From Chain-link Fence to Space-spanning Helical Structures

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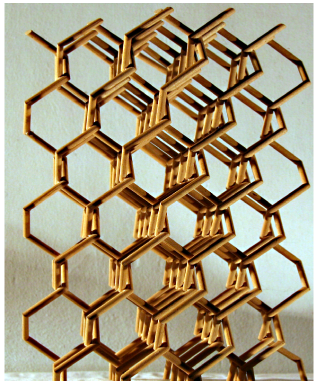

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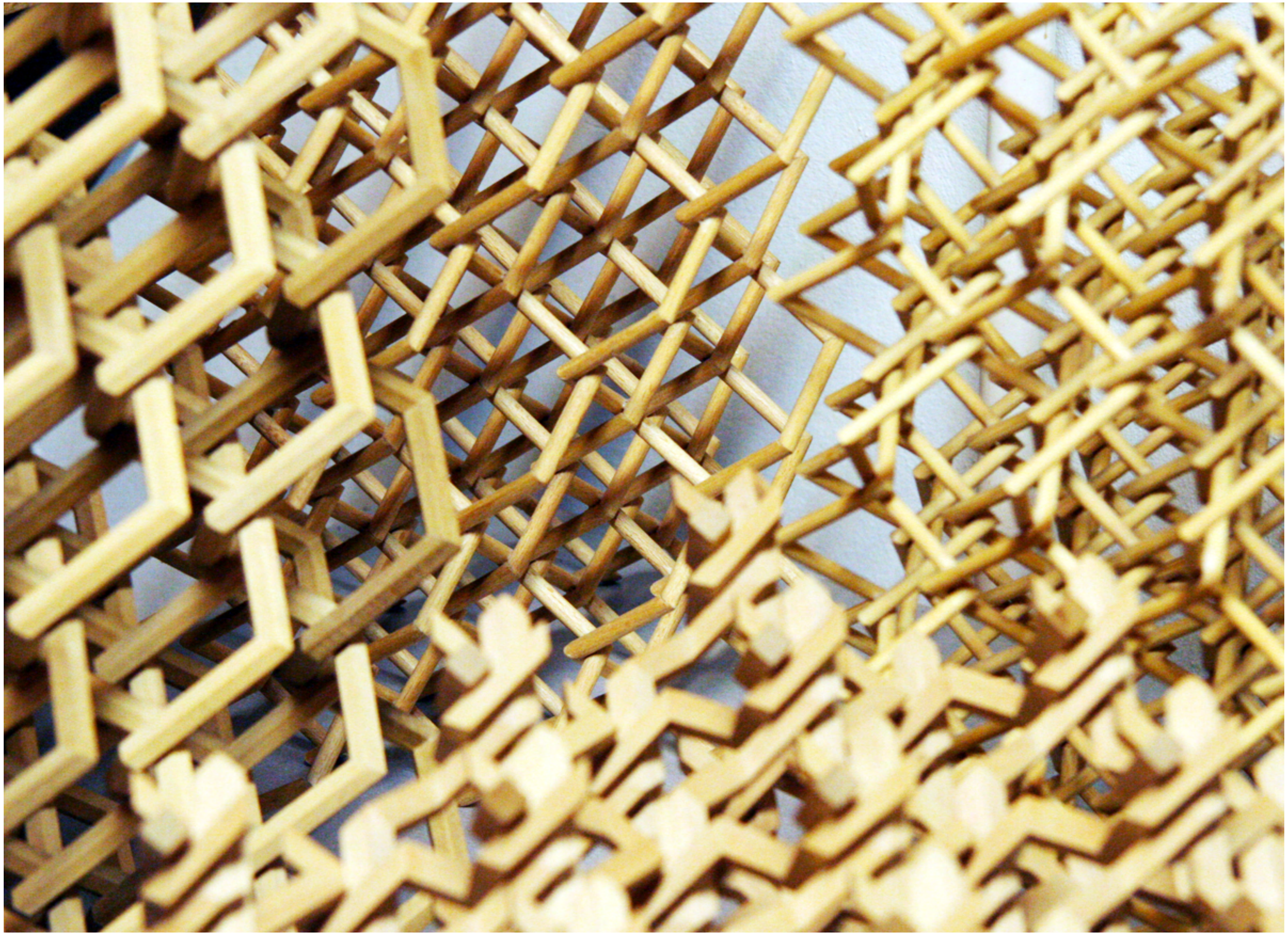
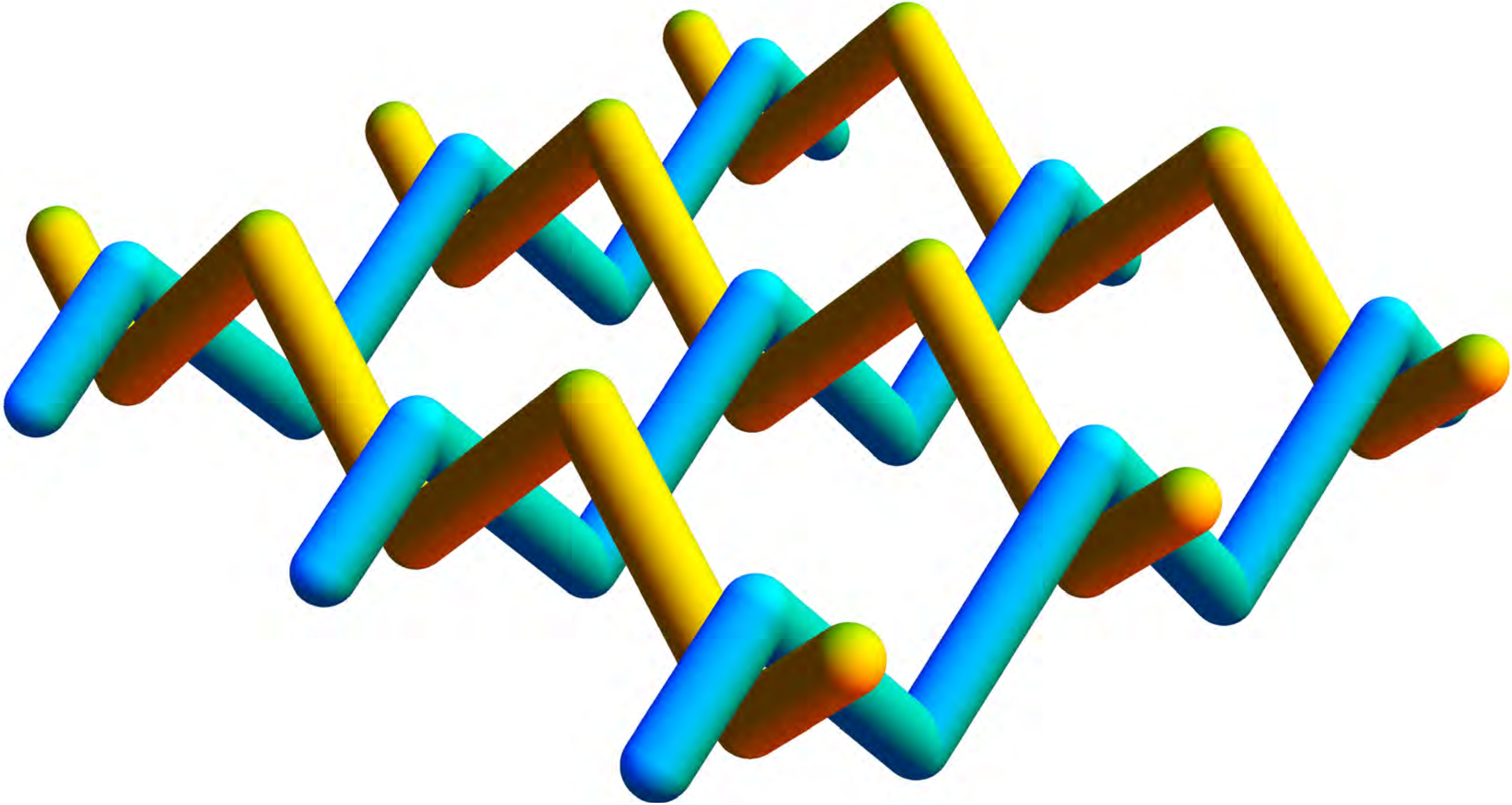
