From Chain-link Fence to Space-Spanning Mathematical Structures

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Abstract

Chain-link fence is constructed from intertwined zigzag wires. Such a fence is basically a two-dimensional structure, which can be extended infinitely in two directions. We describe various ways in which zigzags can be intertwined to produce potentially infinite three-dimensional structures. Next, we generalize the zigzags to helices, and explore their possibilities to span space. These zigzags and helices can be constructed from beams using miter joints. Appropriate choices for the cross section and the kind of miter joint results in artistically appealing sculptures. Several designs were constructed in wood. Along the way, we discovered a nice invariance theorem for helices.

1 Introduction

Chain-link fence, as depicted in Figure 1 (left), consists of *zigzags*, where adjacent zigzags link with each other. Typically, the longitudinal central axes of the zigzags run vertically (Fig. 1, right); thus, all the axes are parallel. Note that strictly speaking these zigzags are not planar, for otherwise they could not interlock.

Figure 1: Chain-link fence, consisting of vertical interlocked zigzags (left); single zigzag with dashed axis

By design, a chain-link fence is a structure that can be extended infinitely both in the horizontal and the vertical direction, thereby spanning a plane. This is precisely what makes the chain-link structure so suitable for fences. Some obvious parameters to vary are thickness of the wire, shape of its cross section, the angle between adjacent segments in a zigzag, and the length of each segment. But those variations are not very exciting.

In this article, we investigate zigzag-like structures that can be extended infinitely into *three* dimensions to span space. Section 2 presents a space-spanning structure made from true zigzags. In Sections 3 and 4, we generalize the planar zigzag to 3D helices, winding around a triangular and square cylinder respectively. In the resulting space-spanning structures, the helices interlock pairwise. Section 5 considers structures where three or more strands interlock at the meeting points. Section 7 concludes the article.

2 Zigzag

To obtain a space-spanning structure made from zigzags, one can drop the property of chain-link fence that the axes of the zigzags are all parallel. Instead, we consider two sets of zigzags with orthogonal axes, as shown on the left in Fig. 2.

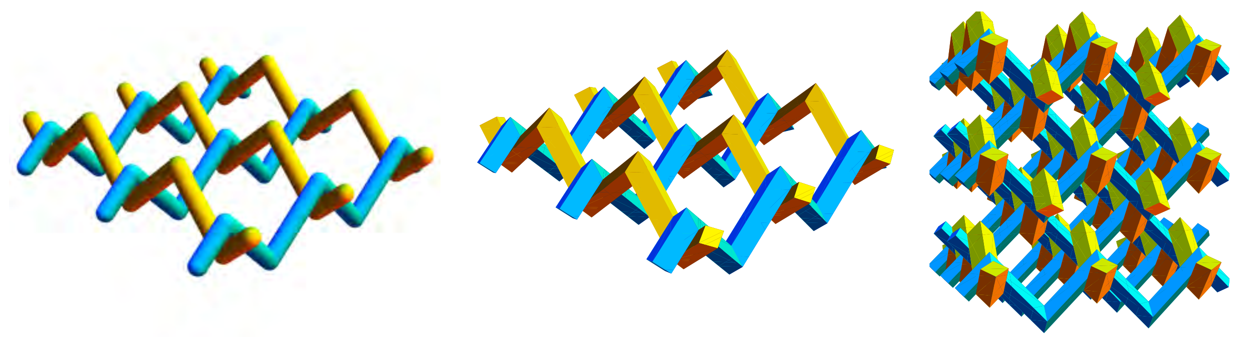


Figure 2: Space-spanning mesh constructed from orthogonally crossing zigzags; left: single layer using bent tubes; center: single layer using rhombic beams and miter joints; right: extension into 3D

For this structure, it is more appealing to make the zigzags from a beam with a $1 : \sqrt{2}$ rhombus as cross section. Cutting such a beam at 45° yields a square cut face. It can then be mitered into a 90° joint [1, 2]. Where two such zigzags cross orthogonally, the square cut face of one joint snugly fits into the 90° joint angle of the other, and vice versa, as can been seen in the center of Fig. 2. Hence, there is no space between two crossing zigzags. They touch along a surface at the crossing; in fact, four faces of each zigzag touch.

