The Mathematics of Mitering and Its Artful Application

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Exposition in Blessum: 24 July – 10 August

Trefoil Knot

Trefoil Knot

Date: approx. 1984

Materials: Steel, painted

Height: approx. 1 m

6 segments in 2 lengths (minimum number required for trefoil knot)

Beam cross section: equilateral triangle

Path symmetries: 3-fold and 2-fold rotational

120° Möbius twist

Bicolored (5, 1) Torus Path

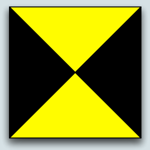
Bicolored (5, 1) Torus Path

Date: approx. 1986

Materials: Ash, Wenge

Height: 44 cm

24 segments; vertices lie on torus

Beam cross section: square 

Path symmetries: 2-fold rotational

180° Möbius twist

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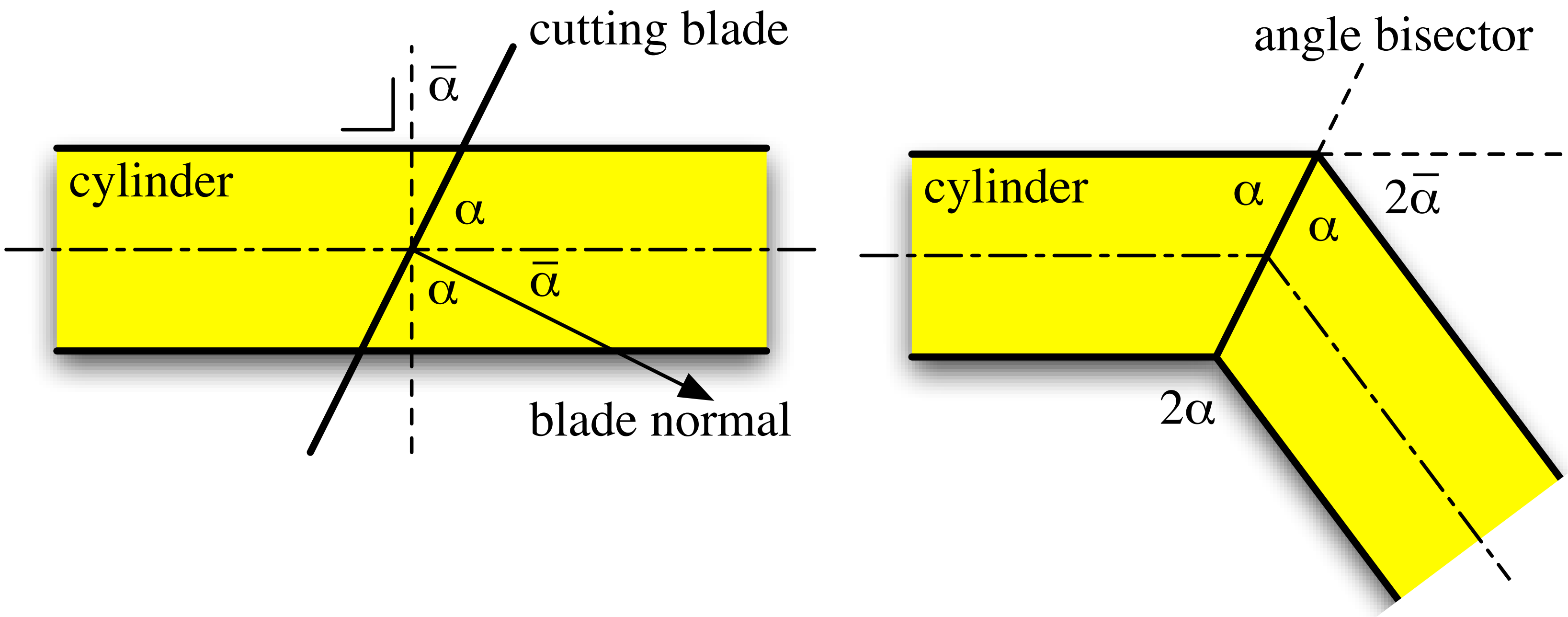
Earliest Work: Characteristics

- “Arbitrary” closed spatial polygonal paths
- “Ad hoc” choice of vertex locations, “ad hoc” joint angles