Intriguing Artworks, Hidden in a Corner



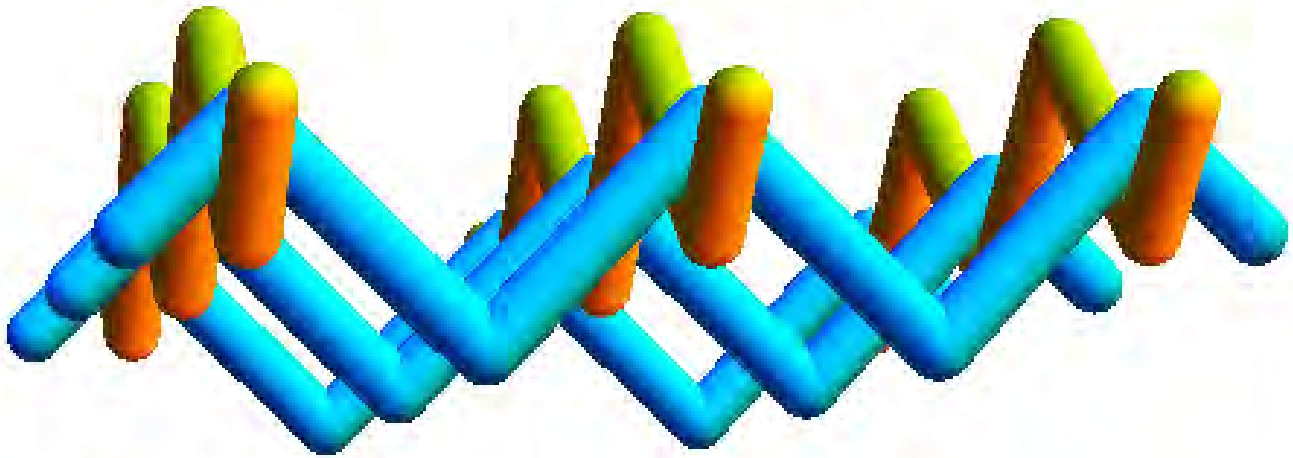
Chain-link Fence

Planar Zigzag

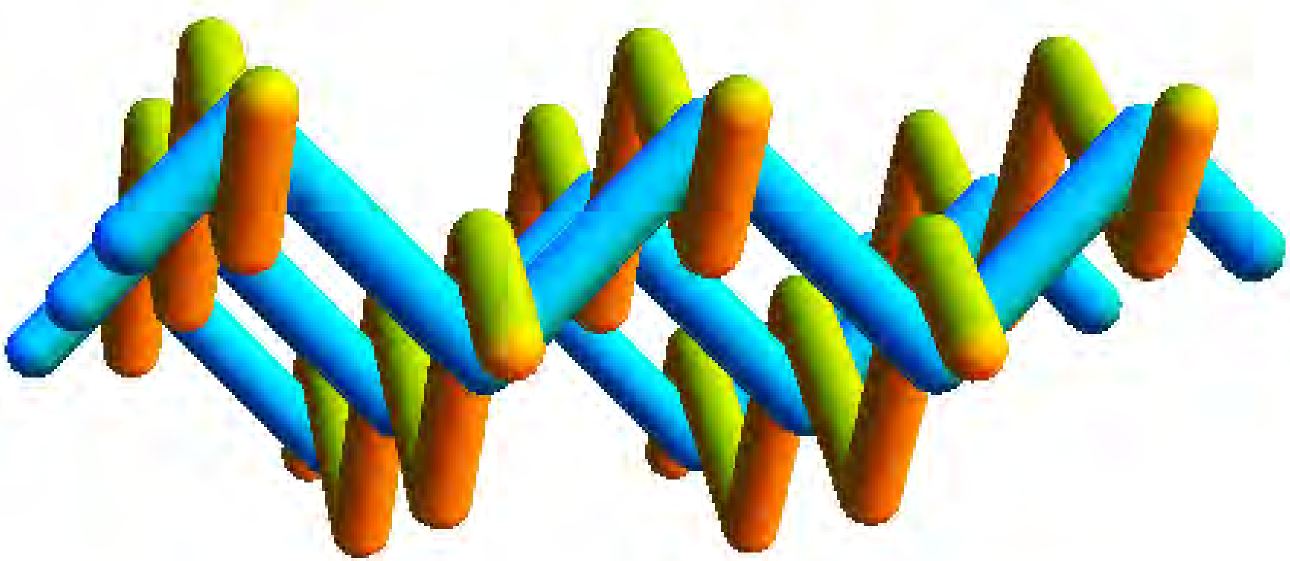
