Branching Miter Joints: Principles and Artwork

Presented at *Bridges 2010*
24 July 2010, Pécs, Hungary

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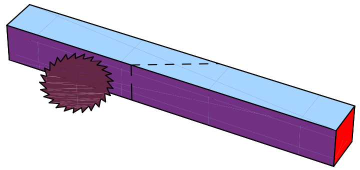
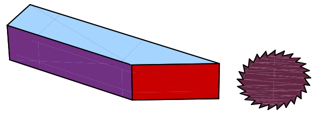
Mathematical Art by Koos Verhoeff

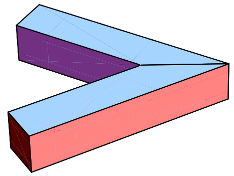
Miter Joints



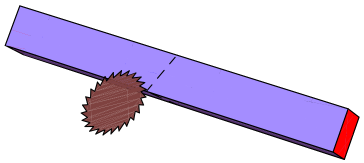
intact beam



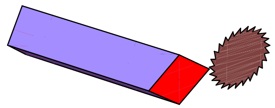
beveled at 30°



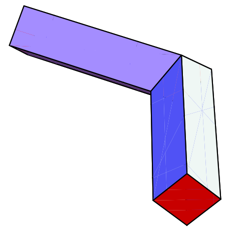
60° miter joint



intact beam
(rolled 45°)



beveled at 60°



120° miter joint

Mathematica Demonstrations Project: *Miter Joint and Fold Joint*

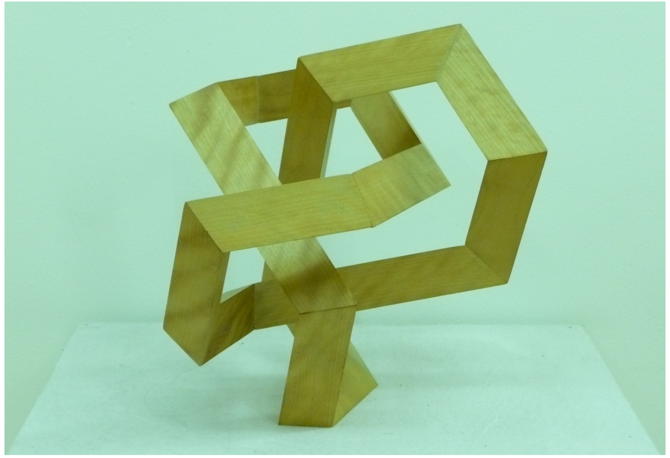
Characteristics of Regular Miter Joints

- Two beams of identical cross section meet at the joint
- The joint face lies in the interior angle bisector plane
- Longitudinal beam edges match up at the joint
- For any *fixed* joint angle, there is one continuous degree of freedom:
  - rotation of the beam's cross section about the longitudinal axis

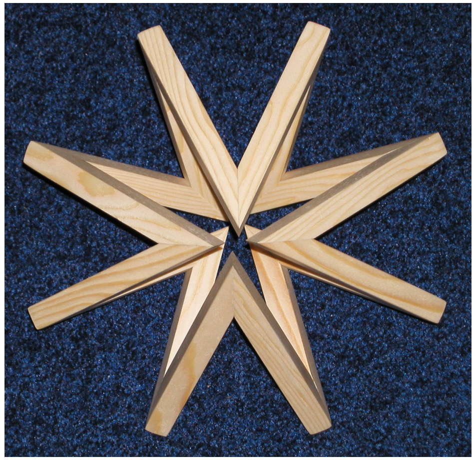
Three Families of Binary Miter Joint Art

'Tinkering'

Lattice walking



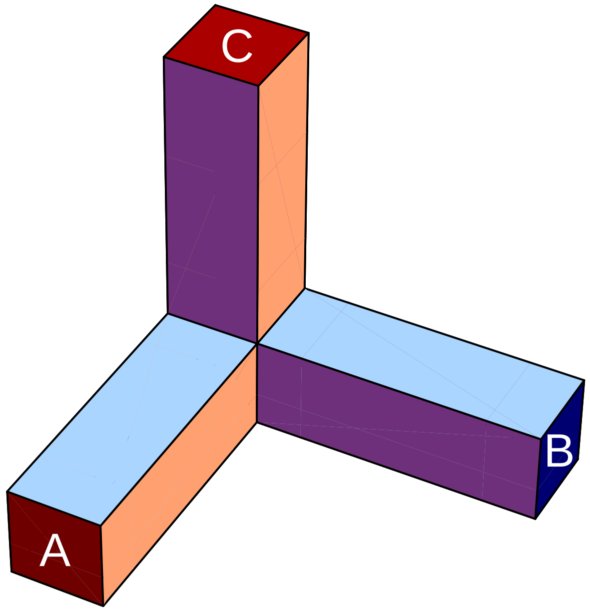
Constant torsion



Problem: make beam edges match all the way round

Mathematica Demonstrations Project: *Mitering A Closed 3D Path*

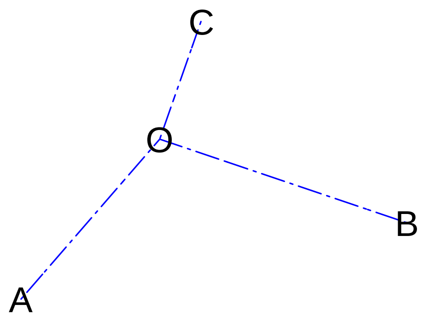
Branching Miter Joints



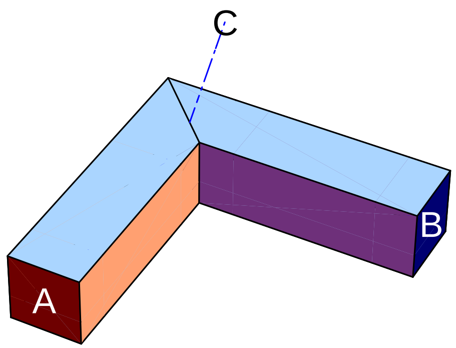
What if we want to connect three or more beams in a single joint?

- Longitudinal edges should nicely match up at the joint

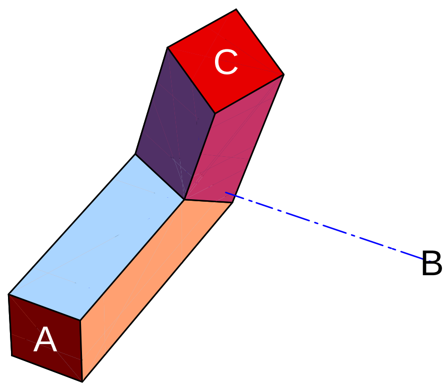
Ternary Meeting Point Induces Three Binary Miter Joints



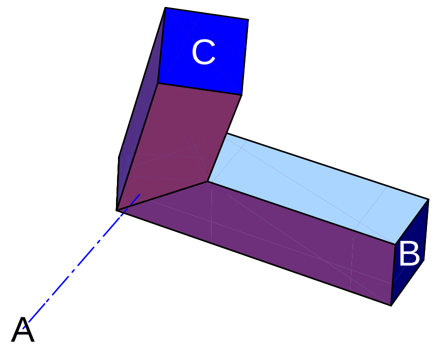
A:	0° W,	0° N
B:	90° W,	0° N
C:	45° W,	61° N



