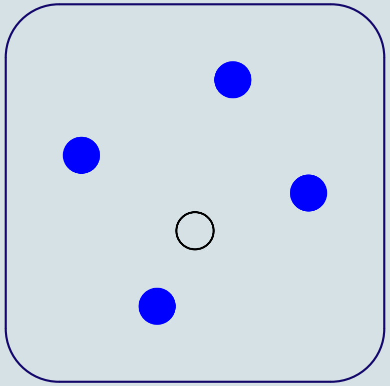
Marble Game 2



Repeatedly take a marble:

- If blue, then remove
- If white, then replace by one blue marble

Marble Game 3



Repeatedly take a marble:

- If blue, then remove it
- If white, then replace by *arbitrary* number of blue marbles

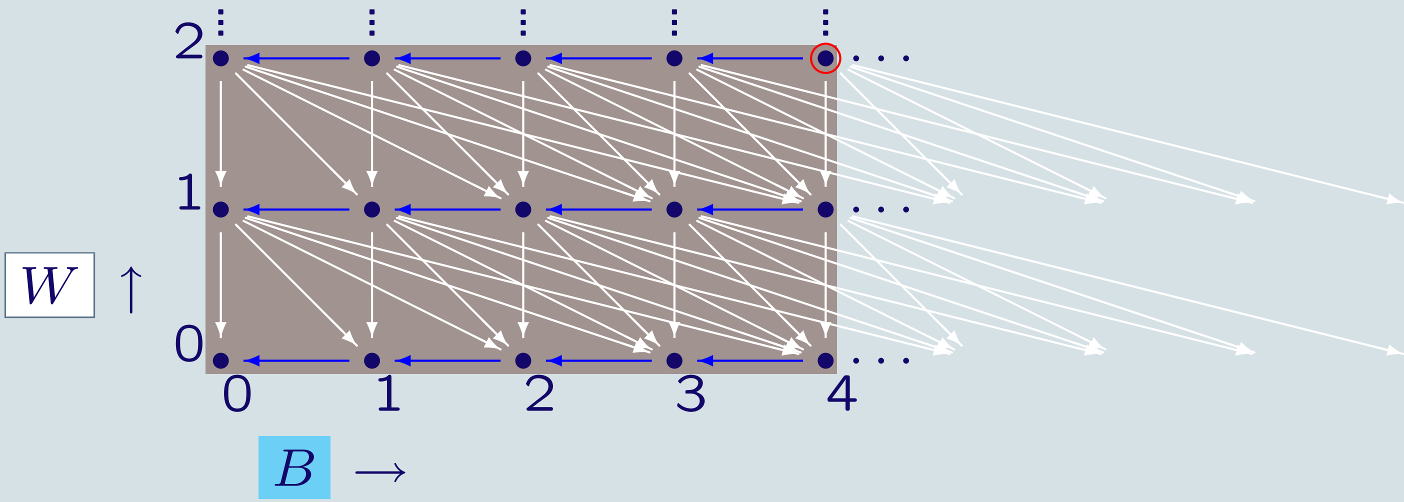
Marble Game Analysis

W white marbles B blue marbles

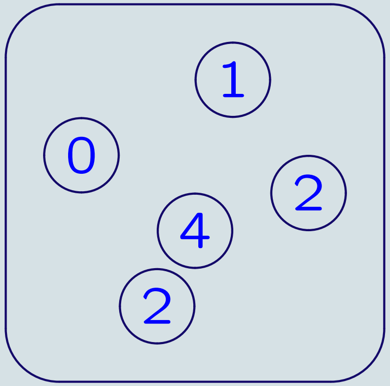
Game 1. Terminates after $f_1(W, B) = W + B$ steps

Game 2. Terminates after $f_2(W, B) = 2 * W + B$ steps

Game 3. Terminates after ten hoogste $f_3(W, B) = \omega * W + B$ steps



Lotto Ball Game



Repeatedly take an $\mathbb{N}$-numbered lotto ball:

- Replace by arbitrary number of balls with smaller numbers

  i.e. (n) replaced by $(<n)$ $(<n)$ $\cdots$ $(<n)$

Operations on Natural Numbers

Successor (1 more): $a|$

Addition (repeated successor): $a + b = a\overbrace{|\cdots|}^{b\times}$

Multiplication (repeated addition): $a * b = \overbrace{a + \cdots + a}^{b\times}$

$a * 0 = 0 \qquad a * 1 = a \qquad a * b| = a * b + a \qquad a * (b + c) = a * b + a * c$

Exponentiation (repeated multiplication): $a^b = \overbrace{a * \cdots * a}^{b\times}$

$a^0 = 1 \qquad a^1 = a \qquad a^{b|} = a^b * a \qquad a^{b+c} = a^b * a^c$

Decimal Expansion

Every natural number is *uniquely* expressible as

sum of powers of 10 with coefficients < 10.

Example:

$$\begin{aligned} 266 &= 200 + 60 + 6 \\ &= 2 * 100 + 6 * 10 + 6 * 1 \\ &= 2 * 10^2 + 6 * 10^1 + 6 * 10^0 \end{aligned}$$

Expansion in Base $B \geq 2$

Every natural number is *uniquely* expressible as

sum of powers of B with coefficients $< B$.