Layer of Orthogonally Crossing Zigzags



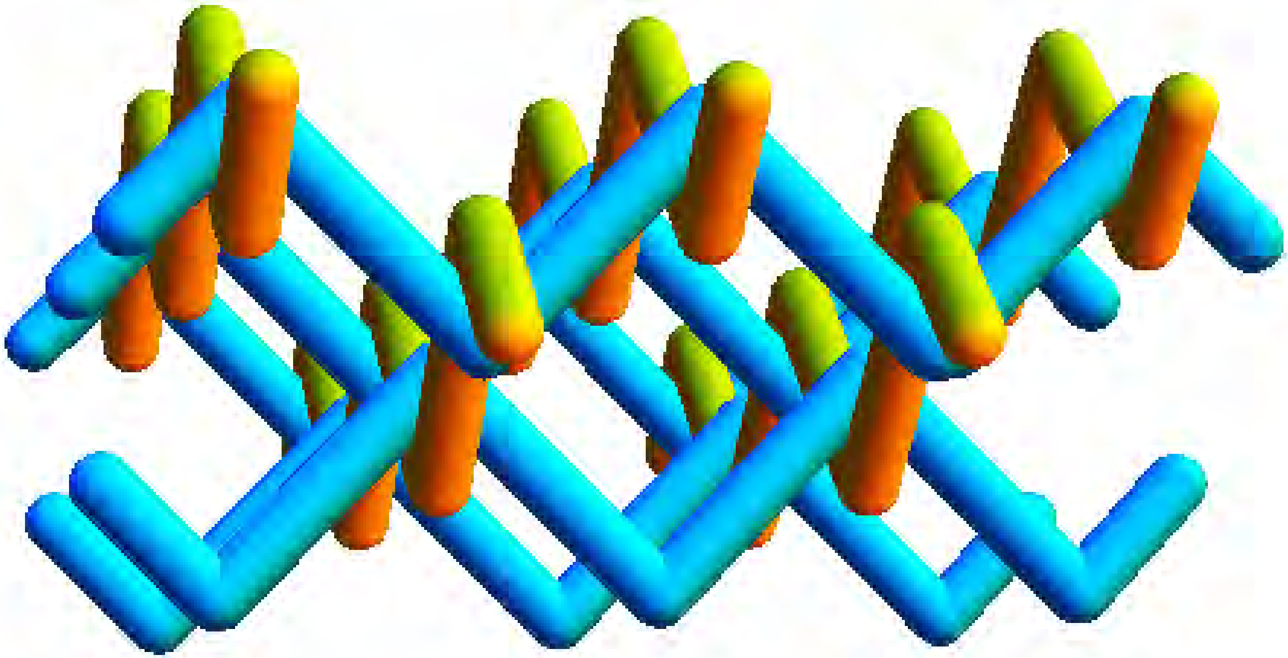
Woven Layers of Orthogonally Crossing Zigzags



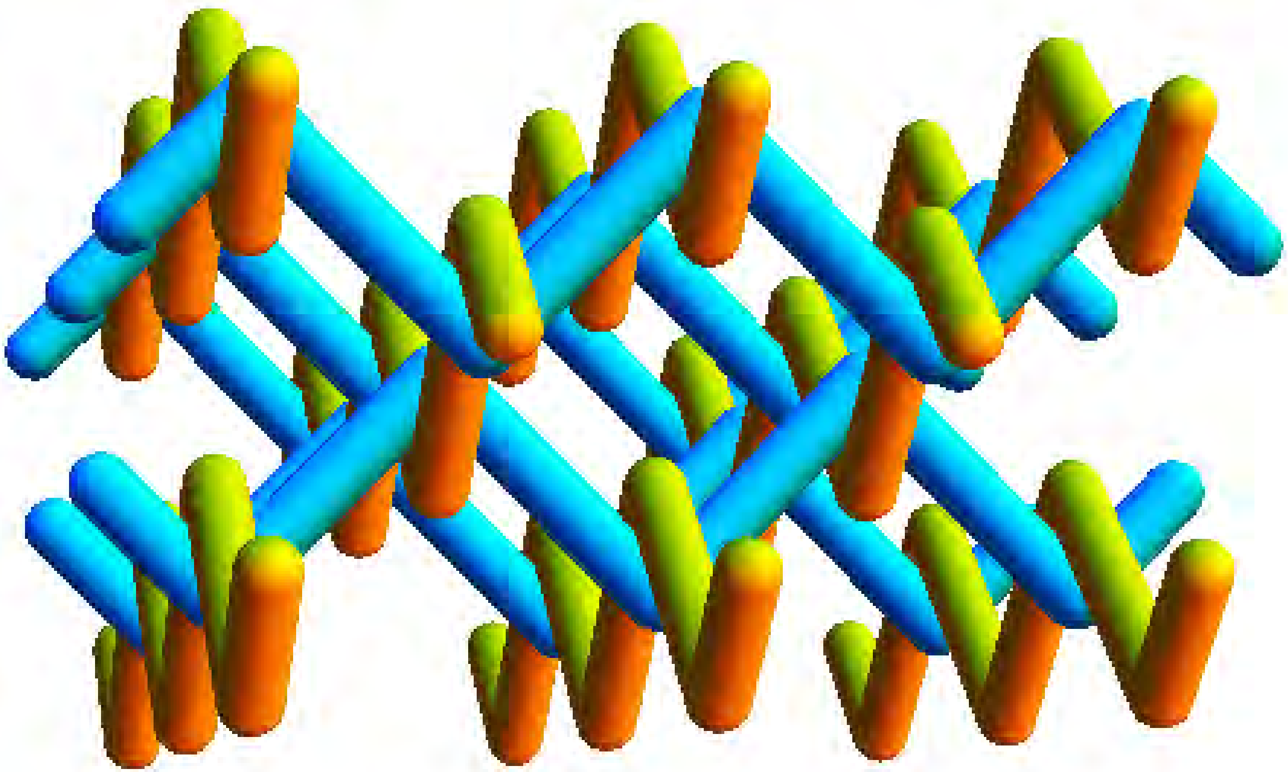