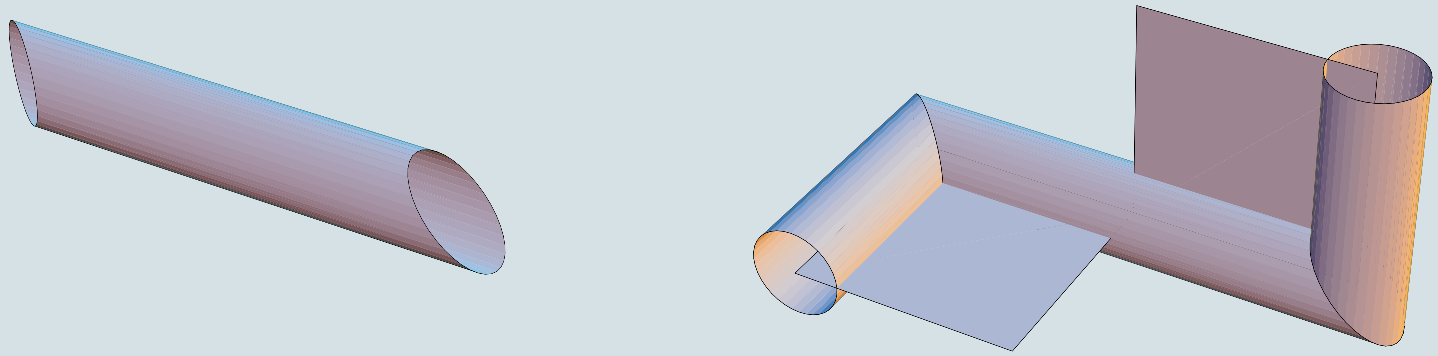
  No a priori restrictions, other than by symmetry
- Classical miter joints
- “Ad hoc” cut faces
- At each joint, beam edges match

The latter is not trivial: the vertex locations need tweaking.

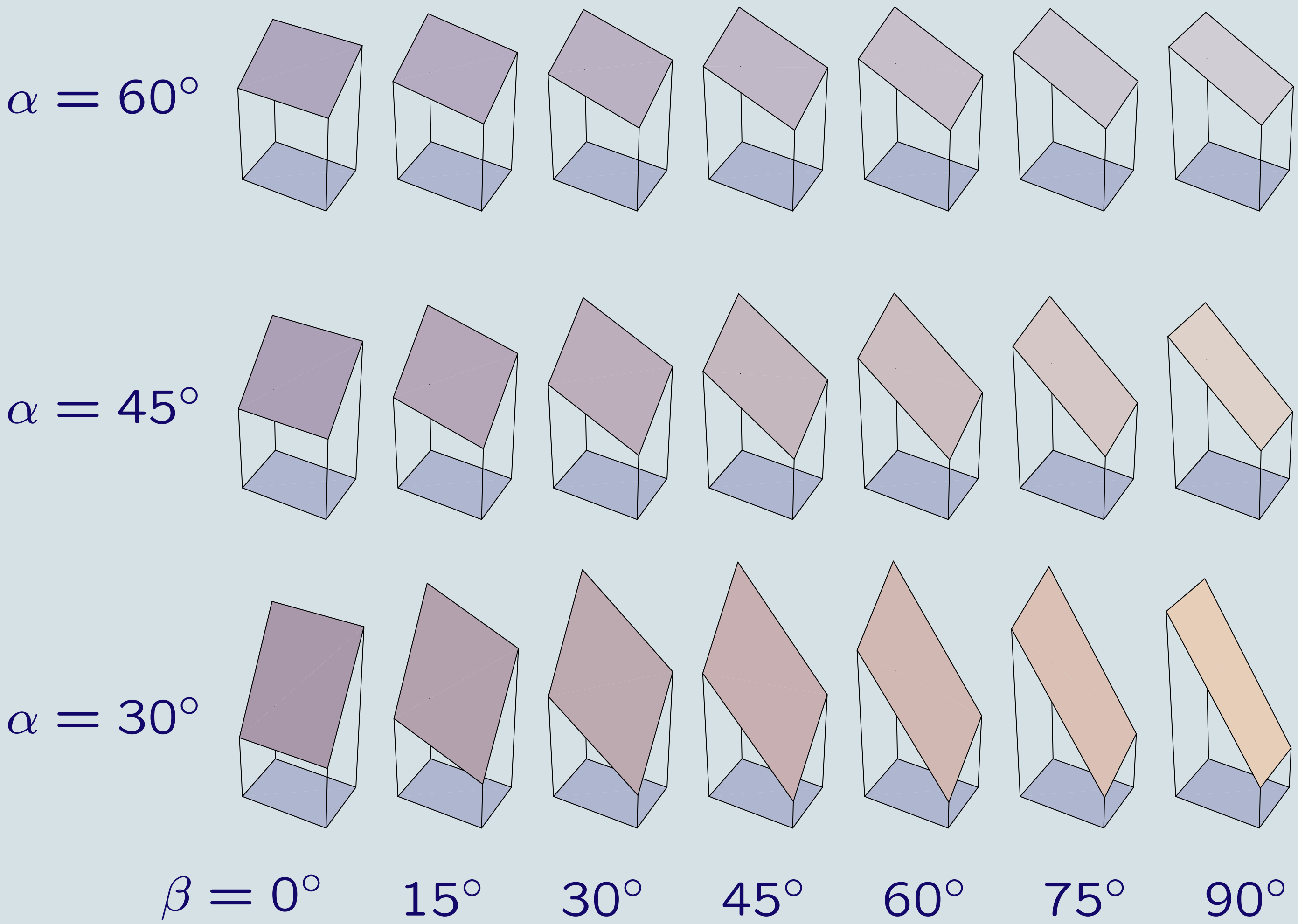
Beveled Beams and (Classical) Miter Joint