The extension into the third dimension is clearly visible on the right in Figure 2. However, it is hard to appreciate the 3D structure from a 2D picture. By appropriately varying the lengths of the zigzags, various shapes can be obtained. Such objects most naturally serve as pendants, for instance, suspended from the ceiling. That way, no glue is needed to hold the structure together. See Figure 3 for some examples.

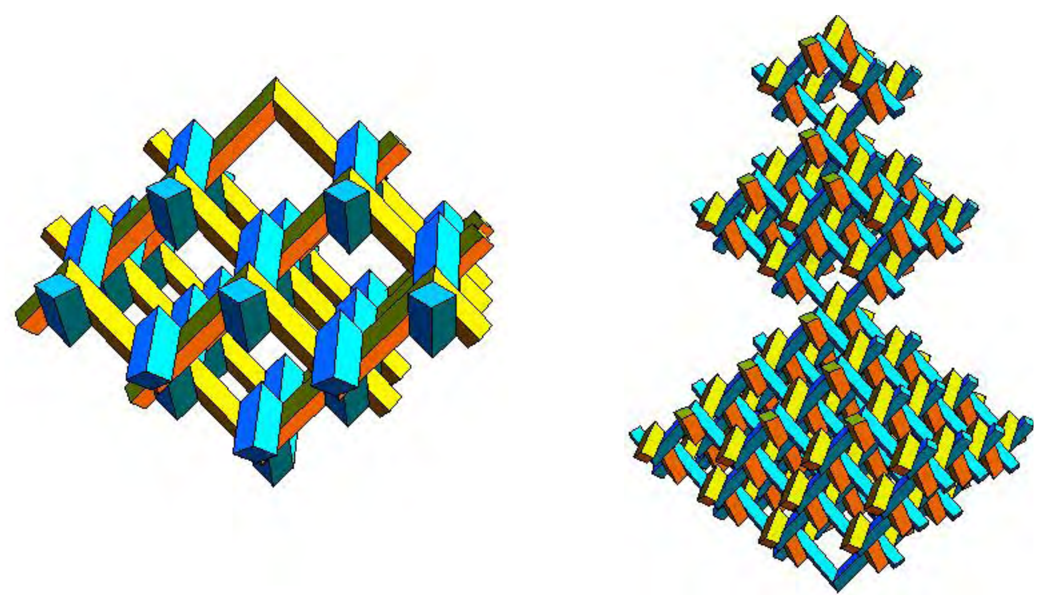


Figure 3: Single octahedron (left) and chain of hanging octahedrons (right) constructed from woven zigzags with a rhombic cross section

3 Zigzagzeg

The planar zigzag used in the preceding section can be described by a *Logo* program of the form

Repeat K [Forward D Left 90 Forward D Right 90] (1)

Using 3D turtle graphics [4], program (1) can be simplified to a program of the form

Repeat K [Forward D RollLeft 180 TurnLeft 90] (2)

where `RollLeft` rotates the turtle about its *heading* vector, and `TurnLeft` rotates it about its *normal* vector. Thus, `RollLeft 180` turns the turtle upside down. Program (2) generalizes to that of a 3D *helix* with parameters D, ψ, and ϕ:

Repeat K [Forward D RollLeft ψ TurnLeft ϕ] (3)

This program generates a shape with *constant torsion* [3], coiling around a central axis.

A so-called x,y,z-helix, which alternates steps in the three primary directions, is generated by the helix program with roll angle $\psi = 90°$ and turn angle $\phi = 90°$ (see Fig. 4, left). We call this helix a *zigzagzeg*. By viewing this helix along its central axis (dashed in Fig. 4, left), you can see that it winds around a *triangular* cylinder; also see Fig. 4, right, which shows a top view of fourteen interlocking zigzagzegs.