Woven Layers of Orthogonally Crossing Zigzags



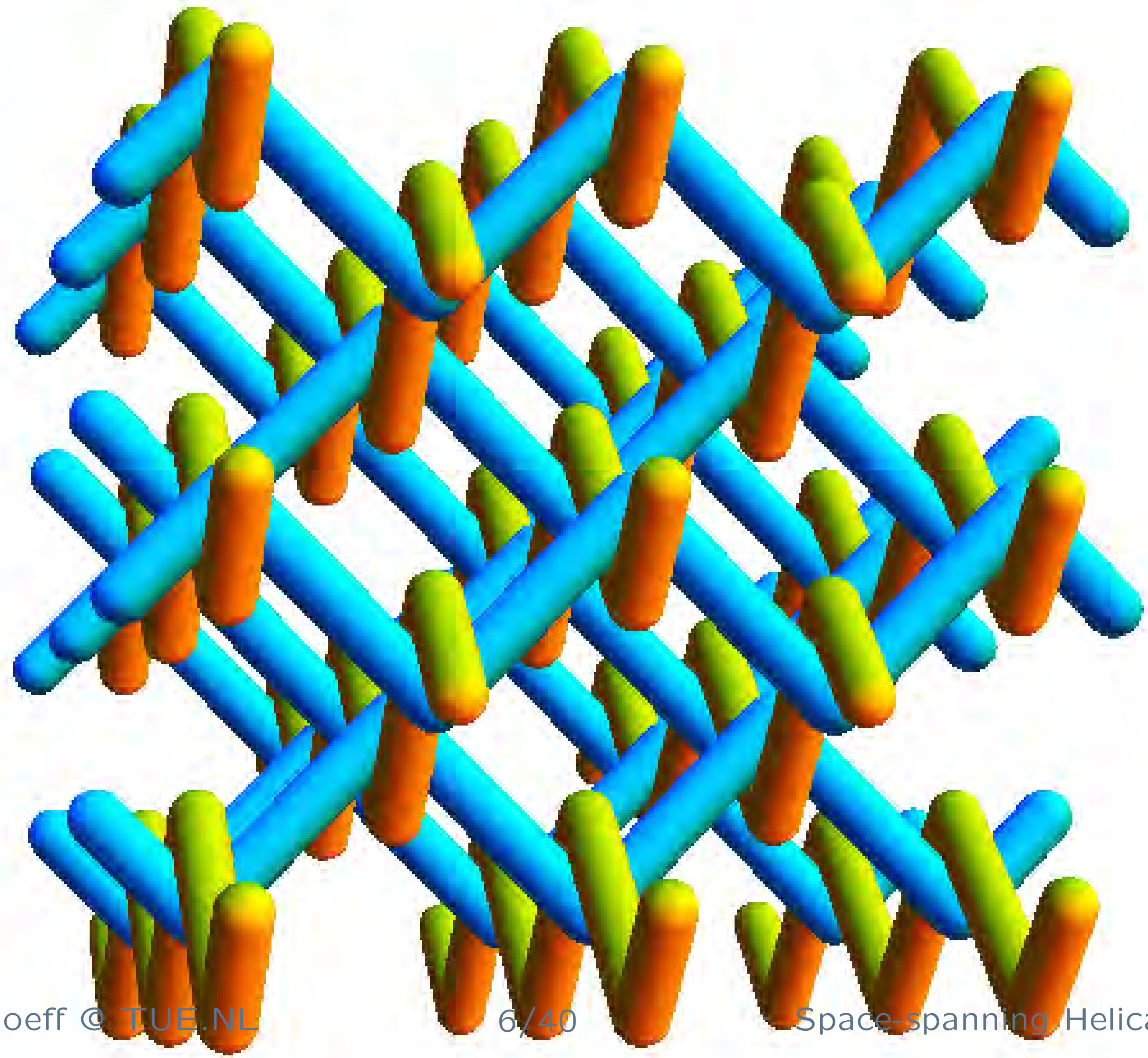
Woven Layers of Orthogonally Crossing Zigzags