A forces B

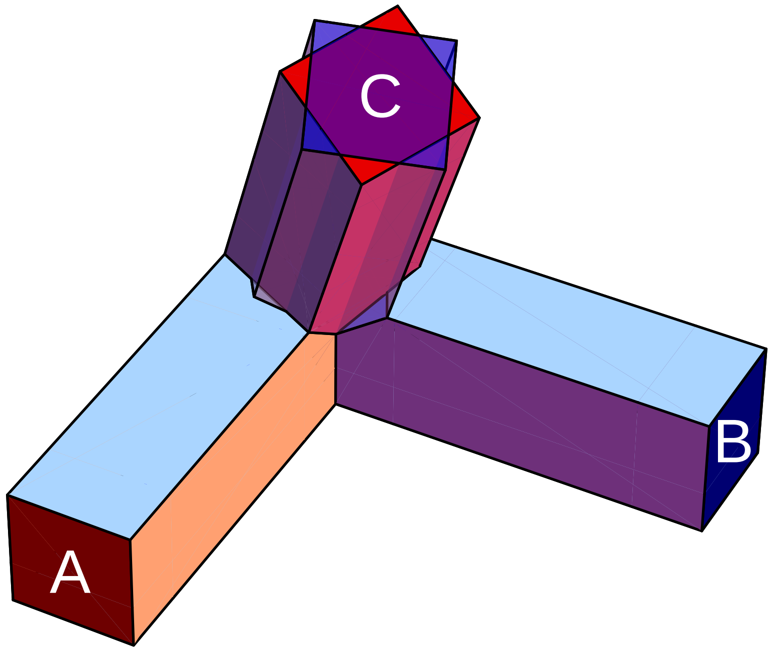


A forces C_A



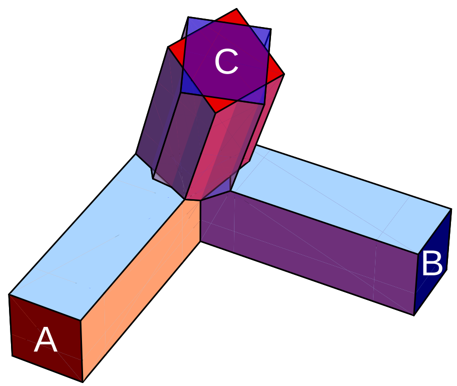
B forces C_B

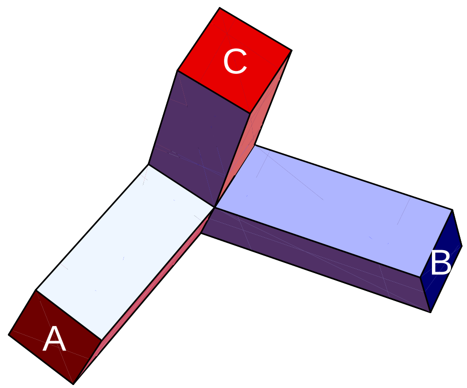
Ternary Miter Joint: Mismatch



Superimposing the binary miter joints reveals a mismatch between beams C_A and C_B

Ternary Miter Joint: Repairing the Mismatch



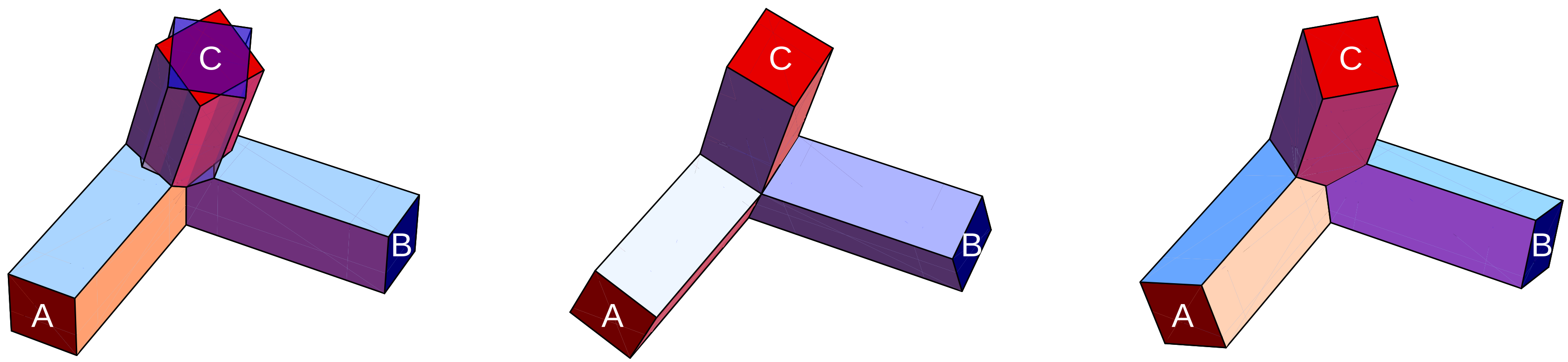


Beam A rotates clockwise $\Rightarrow$ beams B and C_A rotate *counterclockwise*
$\Rightarrow$ beam C_B rotates *clockwise*

C_A and C_B rotate in opposite direction.

Mismatch can be canceled by suitable rotation of beam A.

Ternary Miter Joint: Repairing the Mismatch in Two Ways

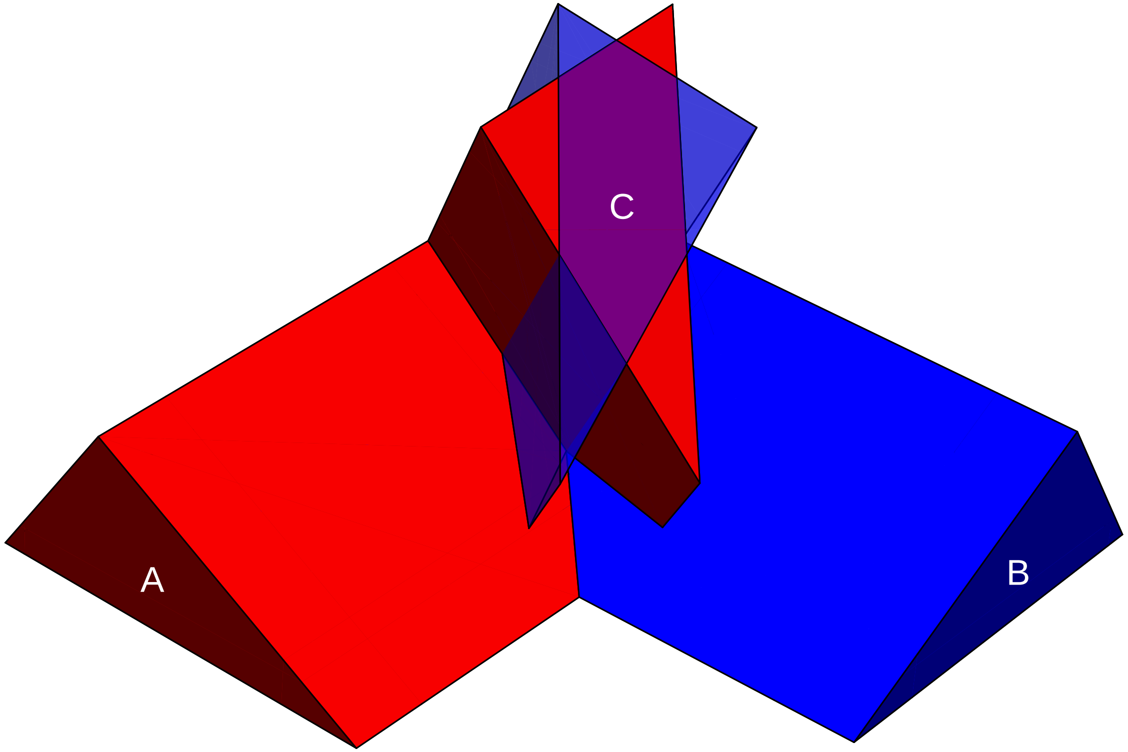


C_A and C_B rotate in opposite direction.

Angle *difference* $C_B - C_A$ changes at double the 'speed' of beam A.

Two proper matchings if cross section is mirror symmetric.

Ternary Miter Joint Not Always Repairable



Matched ternary miter joint is impossible to obtain,
if cross section is *not* mirror symmetric.