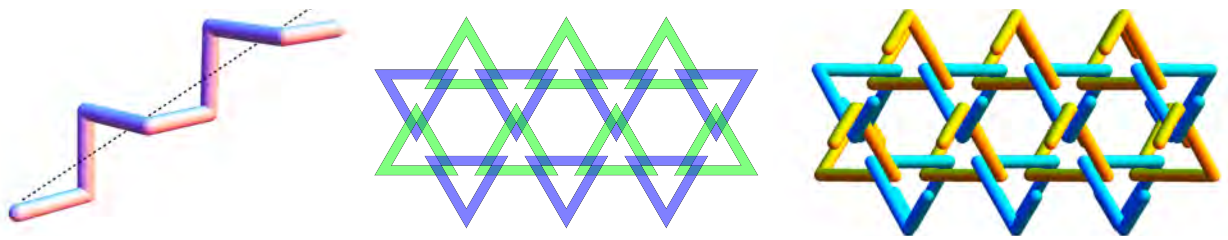


Figure 4: Zigzagzeg (left); pattern of interlocking triangles (center); top view of zigzagzeg mesh (right)

The top view of a bunch of such zigzagzegs, having their axes in parallel, consists of equilateral triangles. Figure 4 (center) shows a nice pattern of pairwise interlocking equilateral triangles. By placing the zigzagzegs with appropriate *phase*, they can be made to intertwine as a space-spanning structure, see Fig. 4 (right). All these helices coil in the same, clockwise, direction. A realization in wood is shown in Figure 5.

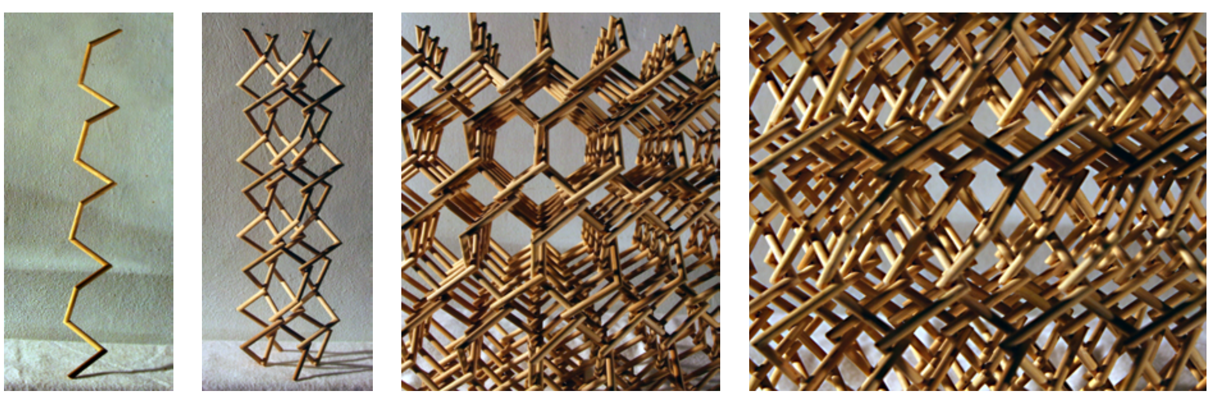


Figure 5: Wooden zigzagzegs: one strand (left); six parallel intertwined strands enclosing a hexagonal cylinder (second from left); corresponding space-spanning mesh (right two); all helix axes run vertically

Using square beams and miter joints, the zigzagzegs can be made to touch along two faces of each zigzagzeg at a crossing. Figure 6 shows two different wooden realizations with six intertwined zigzagzegs.

Figure 6: Six linked wooden zigzagzegs made from square beams and regular miter joints; left: narrower beams; right: wider beams and two types of wood

4 Zigzagzegzug

By using the four main diagonals of a cube as stepping directions, one obtains another helix, aptly called *zigzagzegzug*. The zigzagzegzug has roll angle $\psi = 60^\circ$ and turn angle $\phi = \arccos(1/3) \approx 70.53^\circ$ (see Fig. 7, left). It winds around a *square* cylinder (also see Fig. 7, right, showing a top view of twelve such helices).