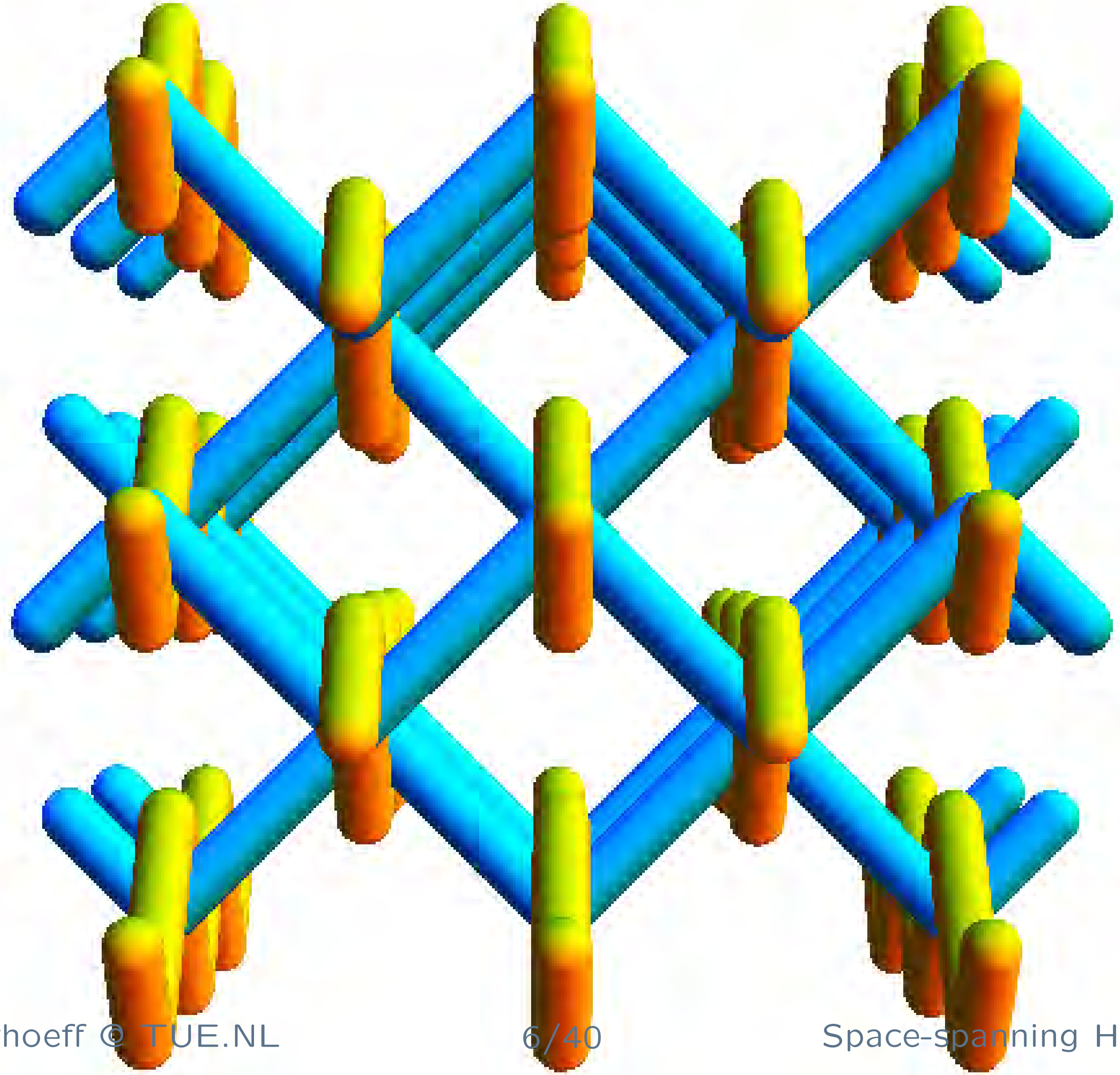
Woven Layers of Orthogonally Crossing Zigzags



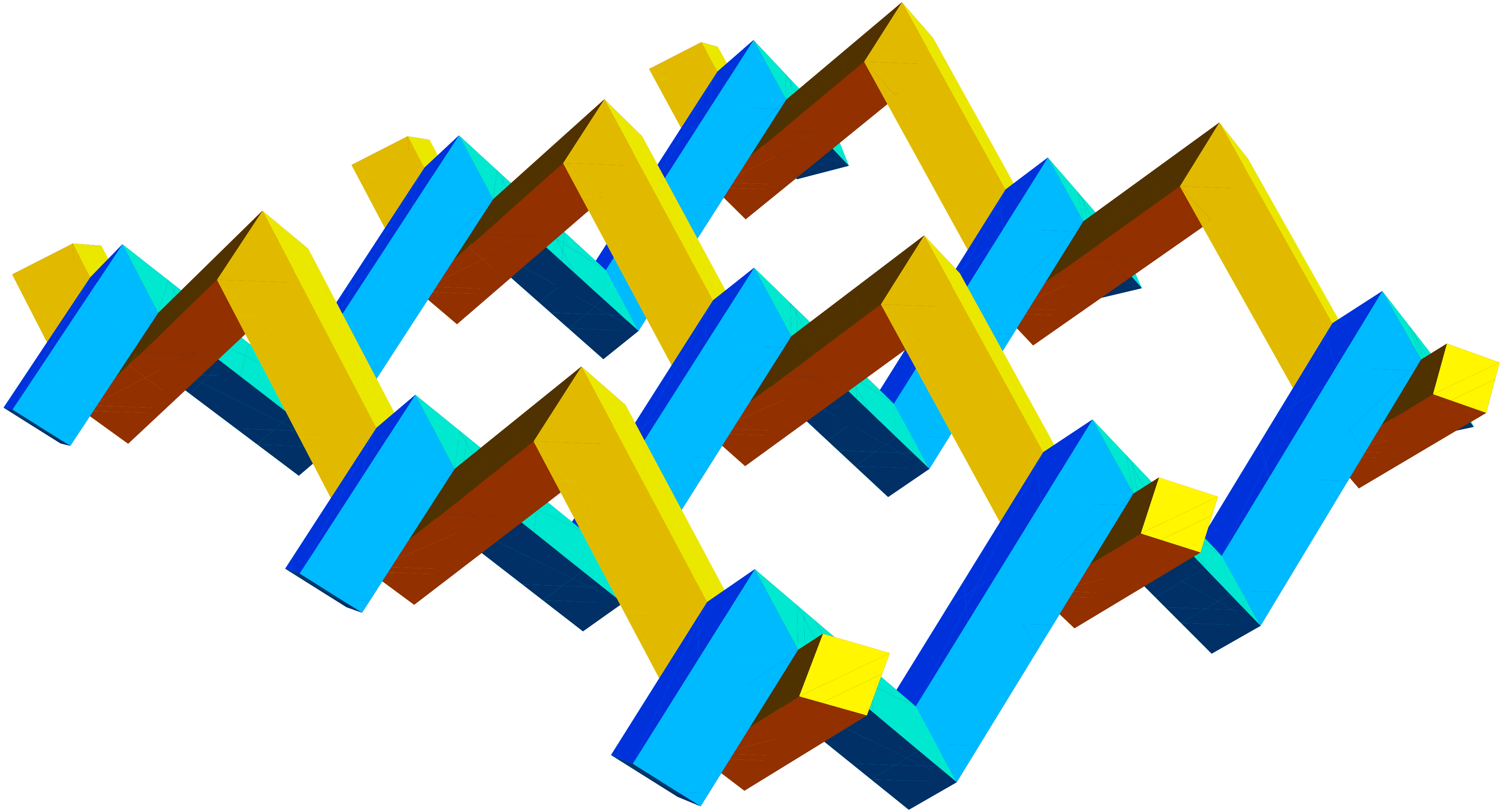