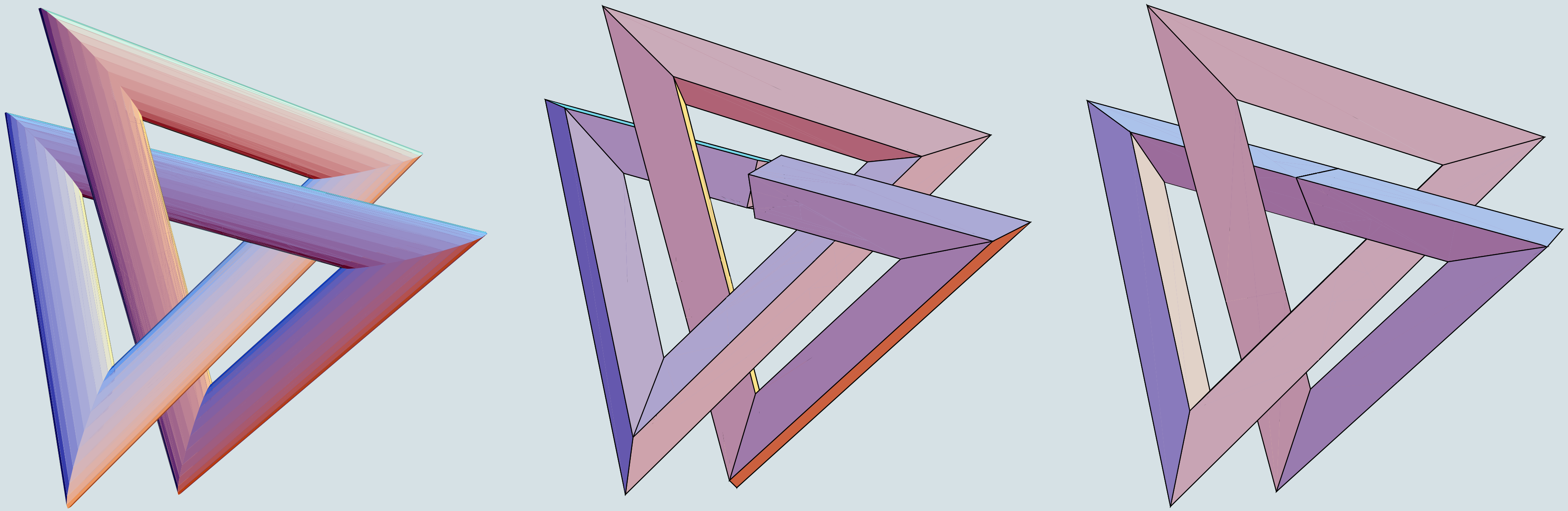
Spatial Mitering



Shape of Cut Face Varies for Given Cross Section



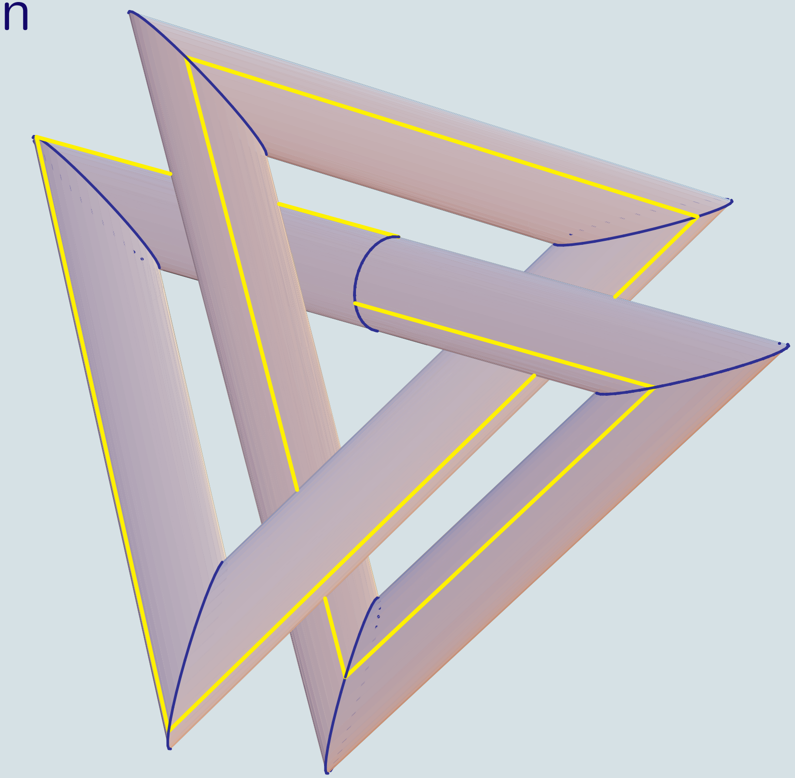
Closing the Path



Miter Joint Rotation Invariance Theorem

The total amount of cross rotation is an inherent property of the polygonal path and does *not* depend on

- choice of initial segment
- initial rotation of cross section around the center line
- the shape of the cross section



Hamilton Path on Truncated Octahedron

Right-angle Champion