Matched Ternary Miter Joint

If the angles between beams A, B, and C are fixed, then

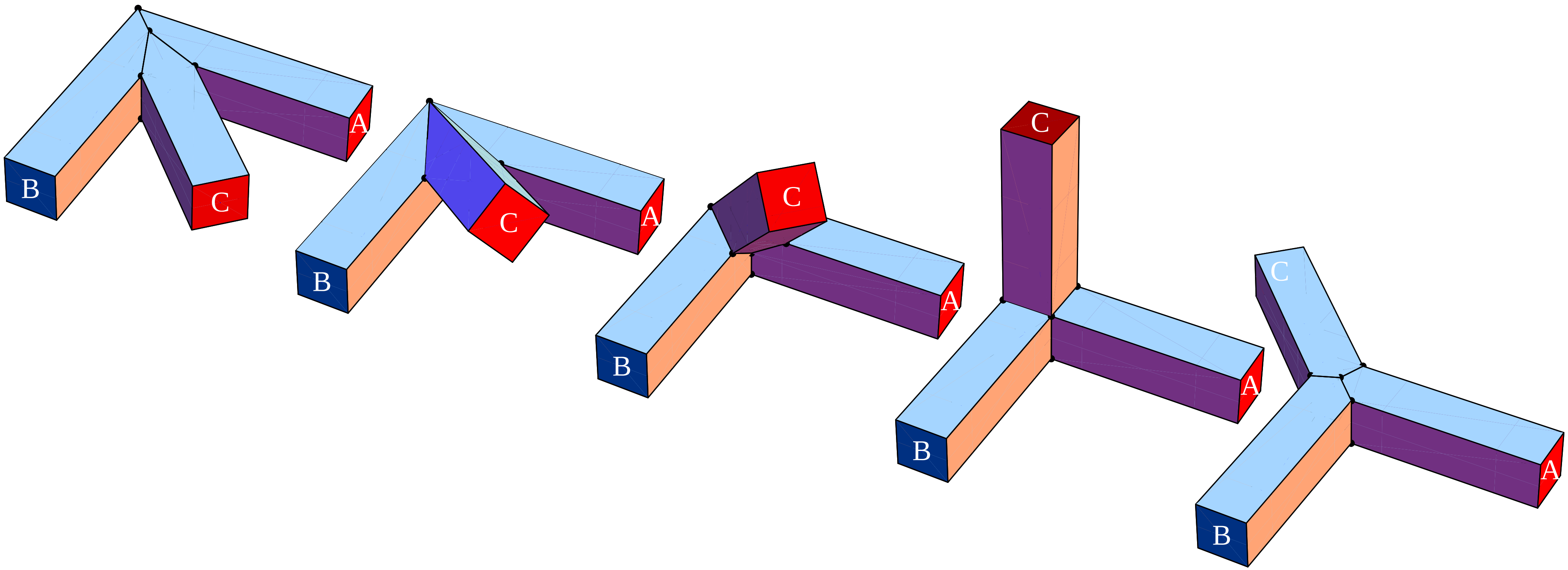
. . . there are 0 or 2 ways to obtain a matched ternary miter joint

. . . by rotating the cross section;

. . . the number depends on the mirror symmetry of the cross section.

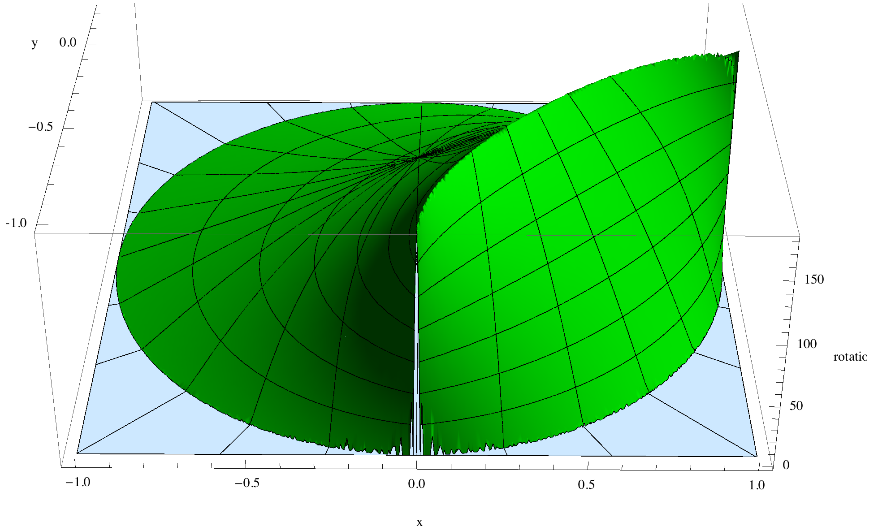
Binary miter joints with fixed angle allow *continuous* beam rotation, while preserving matched edges.

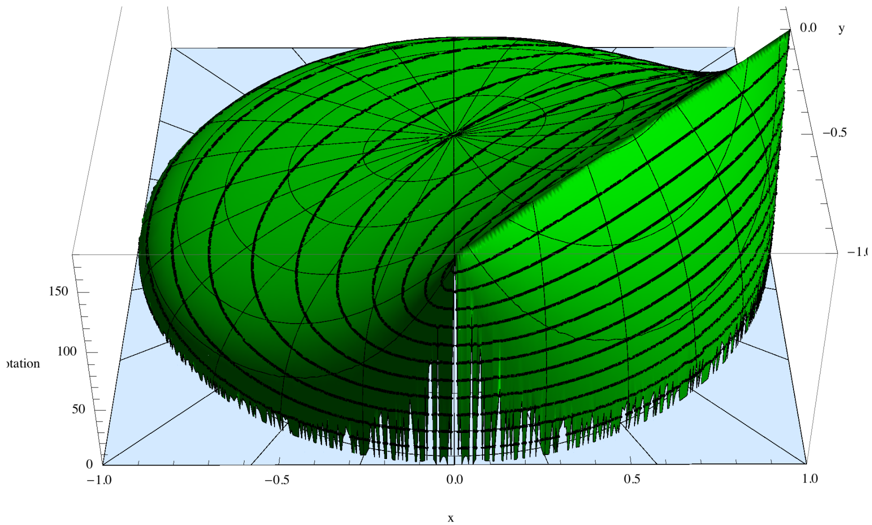
Obtaining a Matched Ternary Miter Joint by Varying Angles

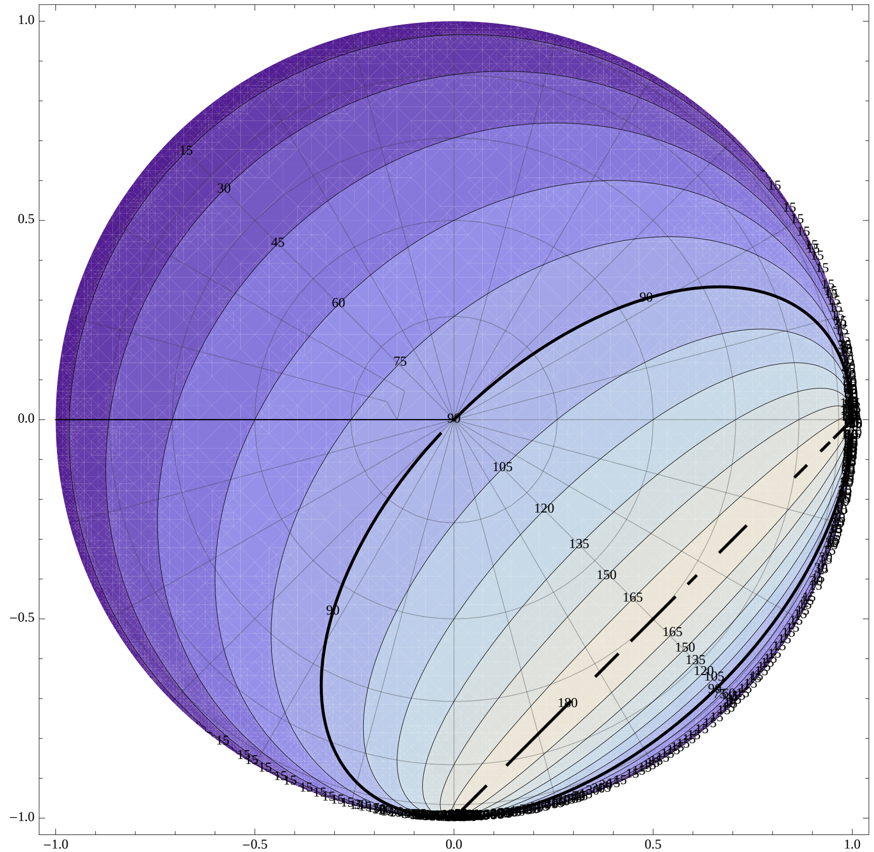


Given a binary miter joint connecting square beams A and B, there are *five* directions for beam C to make a proper ternary miter joint, if it is restricted to the upper-half of the angle bisector plane.

How is Mismatch Related to Position of Beam C?