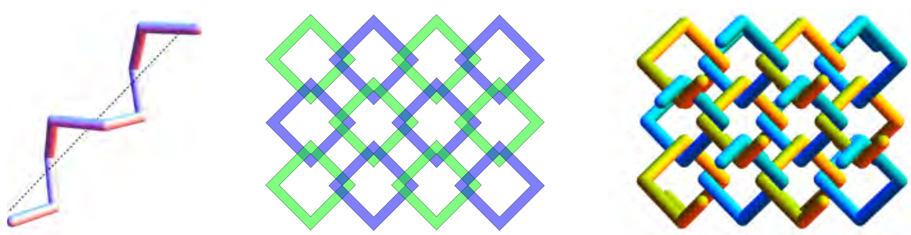


Figure 7: Zigzagzegzug (left); interlocking squares (center); top view of zigzagzegzug mesh (right)

The top view of a bunch of such zigzagzegzugs having their axes in parallel consists of squares. Figure 7 (center) shows a nice pattern of pairwise interlocking squares. By placing the zigzagzegzugs with appropriate *phase* they can be made to intertwine as a space-spanning structure. They all coil clockwise. It is noteworthy to observe that the zigzagzegzugs cross at right angles. Figure 8 shows a realization in wood.

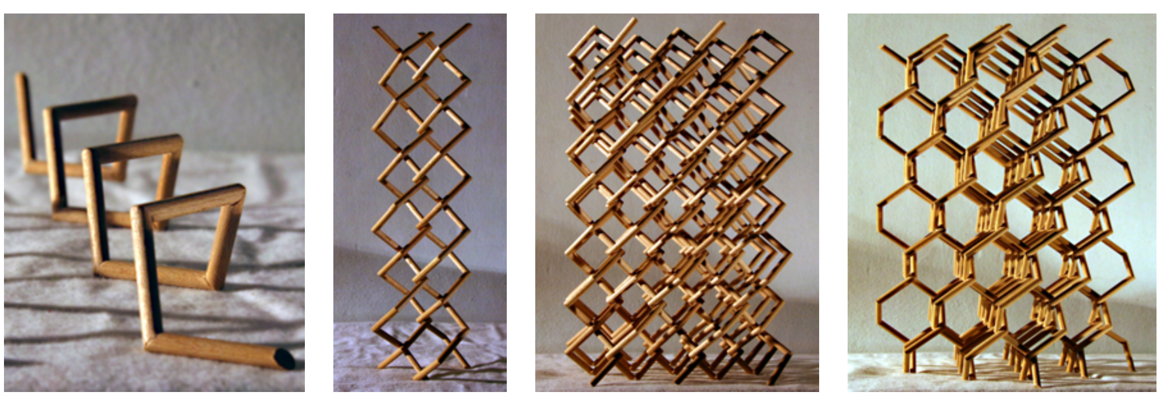


Figure 8: Wooden zigzagzegzugs: one strand (left); four parallel intertwined strands enclosing a square cylinder (center left); corresponding space-spanning mesh (right two); all helix axes run vertically

Zigzagzegzugs can be made conveniently from a beam with hexagonal cross section, because of the 60° roll angle. Since a hexagonal beam cut at 54.74° results in a cut face with a top angle of 109.47°, which matches the internal angle of the zigzagzegzug helix, they intertwine with faces touching each other. See Figure 9.