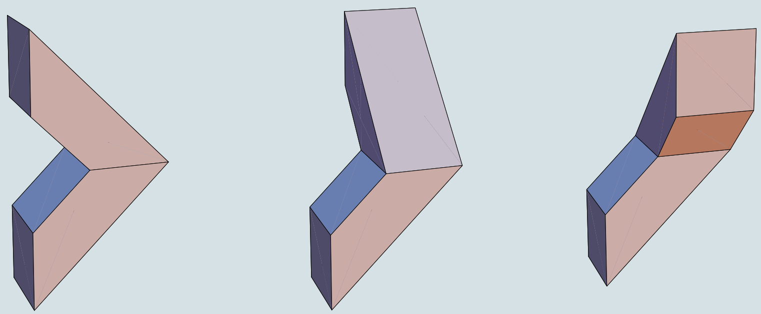
Trefoil Knot

Early Work (late 1980s): Characteristics

- Closed polygonal paths constructed from limited set of pieces

  $1 : \sqrt{2}$-rectangular cross section beveled at 45°

  Trapezoid or parallelogram



- All cut faces the same (square)
- Also using *skew* miter joints
- Vertices restricted to FCC lattice; joint angles are 90° or 120°
- At each joint, beam edges match

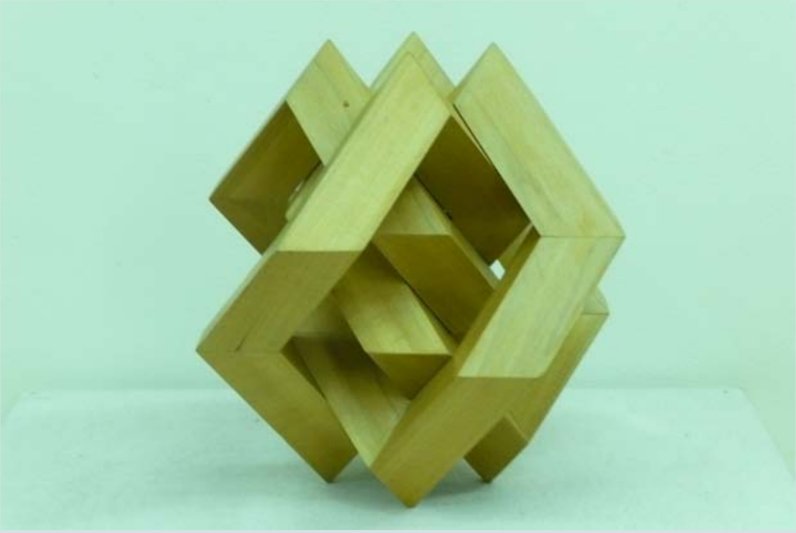
The latter is trivial for closed paths of this kind.