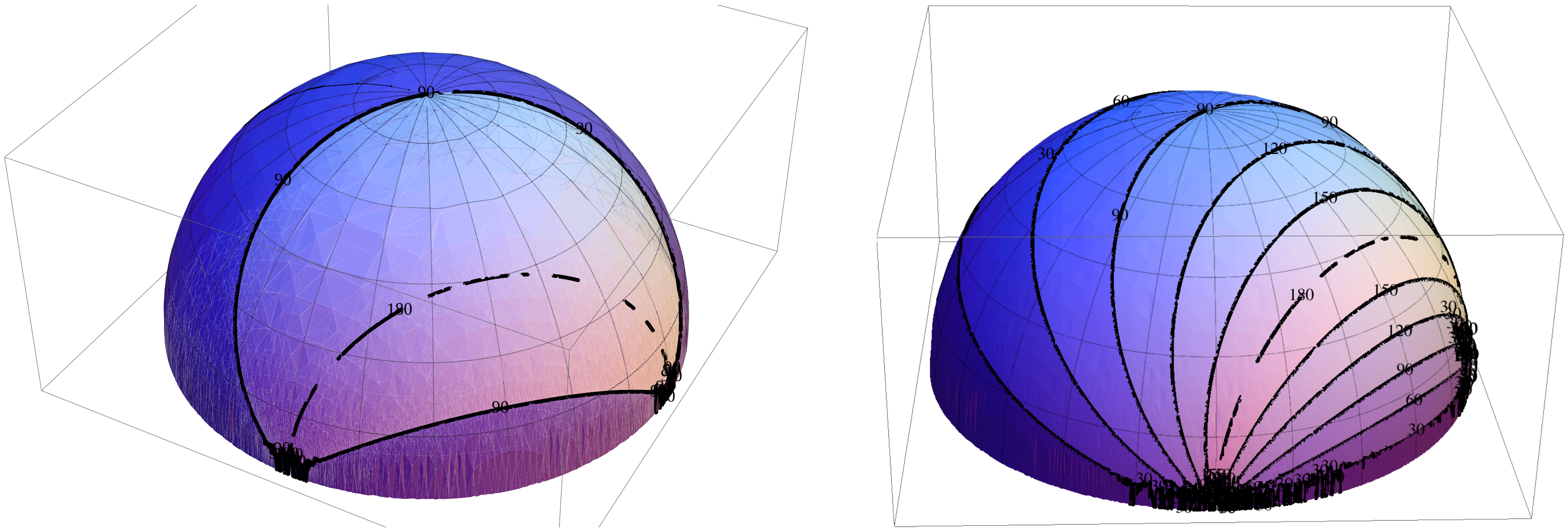






Various ways of plotting the mismatch as function of the position of beam C

Countour Plot of Mismatch as Function of Beam C



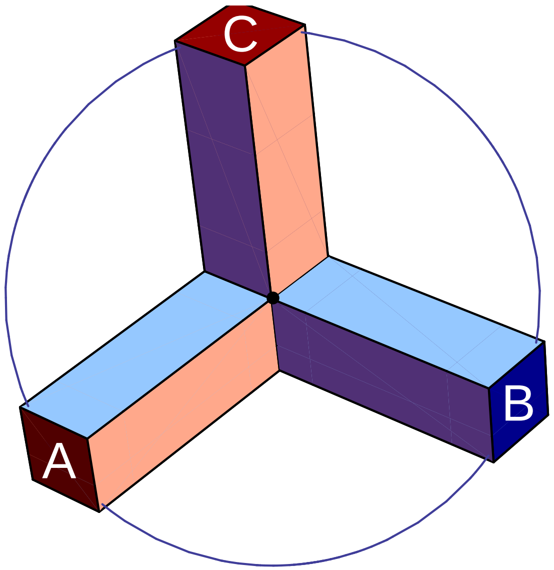
Plot of the rotational mismatch at beam C when square beams A and B are mitered at 90°.

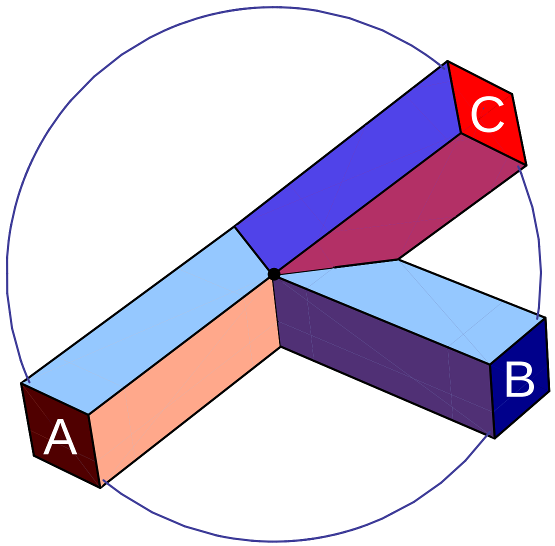
The direction of beam C is determined by its endpoint on the sphere.

Left: mismatches of 90° and 180° have been marked; on the equator the mismatch is 0°; Right: multiples of 30°

Ternary Miter Joint Theorem

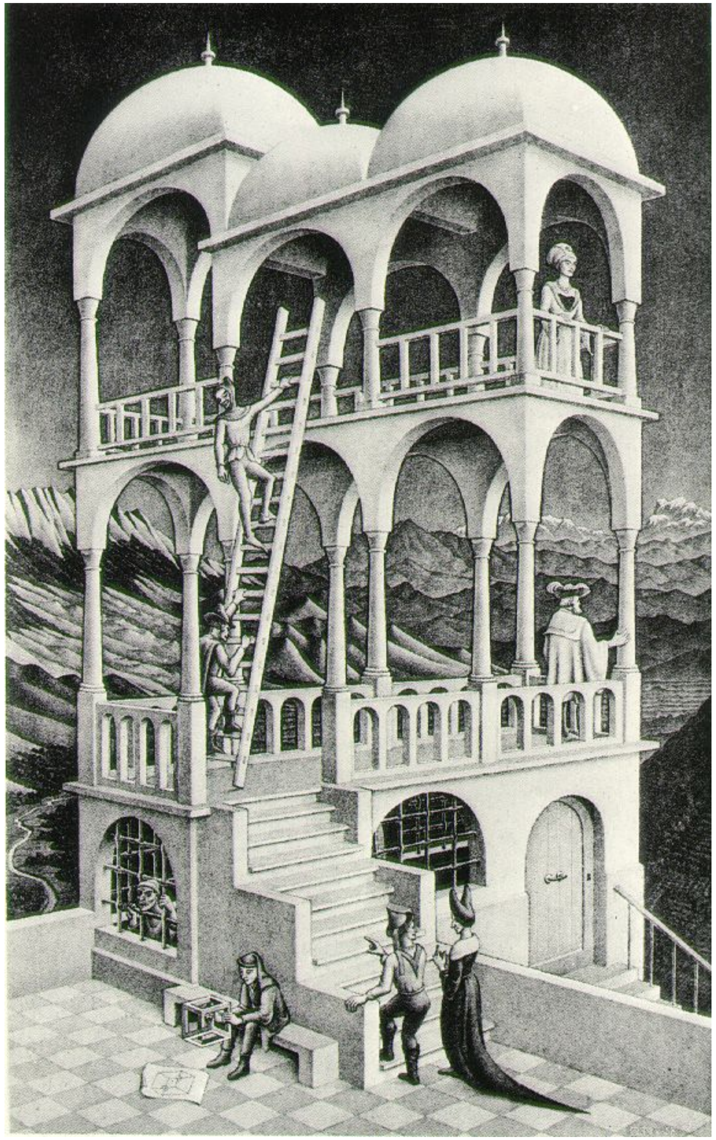
The mismatch is constant
when C moves on the unit hemisphere
along the circle through ABC.