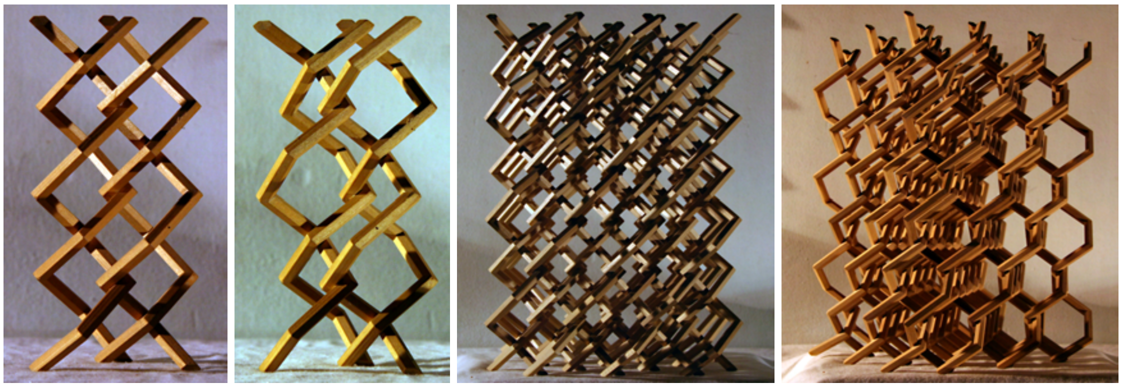


Figure 9: Four parallel zigzagzegzug strands with hexagonal cross section (left two); corresponding space-spanning mesh (right two)

5 Three or More Strands Link

In the three preceding sections, all structures have strands that link in pairs. We now present various ways to make three or four strands embrace each other at the meeting point. This is illustrated by Figures 10, 11, 12, 13, and 14. The characteristics of these structures are summarized in the following table.

Figure	Roll ψ	Turn ϕ	Projection	Degree	Directions	Cross section	Miter joint
10	90°	90°	△	3	1	square	regular
11	$\arccos(-1/3) \approx 109.5°$	60°	△	4	4	rhombic	skew
12	$\arccos(1/3) \approx 70.5°$	60°	□	3	3	rhombic	skew
13 left	180°	60°	–	4	4	rhombic	skew
13 right	180°	60°	–	3	3	rhombic	skew
14	180°	90°	–	3	3	square	regular

Note that the helices in Figure 14 are zigzags, those in Figure 10 are zigzagzegs, and those in Figure 12 zigzagzegzugs. The helices in Figures 11 and 13 have not appeared in the preceding sections.

The *degree* is the number of helices that link at each meeting point. In the first structure, the central axes of the helices are all parallel, but in the other structures they occur in a limited number of distinct directions.

Rhombic cross sections with skew miter joints [1] are used to allow faces to touch each other flat on where helices cross. These helices all have $\phi = 60°$ corresponding to the skew miter joint for this cross section.

Where the helices link, they 'twirl' around each other. This can be seen clearly in Fig. 10 (center), where three zigzagzegs interlock with each other. Note that these linking 'twirls' have a direction of themselves. In these structures the twirl directions are all parallel. In Figure 10, the direction of the twirls is the same as the direction of helix axes. In the other cases, the twirl directions differ from the helix directions.