Exhaustive Investigation

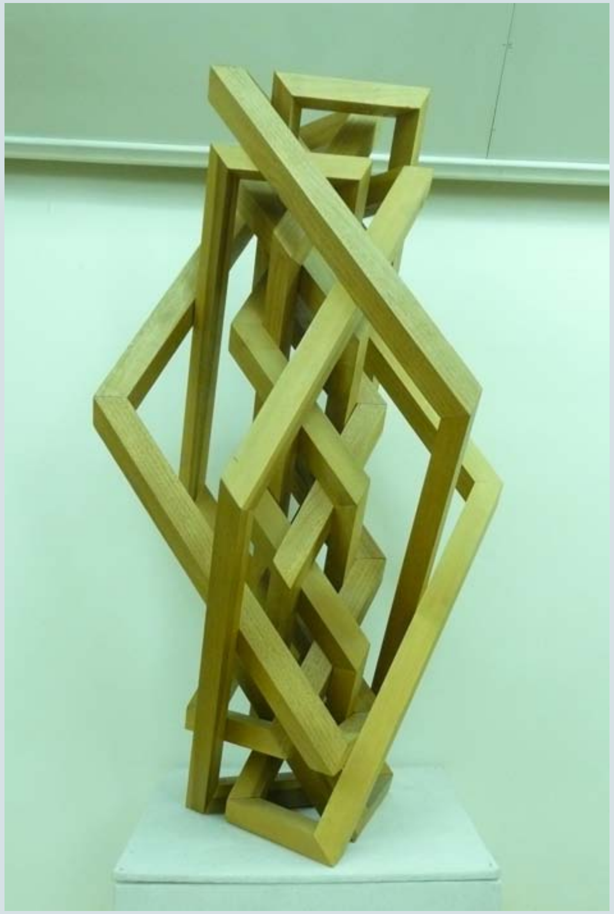
Enumerate all closed paths using N trapezoids with $1 : \sqrt{2}$-rectangular cross section and square cut faces (cf. *MathMaker*):

# Pieces	# Paths	Remarks
4	1	□ picture frame (planar)
6	2	incl. regular hexagon (planar)!
8	1	has many symmetries
10	0	why?
12	16	of which 1 “without” symmetry
14	10	
16	44	
⋮	⋮	
24	62 688	no knots; max. # right angles: 14

Trinity, Four-Unity, Hopeless Love I & II



Three-stranded Up-Down Spirals



Characteristics

Includes polylinks

Beam cross section: $1 : \sqrt{2}$-rhombus

Bevel angle: 45°

Cut face: square

Joint angles: 90° (regular), 120° (skew)

Allows segments to lie flush with each other

Lambiek

Lambiek

Date: approx. 2000

Material: Stainless steel

24 segments

Beam cross section: line segment (beam is a strip)