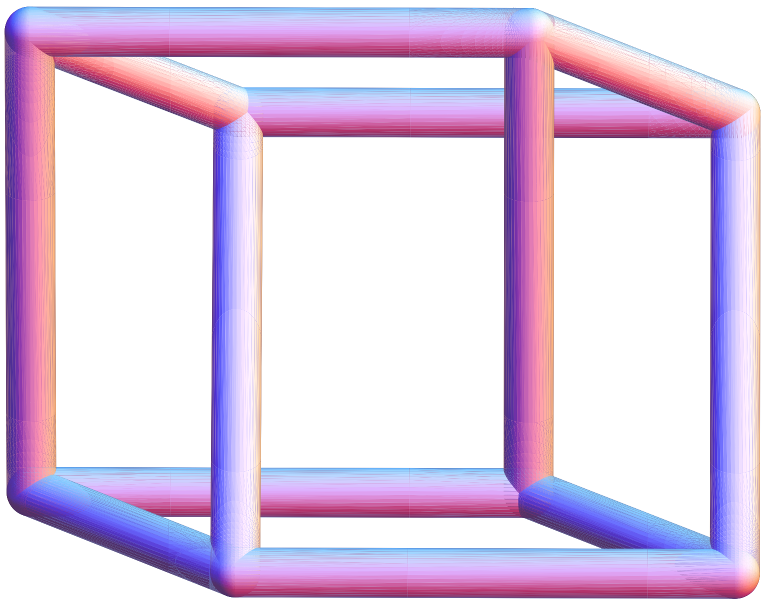
Proof in appendix.

Escher's Belvedere Lithograph

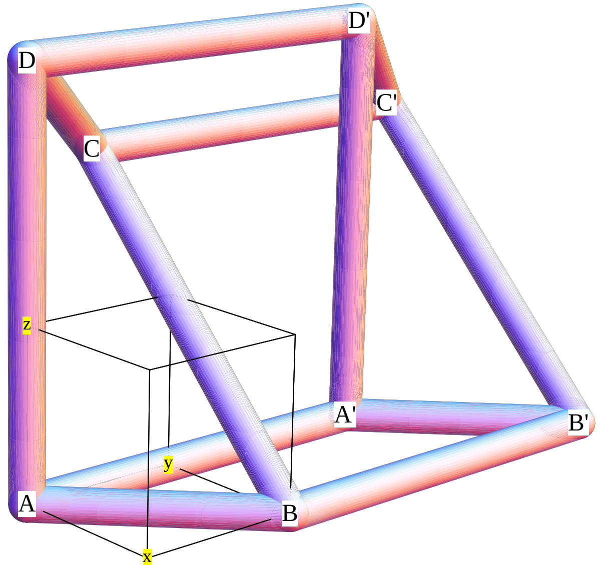


Ternary Miter Joint Artwork: Impossible Cuboid

- Idea by Dick Baas Becking
- Design with ternary miter joints by Koos Verhoeff
- First wooden sculpture by Popke Bakker

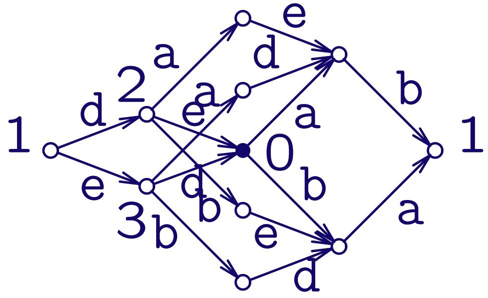


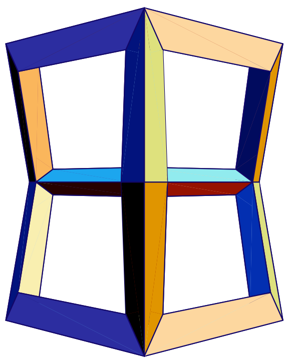
Impossible Cuboid Design Parameters

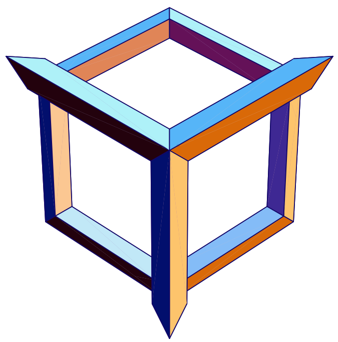


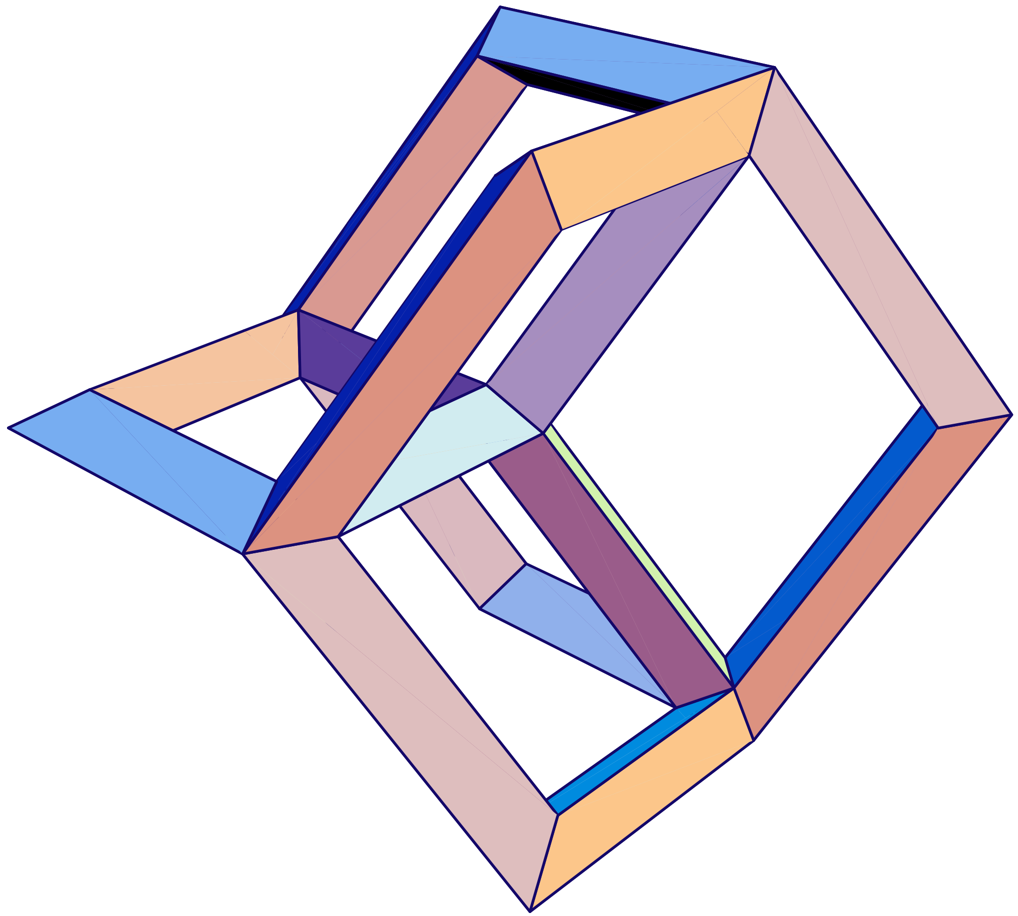
12 square beams; $6 + 2 = 8$ ternary miter joints;
beam lengths $AB : BC = 1 : 1 + 1/\sqrt{2} \approx 7 : 12$;
beams rotated over $\arctan(\sqrt{2} - 1) = 22.5°$;
6 'faces': 2 squares ($AA'D'D$, $BB'C'C$),
2 parallelograms ($AA'B'B$, $CC'D'D$ with $\angle BAA' = 45°$),
2 non-planar quadrangles ($ABCD$, $A'B'C'D'$: $\angle ABC = \angle CDA = 60°$, and $\angle DAB = \angle BCD = 90°$)

Miter Joints with Four Branches





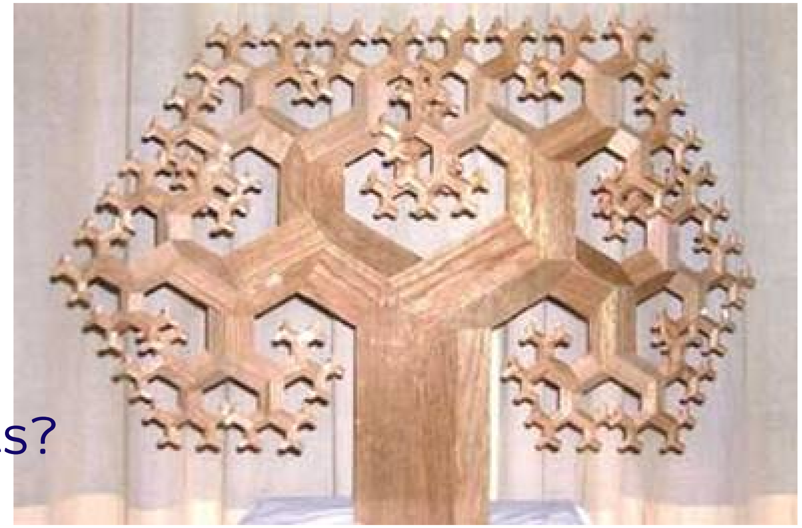




Conclusion

- Ternary miter joint: condition for proper matching of edges
- Theorem about matching ternary miter joint
- Artwork involving ternary and quaternary miter joints

What else:



- Fractal trees by Koos Verhoeff do *not* involve ternary miter joints.
- How about skew ternary miter joints? (cut not in bisector plane)

Related Work

- Tom Verhoeff & Koos Verhoeff.
  "The Mathematics of Mitering and Its Artful Application",
  Bridges 2008, pp.225–234.