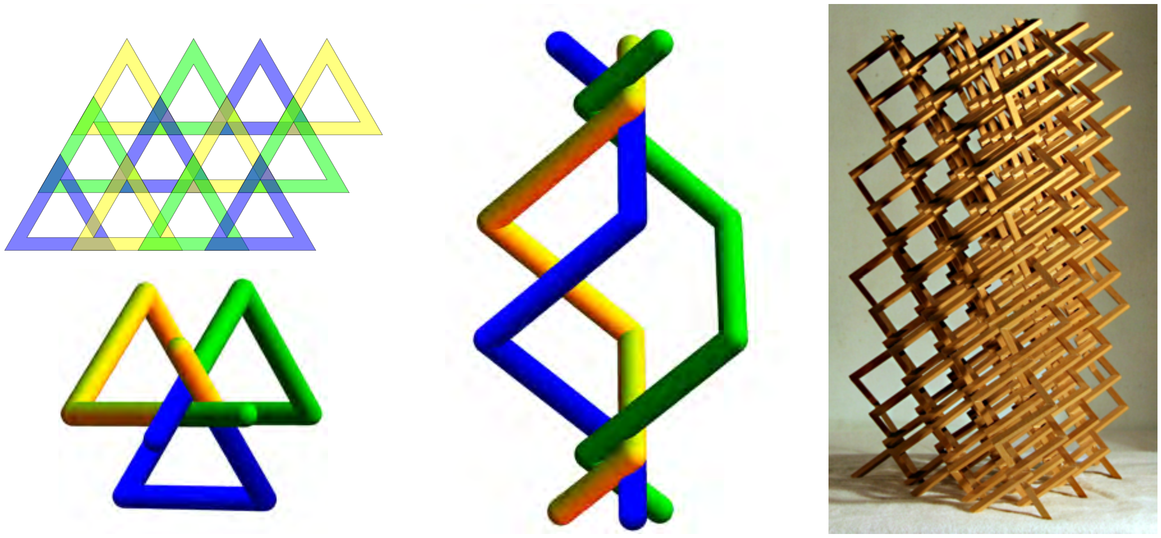


Figure 10: Triangles interlocking in triples (top left); three zigzagzegs in ternary embrace (bottom left: top view; center: side view); space-spanning mesh from triple-linked parallel zigzagzegs with square cross section and regular miter joints in wood (right)

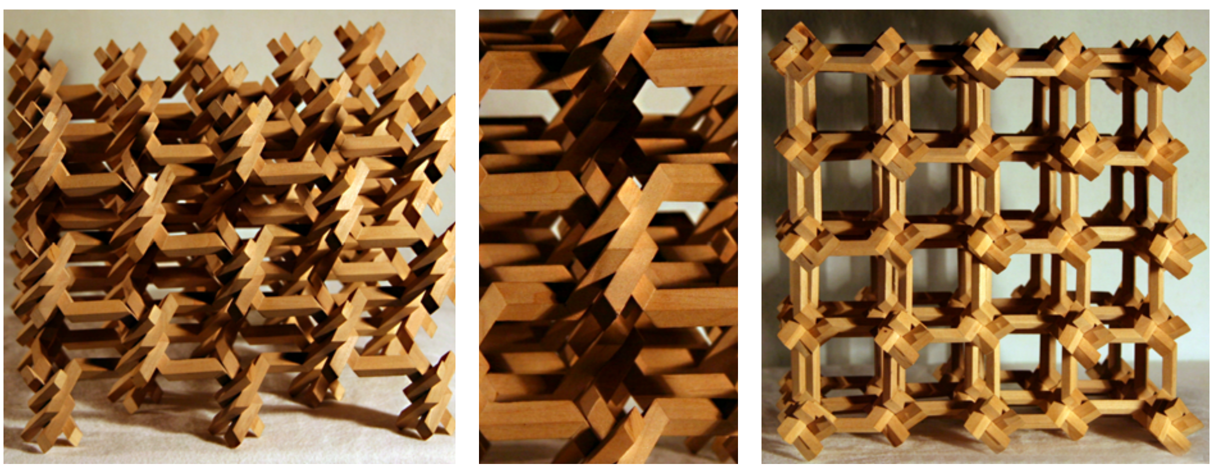
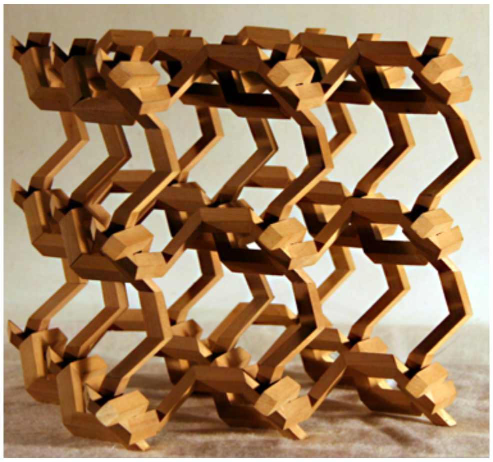


Figure 11: Space-spanning mesh from non-parallel zigzagzegs with rhombic cross section and skew miter joints (square cut faces), four strands link at each meeting point, each strand having its "own" axis direction (main diagonals of the cube)

6 An Invariance Theorem for Turtle Helices

While preparing the illustrations for this article (in particular, Fig. 7, right), the turtle angles for the zigzagzegzug were accidentally interchanged at some point. That is, instead of taking roll angle $\psi = 60^\circ$ and turn angle $\phi = \arccos 1/3 \approx 70.53^\circ$, we had used roll angle $\psi = \arccos 1/3 \approx 70.53^\circ$ and turn angle $\phi = 60^\circ$. At first, this mistake was not noticed because the projection of the helix with interchanged angles also happens to be a square. The stepping directions of this helix are face diagonals of the four vertical faces of the cube.