Woven Layers of Orthogonally Crossing Zigzags



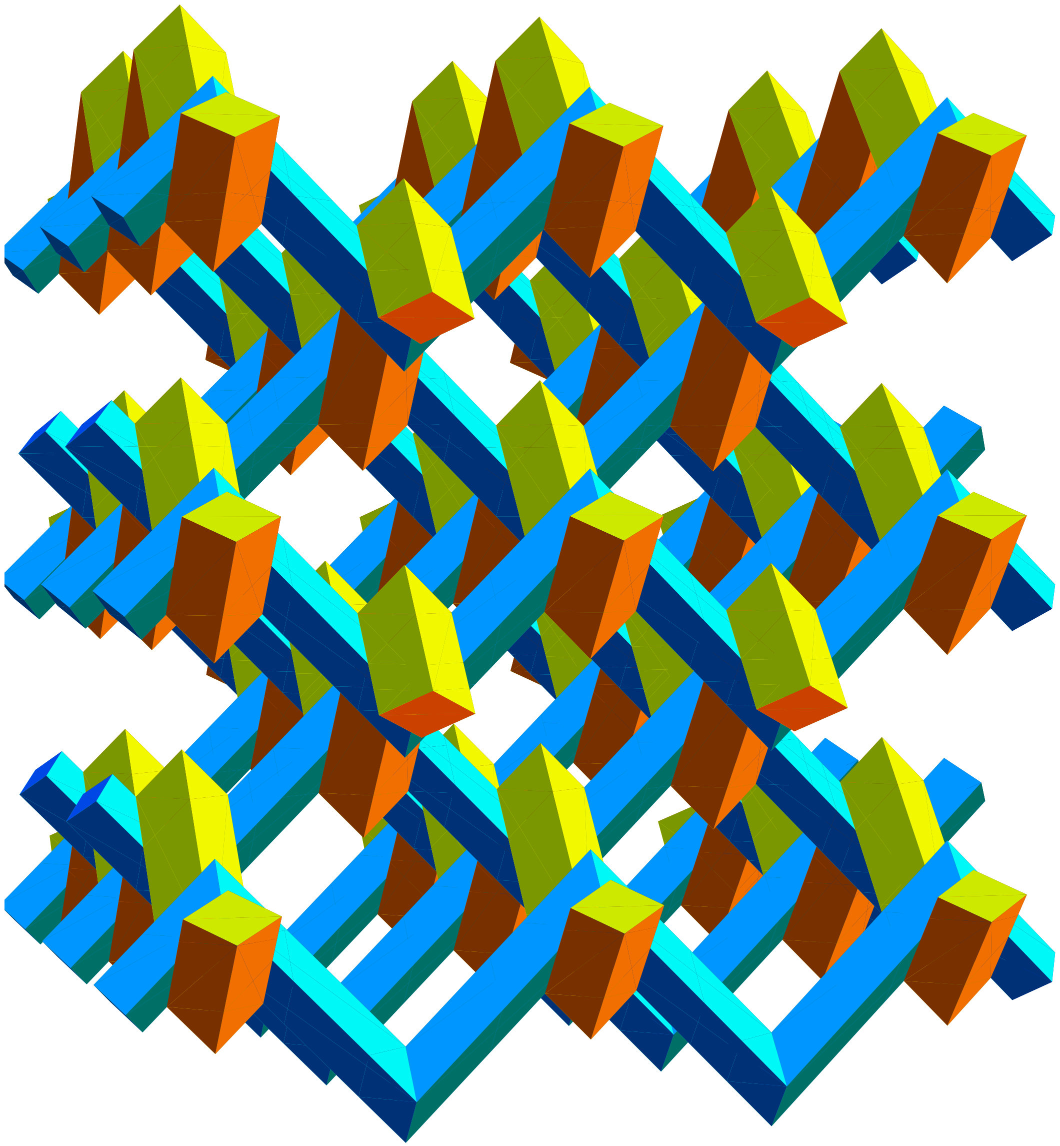
Woven Layers of Orthogonally Crossing Zigzags



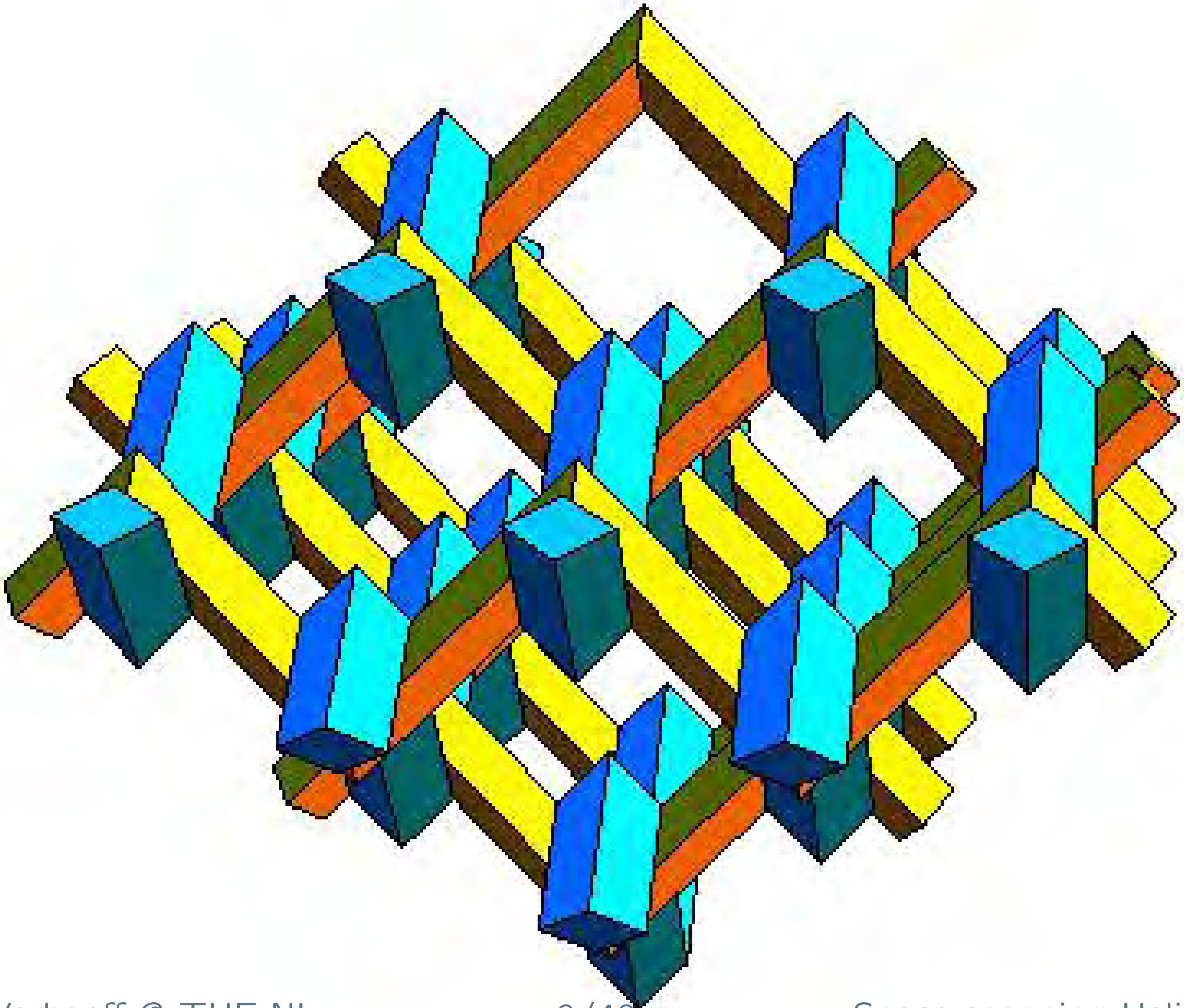