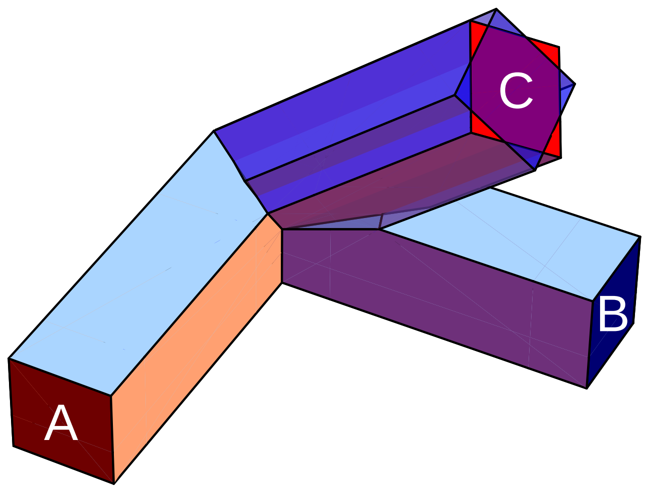
- Tom Verhoeff & Koos Verhoeff.
  "Regular 3D Polygonal Circuits of Constant Torsion",
  Bridges 2009, anada, pp.223–230.

- Tom Verhoeff.
  "3D Turtle Geometry: Artwork, Theory, Program Equivalence and Symmetry".
  Int. Journal of Arts and Techology, **3**(2/3):288-319 (2010).

Also see: `http://www.win.tue.nl/~wstomv/publications/`

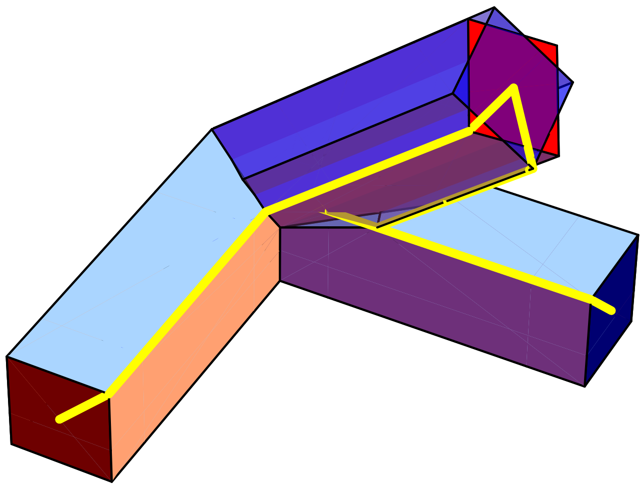
Proof of Ternary Miter Joint Theorem

The mismatch is constant
when C moves on the unit hemisphere
along the circle through ABC.



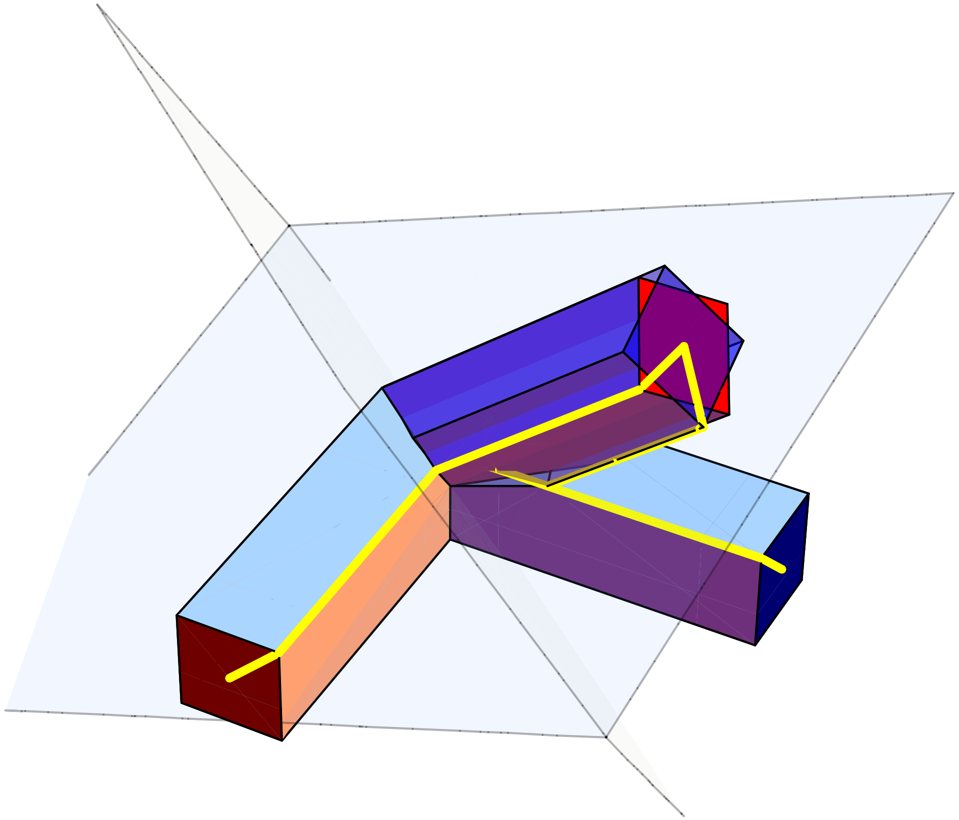
Proof of Ternary Miter Joint Theorem (2)

Consider the seams that determine the mismatch.



Proof of Ternary Miter Joint Theorem (3)

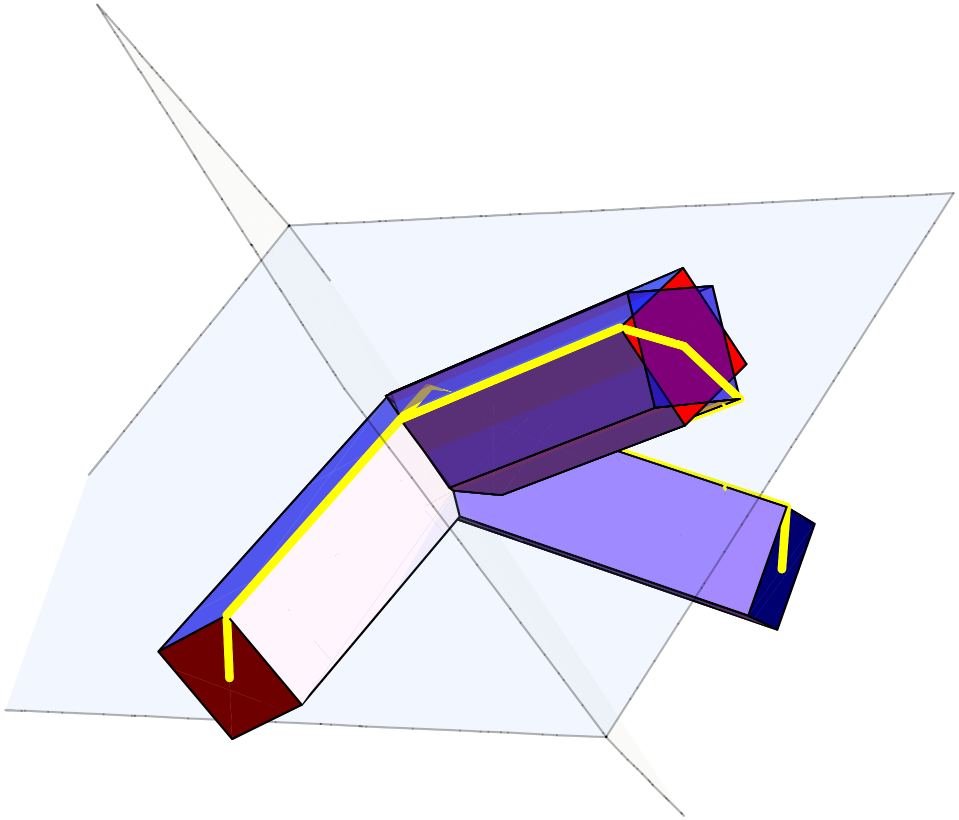
Seams are obtained by reflection in the interior bisector planes of angles AOC and BOC.