Joints: Regular fold

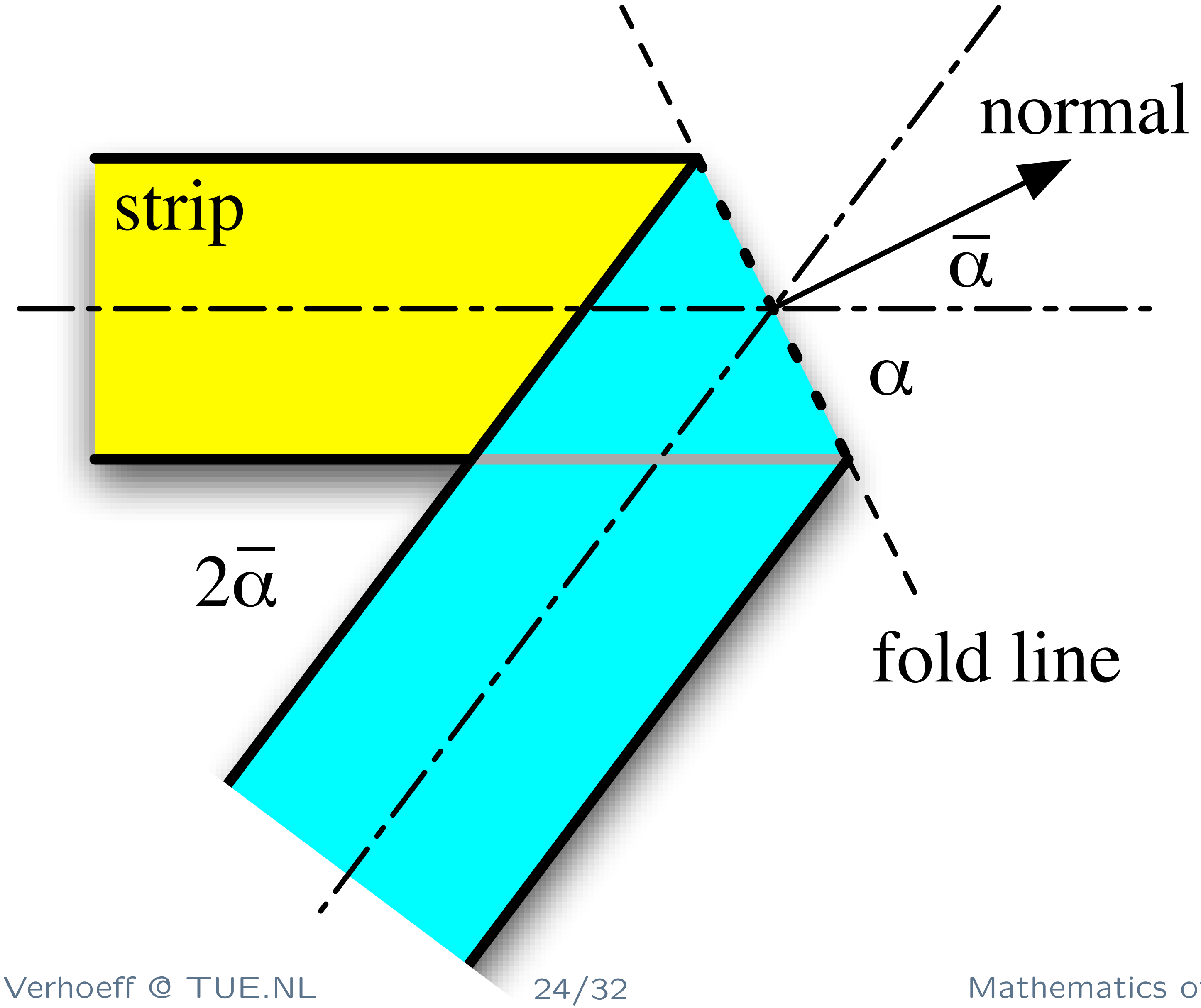
Topology: Trefoil knot, 180° Möbius twist

“Later” Work: Characteristics

- Closed polygonal paths
- Using a strip (line segment as cross section)
- Classical fold joints
- At each joint, (the two) beam edges match

The latter is not trivial: the vertex locations need tweaking.

