# Settling Multiple Debts Efficiently: Problems

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**Abstract**

I present a series of problems related to [1] for use in the class room.

## Introduction

A group of friends lend each other money throughout the year. They carefully record each transaction. When Alice lends 10 euro to Bob, this is recorded as Alice $\xrightarrow{10}$ Bob.

At the end of the year they wish to settle their debts. How should they transfer money so as to settle all debts? Of course, they could reverse the action of each recorded loan separately. However, often there is a better way.

The problems posed below concern efficient ways of settling all debts. Try to minimize the number of transfers and the total amount transferred. We consider only settling schemes where money is transferred between a pair of persons.

# Problems

1. After the loans

$$\begin{array}{lcl} \text{Alice} & \xrightarrow{10} & \text{Bob} \\ \text{Carol} & \xrightarrow{10} & \text{Dick} \end{array}$$

   is this a proper settlement:

$$\begin{array}{lcl} \text{Bob} & \xrightarrow{10} & \text{Carol} \\ \text{Dick} & \xrightarrow{10} & \text{Alice} \end{array}$$

   (involving 2 transfers for a total amount of 20 euro)?

2. How to settle efficiently the loans:

$$\begin{array}{lcl} \text{Alice} & \xrightarrow{10} & \text{Bob} \\ \text{Alice} & \xrightarrow{10} & \text{Carol} \\ \text{Bob} & \xrightarrow{10} & \text{Carol} \end{array}$$

3. How to settle efficiently the loans:

$$\begin{array}{lcl} \text{Alice} & \xrightarrow{10} & \text{Bob} \\ \text{Alice} & \xrightarrow{20} & \text{Carol} \\ \text{Bob} & \xrightarrow{15} & \text{Carol} \\ \text{Carol} & \xrightarrow{25} & \text{Alice} \end{array}$$

4. How to settle efficiently the loans:

$$\begin{array}{lcl} \text{Alice} & \xrightarrow{10} & \text{Bob} \\ \text{Alice} & \xrightarrow{10} & \text{Carol} \\ \text{Bob} & \xrightarrow{10} & \text{Carol} \\ \text{Carol} & \xrightarrow{20} & \text{Alice} \end{array}$$

5. How to settle efficiently the loans:

$$\begin{array}{lcl} \text{Alice} & \xrightarrow{10} & \text{Bob} \\ \text{Alice} & \xrightarrow{20} & \text{Dick} \\ \text{Carol} & \xrightarrow{30} & \text{Bob} \\ \text{Carol} & \xrightarrow{40} & \text{Dick} \end{array}$$

   Can you find another (optimal) way?

6. How to settle efficiently the loans:

$$\begin{array}{lcl} \text{Alice} & \xrightarrow{20} & \text{Bob} \\ \text{Bob} & \xrightarrow{10} & \text{Carol} \\ \text{Alice} & \xrightarrow{60} & \text{Dick} \\ \text{Dick} & \xrightarrow{30} & \text{Carol} \\ \text{Carol} & \xrightarrow{20} & \text{Alice} \end{array}$$

7. How can the settlement

$$\begin{array}{lcl} \text{Alice} & \xrightarrow{10} & \text{Carol} \\ \text{Carol} & \xrightarrow{60} & \text{Dick} \\ \text{Dick} & \xrightarrow{20} & \text{Bob} \end{array}$$

   be improved to minimize the total amount transferred?

8. How can the settlement

$$\begin{array}{lcl} \text{Alice} & \xrightarrow{20} & \text{Bob} \\ \text{Alice} & \xrightarrow{30} & \text{Dick} \\ \text{Carol} & \xrightarrow{30} & \text{Bob} \end{array}$$

   be improved to minimize the number of transfers?

9. Describe a general scheme for settling all debts among $N$ persons in at most $N-1$ transfers and with a minimal total amount transferred. It is not necessary to minimize the number of transfers.

10. How can the six balances

$$\begin{array}{rr} -120 & +110 \\ -60 & +90 \\ -50 & +30 \end{array}$$

    be settled optimally?

11. How can the twelve balances

$$\begin{array}{rr} -301 & +900 \\ -300 & +203 \\ -299 & +100 \\ -297 & +99 \\ -202 & +98 \\ -2 & +1 \end{array}$$

    be settled optimally?