Proof of Ternary Miter Joint Theorem (4)

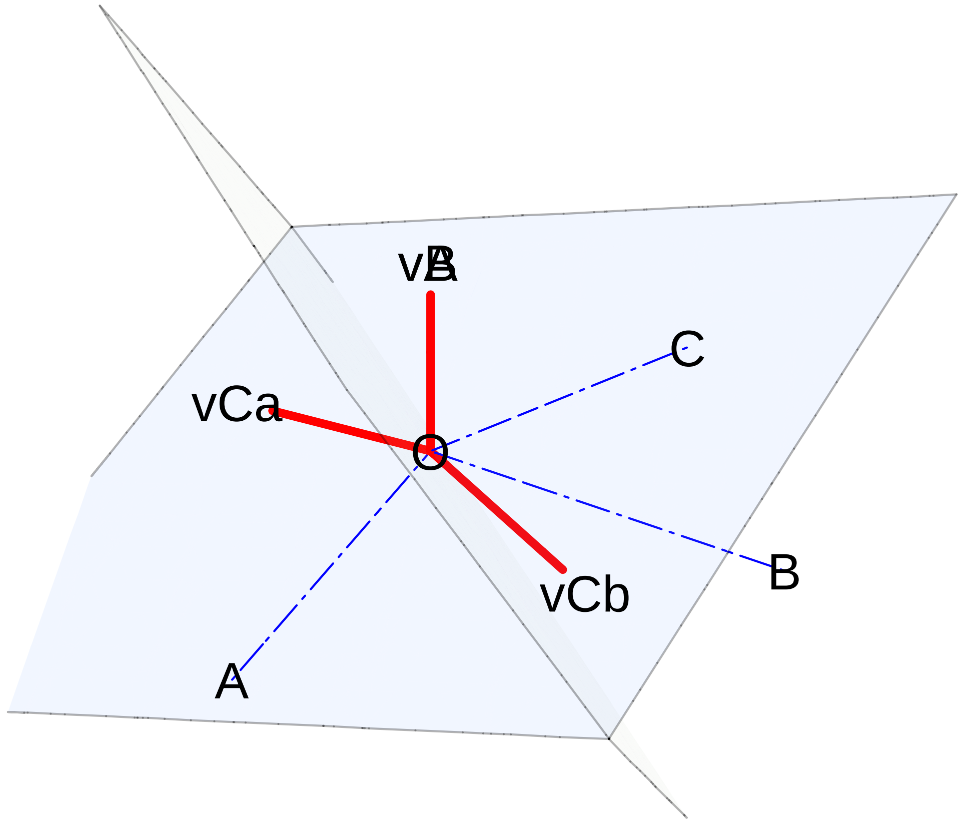
Beam rotation changes the mismatch by a constant.

Rotate until seam pointing vectors v_A and v_B are aligned.



Proof of Ternary Miter Joint Theorem (5)

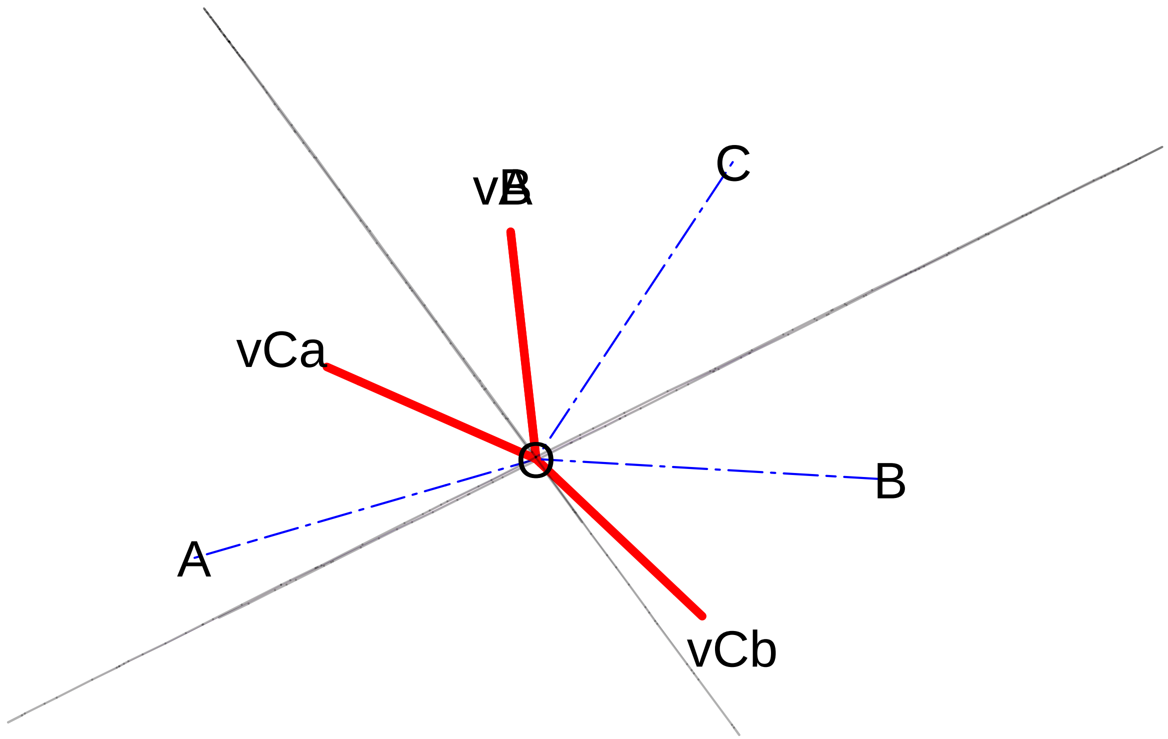
Translate seam pointing vectors to the origin: $v_A = v_B$.



Proof of Ternary Miter Joint Theorem (6)

Change view to look along intersection of bisector planes.

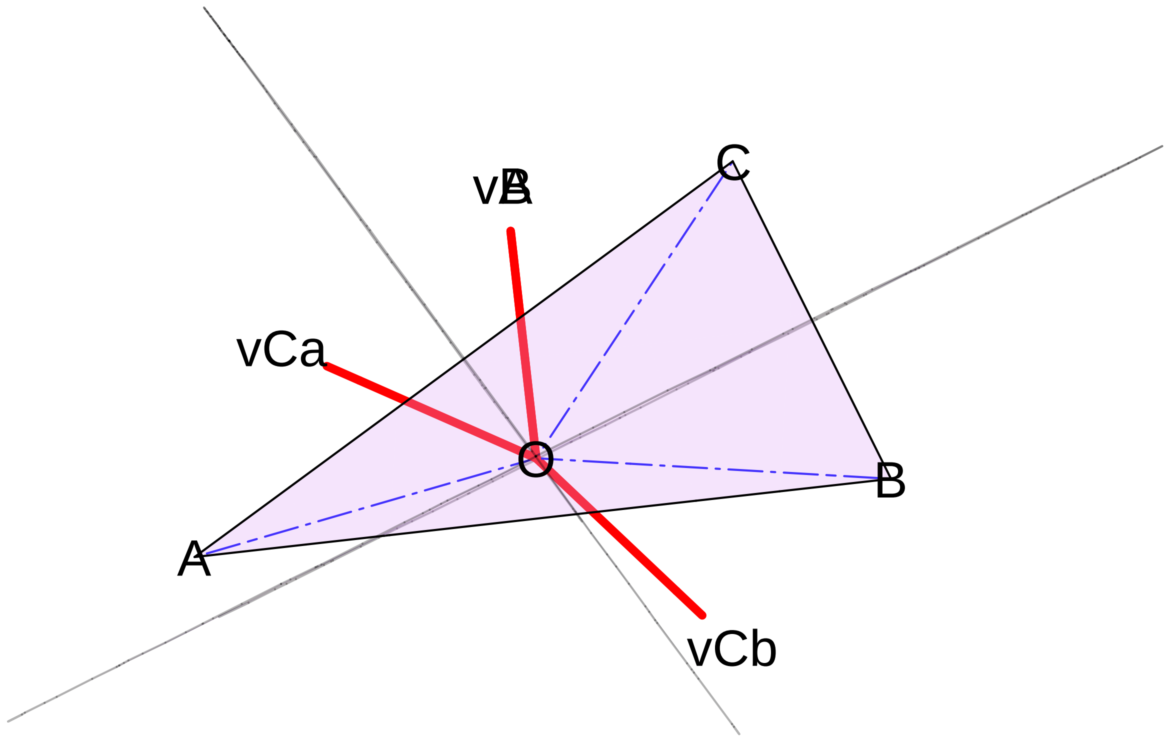
Observe that $\angle v_A v_C a + \angle v_B v_C b = 2\angle$bisector planes of $\angle AOC, \angle BOC$.