When discovering the mistake, we were surprised that the shape of the projection was not affected by the interchange of roll and turn angle. This made us wonder whether this is accidental for these special angles, or a general property of turtle helices.



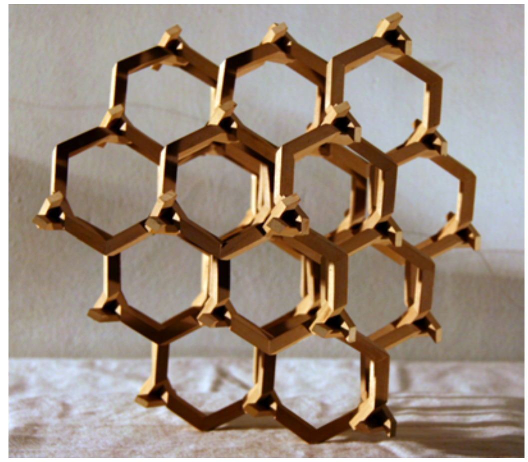
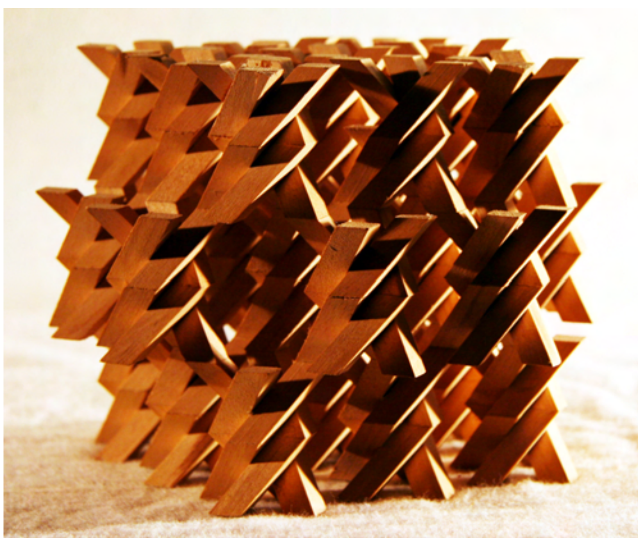


Figure 12: Space-spanning mesh from non-parallel zigzagzegzugs with rhombic cross section and skew miter joints (square cut faces), three strands link at each meeting point, each strand having its "own" axis direction (orthogonal)



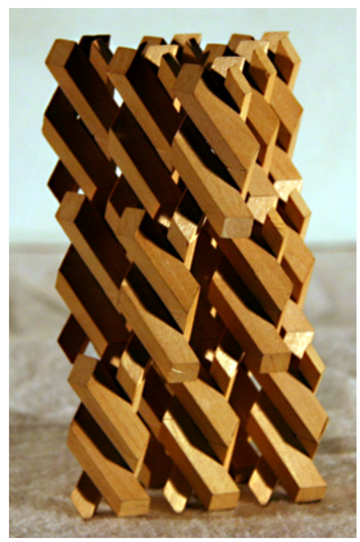


Figure 13: Space-spanning mesh from (non-parallel) zigzags with rhombic cross section and skew miter joints (square cut faces), where four (left) and three (right) strands link at each meeting point