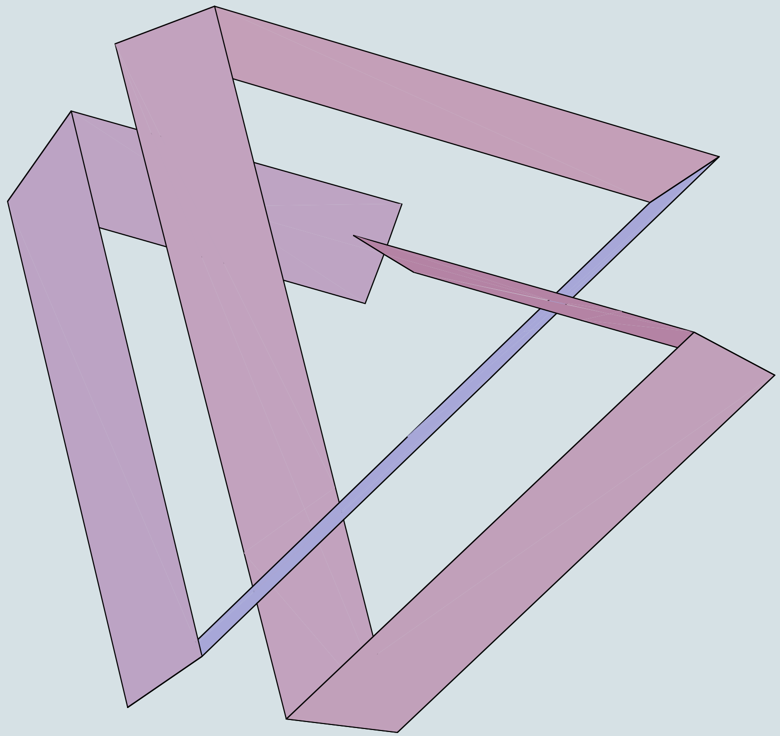
(Classical) Fold Joint



Even Fold Rotation Invariance Theorem

For an even number of vertices, the total amount of cross rotation is an inherent property of the polygonal path and does *not* depend on

- choice of initial segment
- initial rotation of strip around the center line
- strip width



Odd Fold Matching Theorem

For an odd number of vertices, there exist two rotations of the initial segment such that all fold joints match.

One yields a two-sided strip, the other a one-sided (Möbius) strip.

Hamilton Path on Cuboctahedron

Hamilton Path on Cuboctahedron

Date: mid 1990s

Materials: Maple (?)

12 segments; vertices form cuboctahedron

Beam cross section: parallelogram

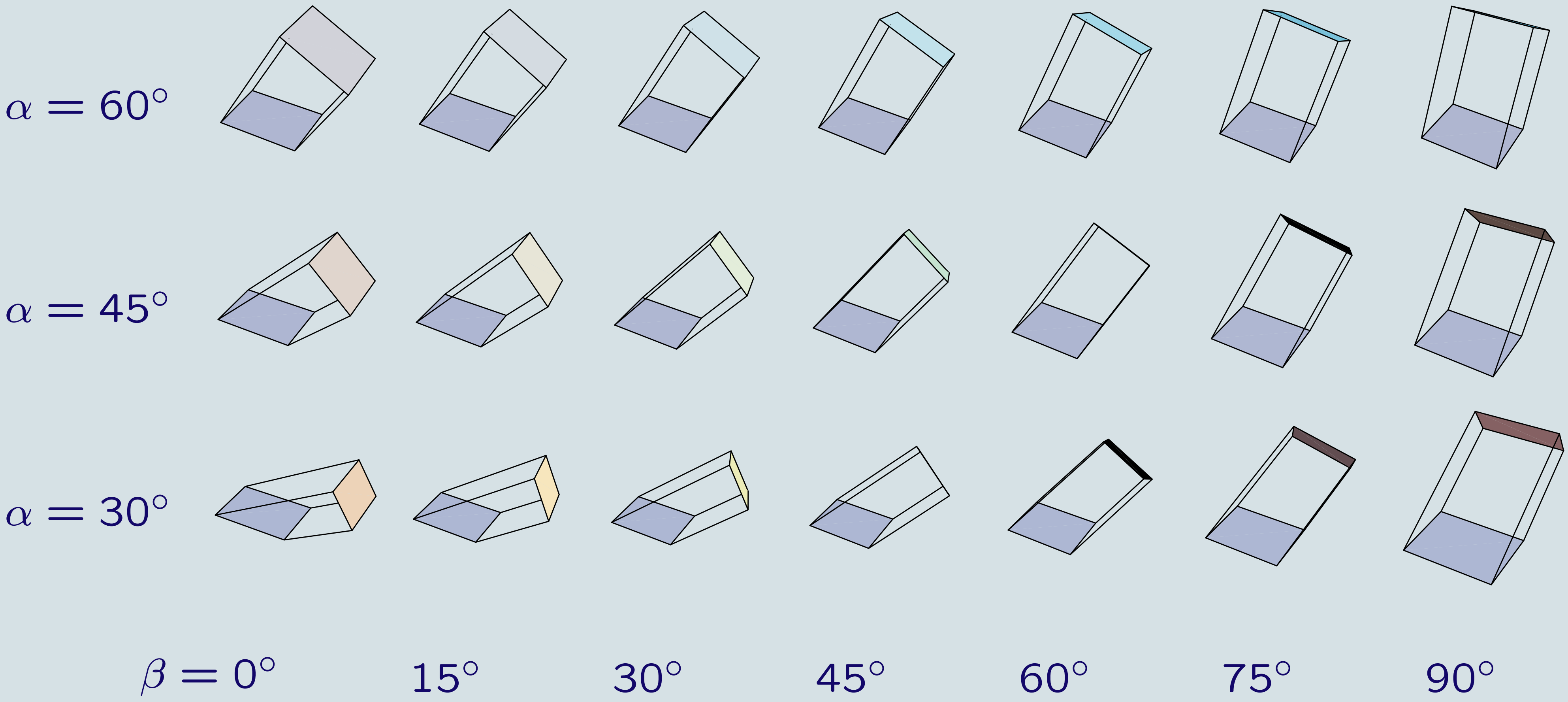
Bevel angles: $\alpha = 30^\circ$, $\beta = \arctan \frac{1}{\sqrt{2}} \approx 35.26\cdots^\circ$

