A Beautiful Characterization of Equivalence Relations

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While recording a proof today, I found myself deriving something like:

$$\begin{array}{ll} & x \sim z \\ \equiv & \quad \{ \sim \text{ is an equivalence relation and } x \sim y \text{ is assumed } \} \\ & y \sim z \end{array}$$

Having read EWD 1102 ("Why preorders are beautiful") the other day, I still had a heightened awareness of beauty. Thus, I started wondering.

Apparently, in my derivation I use the following property of an equivalence relation $\sim$:

$$(\forall x, y :: x \sim y \;\Rightarrow\; (\forall z :: x \sim z \equiv y \sim z)). \tag{0}$$

This property follows immediately from the transitivity and symmetry of $\sim$. From the reflexivity of $\sim$ one can infer

$$(\forall x, y :: x \sim y \;\Leftarrow\; (\forall z :: x \sim z \equiv y \sim z)), \tag{1}$$

by instantiating the z-quantification with $z := y$. Hence, an equivalence relation $\sim$ satisfies the conjunction of (0) and (1), which is equivalent to

$$(\forall x, y :: x \sim y \;\equiv\; (\forall z :: x \sim z \equiv y \sim z)). \tag{2}$$

The beautiful thing now is that (2) completely characterizes equivalence relations, that is, relation $\sim$ is an equivalence relation *if and only if* it satisfies (2).

From EWD 1102 we know that (2) implies "$\sim$ is a preorder", that is, "$\sim$ is reflexive and transitive". In fact, (0) implies "$\sim$ is transitive" and (1) is equivalent to "$\sim$ is reflexive". The reader can easily verify this without reference to EWD 1102. Symmetry of $\sim$ follows immediately from (2) and the symmetry of $\equiv$.