Let us define $\theta(\psi,\phi)$ as the exterior angle of the regular polygon obtained by projecting the helix with roll angle ψ and turn angle ϕ along its central axis. That is, $\theta(\psi,\phi)$ is the turtle's turn angle when drawing the projection. For example, $\theta(90°,90°) = 120°$, since the zigzagzeg with $\psi = \phi = 90°$ projects onto an equilateral triangle. It turns out that the following *invariance theorem* for turtle helices holds:

$$\theta(\psi,\phi) \quad = \quad \theta(\phi,\psi) \tag{4}$$

The proof is beyond the scope of this article. It is a consequence of the following beautiful relationship between roll angle ψ, turn angle ϕ, and exterior projection angle $\theta = \theta(\psi,\phi)$:

$$\cos(\theta/2) \quad = \quad \cos(\psi/2)\cos(\phi/2) \tag{5}$$

Note that the right-hand side of (5) is symmetric in ψ and ϕ, thus yielding (4). We have not encountered there properties in the literature, and believe they are new.

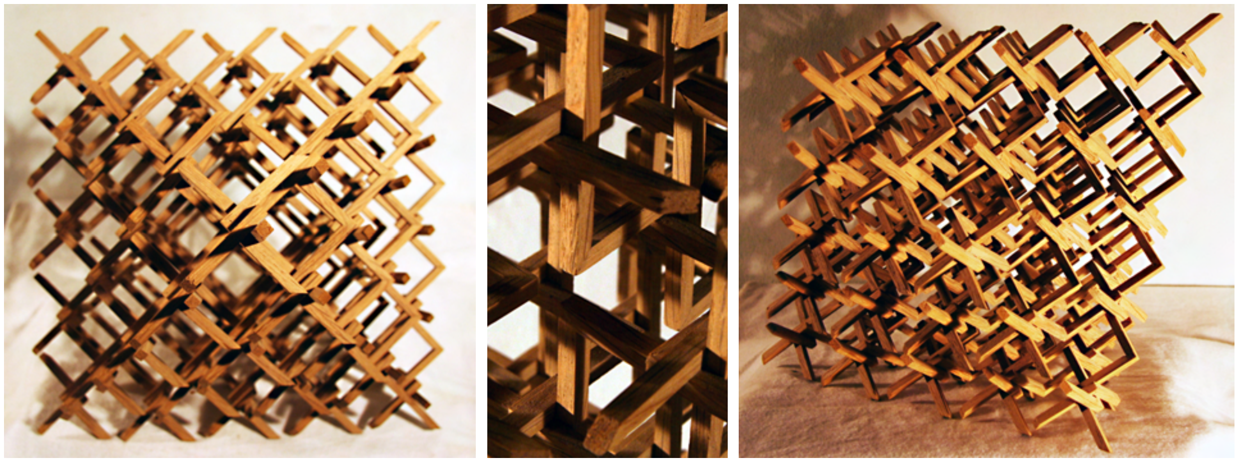


Figure 14: Space-spanning mesh from (non-parallel) zigzags with square cross section, three strands link at each meeting point; each zigzag having its "own" direction (the face diagonals of a cube meeting in one corner)

7 Conclusion

We have shown various ways to construct space-spanning structures from zigzags and other helices, in particular, zigzagzegs and zigzagzegzugs. By using appropriate cross sections and miter joints, the linking helices can be made to meet snugly. First, we presented several families of structures where helices link in pairs. But we have also discovered structures where three or four helices link together. Accidentally, we stumbled upon a nice invariance theorem for helices defined by turtle programs.

These structures gave rise to appealing artwork, of which we have shown numerous examples. They can possibly also be applied in nanoscience and nanoengineering to construct new fabrics.

There are obvious ways to pursue these ideas further. For instance, one can consider helices that wind around regular polygons other than an equilateral triangle (zigzagzeg) or square (zigzagzegzug), such as a regular hexagon or octagon, which can be interlocked in nice patterns. In the structures presented here, all helices coil in the same direction, but it is possible to mix left- and right-handed helices.

Acknowledgments Koos Verhoeff constructed all wooden artwork. We would like to thank Tom Goris for the pictures. The illustrations were made with *Mathematica*.

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