Example with $B = 2$ (*binary*):

$$\begin{aligned} 266 &= 256 + 8 + 2 \\ &= 2^8 + 2^3 + 2^1 \end{aligned}$$

Example with $B = 3$ (*ternary*):

$$\begin{aligned} 266 &= 243 + 18 + 3 + 2 \\ &= 3^5 + 2 * 3^2 + 3^1 + 2 * 3^0 \end{aligned}$$

Super-Expansion in Base $B \geq 2$

1. Expand in base B.

2. Repeatedly expand the *exponents* in base B as well.

3. Stop when all numbers $\leq B$.

Example:

$B = 2$	$B = 3$
$266 = 2^8 + 2^3 + 2$	$266 = 3^5 + 2 * 3^2 + 3^1 + 2$
$= 2^{2^3} + 2^{2+1} + 2$	$= 3^{3+2} + 2 * 3^2 + 3^1 + 2$
$= 2^{2^{2+1}} + 2^{2+1} + 2$	

Goodstein Sequence of $N > 0$ and $B \geq 2$

$N = 8 \quad B = 2$

1. Super-expand N in base B. — $8 = 2^{2+1}$
2. Replace each B by $B + 1$. — $3^{3+1} = 81$
3. Decrease by 1; yields new N. — $N' = 80$
4. Increase B by 1; yields new B. — $B' = 3$
5. Stop when $N = 0$, otherwise repeat from step 1.

Goodstein Sequence for $N = 266$ and $B = 2$

Step	N	B
1	266	2
	$2^{2^{2+1}} + 2^{2+1} + 2$	
	$3^{3^{3+1}} + 3^{3+1} + 3 - 1$	
2	$443\ldots886$ (39 digits)	3
	$3^{3^{3+1}} + 3^{3+1} + 2$	
	$4^{4^{4+1}} + 4^{4+1} + 2 - 1$	
3	$323\ldots681$ (617 digits)	4
	$4^{4^{4+1}} + 4^{4+1} + 1$	
	$5^{5^{5+1}} + 5^{5+1} + 1 - 1$	
4	$\ldots$ ($>$ 10 000 digits)	5
	$5^{5^{5+1}} + 5^{5+1}$	

Goodstein's Theorem (1944)

Every Goodstein sequence terminates with $N = 0$.

It can take a while:

- $N = 3$, $B = 2$ terminates after 5 steps
- $N = 4$, $B = 2$ terminates after $3 * 2^{402\,653\,211} - 3 \approx 10^{10^8}$ steps

Provability of Goodstein's Theorem

Theorem of Kirby and Paris (1982):

> Goodstein's Theorem can*not* be proven from Peano's Axiomas.

'Ordinary' induction does not suffice: the sequence 'grows too fast'.

Every proof of Goodstein's Theorem involves (a form of) *transfinite* induction such as over the *ordinal numbers*.

Ordinal Numbers

Extend operations on numbers with a limit operation . . .

$$0, \quad 1, \quad 2, \quad 3, \quad \overset{1}{\cdots}, \quad \omega$$

$$\omega + 1, \quad \omega + 2, \quad \overset{1}{\cdots}, \quad \omega + \omega = \omega * 2$$

$$\omega * 2 + 1, \quad \omega * 2 + 2, \quad \overset{1}{\cdots}, \quad \omega * 2 + \omega = \omega * 3$$

$$\overset{1}{\cdots}, \quad \omega * 4, \quad \overset{1}{\cdots}, \quad \omega * 5, \quad \overset{2}{\cdots}, \quad \omega * \omega = \omega^2$$

$$\omega^2 + 1, \quad \overset{1}{\cdots}, \quad \omega^2 + \omega, \quad \overset{2}{\cdots}, \quad \omega^2 + \omega^2 = \omega^2 * 2$$

$$\overset{2}{\cdots}, \quad \omega^2 * 3, \quad \overset{2}{\cdots}, \quad \omega^2 * 4, \quad \overset{3}{\cdots}, \quad \omega^2 * \omega = \omega^3$$

$$\overset{4}{\cdots}, \quad \omega^4, \quad \overset{5}{\cdots}, \quad \omega^5, \quad \overset{\omega}{\cdots}, \quad \omega^\omega$$

Normal form of ordinal numbers $< \omega^\omega$

Generalize base-B expansion, taking $B = \omega$:

$$N = B^k * c_k + B^{k-1} * c_{k-1} + \cdots + B^2 * c_2 + B * c_1 + c_0$$

where $0 \le c_i < B$, so now unbounded coefficients.

Normal form of $\alpha < \omega^\omega$:

$$\alpha = \omega^k * c_k + \omega^{k-1} * c_{k-1} + \cdots + \omega^2 * c_2 + \omega * c_1 + c_0$$

where k and all c_i are *finite*.

Solution to Lotto Ball Game: c_i = number of balls with value i

Proof of Goodstein's Theorem

Super-expand N in base B.

Replace every B by ω.

The result $f_G(N, B)$ is an ordinal number $< \underbrace{\omega^{\omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}}}_{\omega\times} = \epsilon_0$.

For example: $f_G(266, 2) = \omega^{\omega^{\omega+1}} + \omega^{\omega+1} + \omega$

Claim: If $N, B \to N', B'$ in the Goodstein sequence, then

$$f_G(N', B') < f_G(N, B)$$

Ordinal numbers are well ordered: every decreasing sequence ends.

Be Successful in Your Endeavours: Can You Grow Too Fast?