Cut faces: $1 : \sqrt{2}$-rectangle

Joint angles: 60° (regular miter) and 120° (skew miter, flush)

0° Möbius twist

Shape of Cross Section Varies for Given Cut Face



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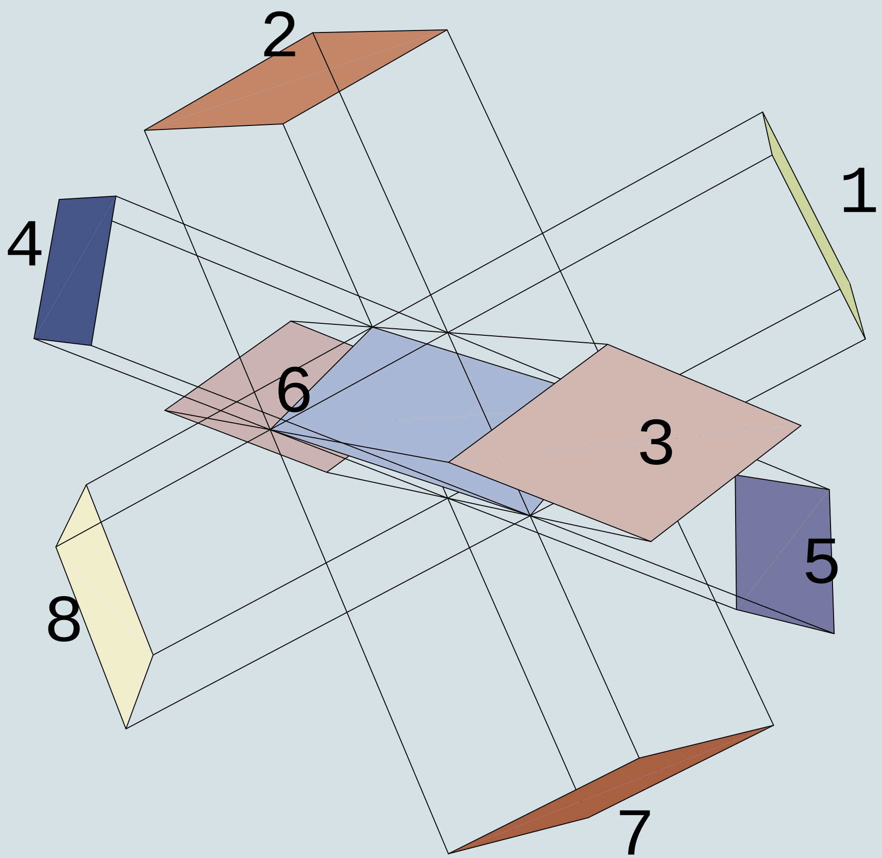
Miter Joint Angle Characterization Theorem

Let beam B have cross section S and cut face F.

> The set of all beams with cross section S and cut face F that form a matched (miter or fold) joint with B at F
>
> consists exactly of
>
> those beams obtained from B and its extension through F by applying a symmetry of F.

These symmetries are in 3D, including reflection in the plane that contains F.

$1 : \sqrt{2}$ Rectangle as Cut Face



Joint	Type	Angle
7–3	reg. miter	60°
7–1	skew miter	90°
7–4	skew miter	120°
7–6	reg. fold	120°
7–5	skew fold	60°
7–8	skew fold	90°
7–2	straight	180°

What's Next

The theme in this talk:

- Closed linear structures

- Beam cross sections all the same

Branching, scaling, continuous, . . .