Proof of Ternary Miter Joint Theorem (7)

Consider triangle ABC. Triangles AOC and BOC are isosceles.

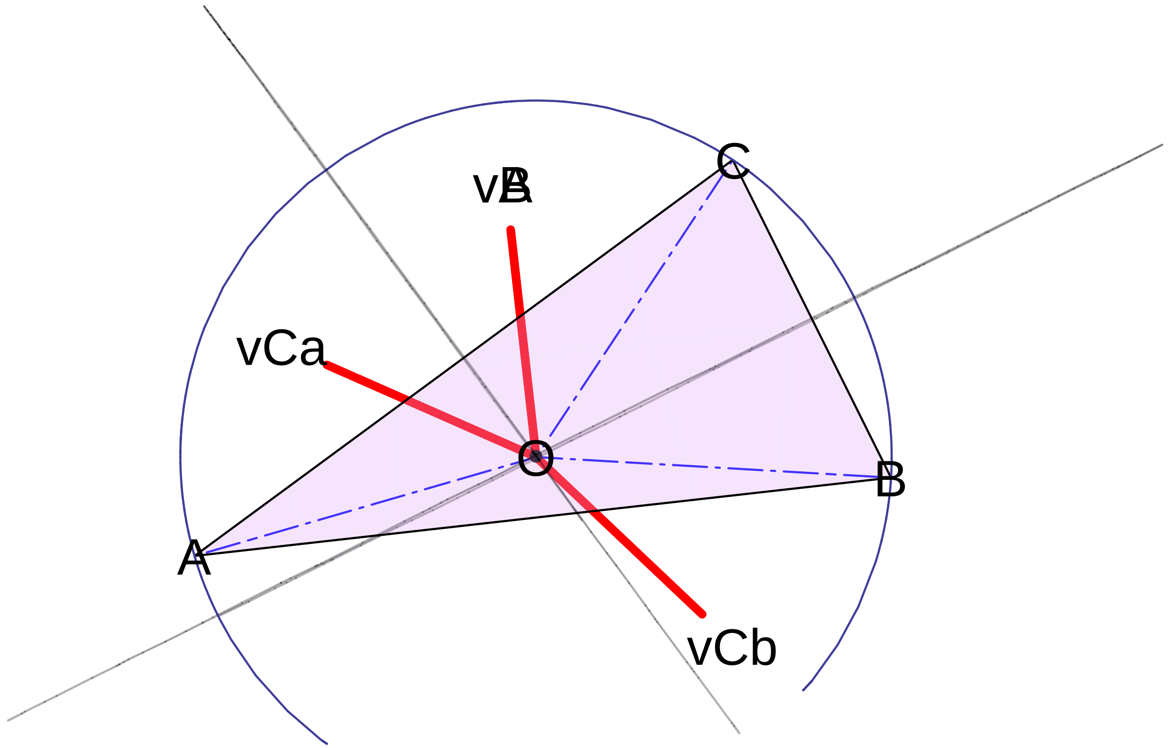
The bisector planes intersect $\triangle ABC$ at its perpendicular bisectors.



Proof of Ternary Miter Joint Theorem (8)

The perpendicular bisectors intersect at the center of the circumcircle.

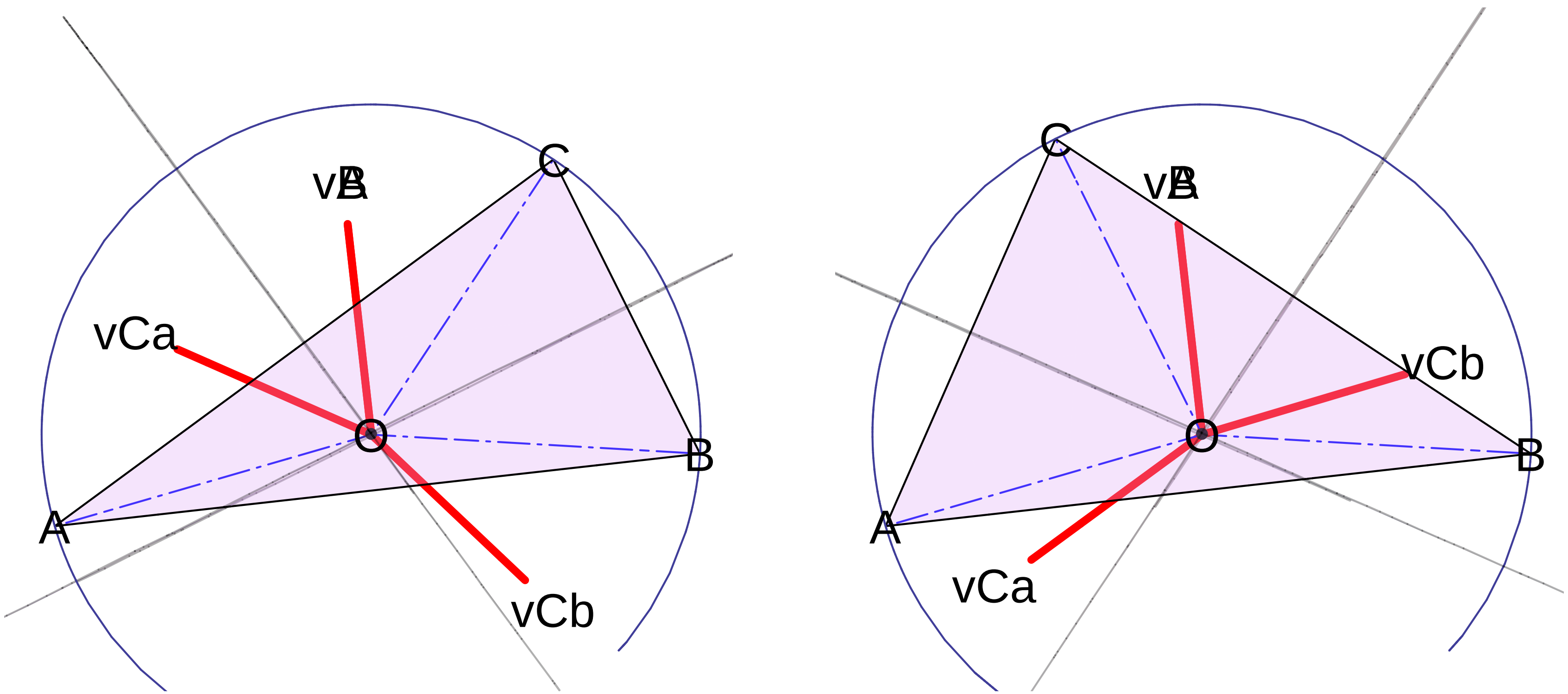
Goal: To prove that mismatch is does not change along this circle.



Proof of Ternary Miter Joint Theorem (9)

Moving C along the circumcircle, preserves circumcenter and $\angle ACB$.

Hence, angle between bisector planes of AC and BC is preserved.



Proof of Ternary Miter Joint Theorem (10)

If angle between bisector planes of $\angle AC$ and $\angle BC$ is preserved,
and direction of their intersection line is preserved,
then angle between seam pointing vectors $v_C a$, $v_C b$ is preserved.

Hence, mismatch is preserved. Q.E.D.