$1:\sqrt{2}$-Rhombus Cross Section & Miter Joints Give Snug Fit



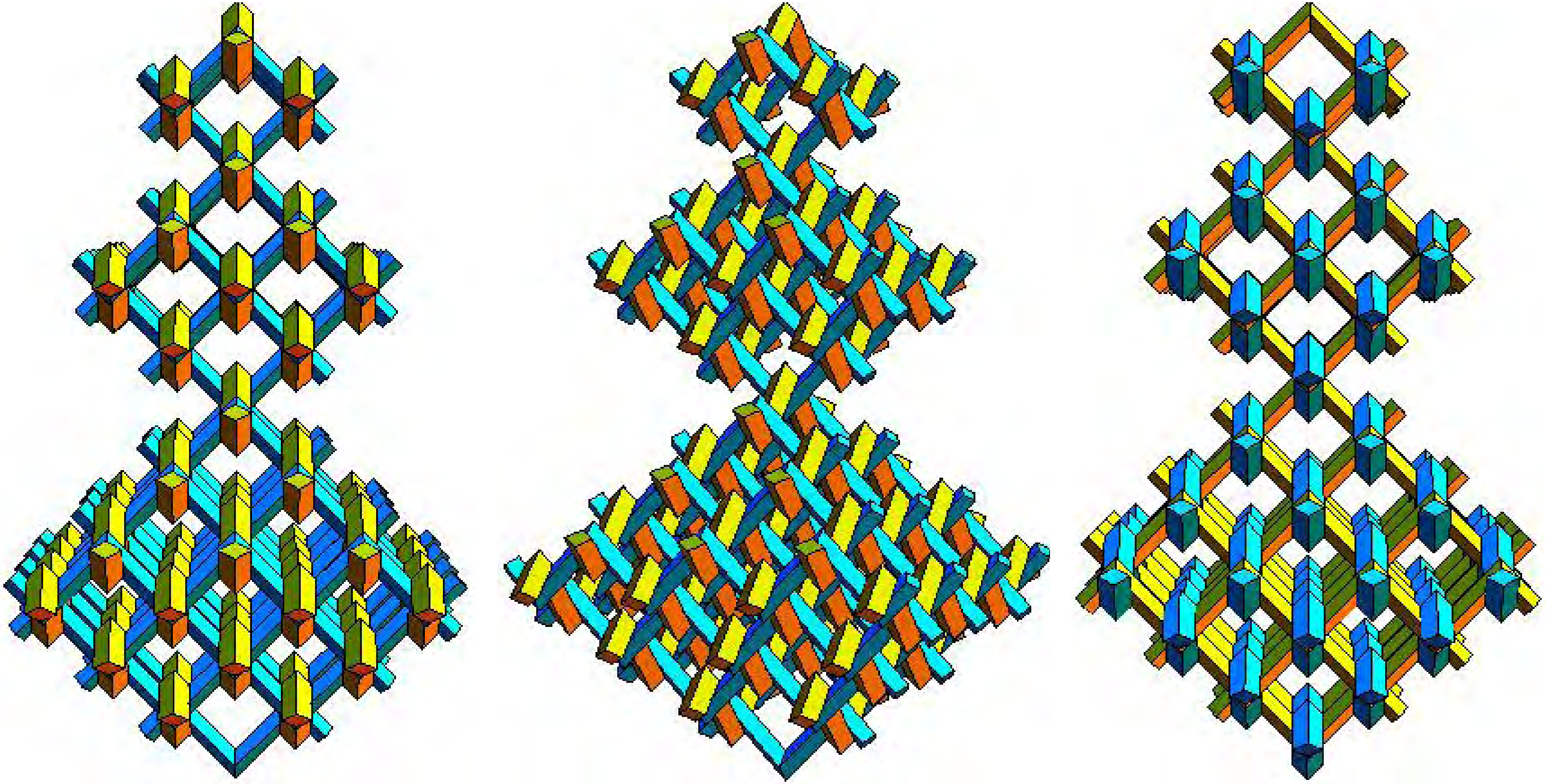
Woven Layers of Rhombic Mitered Zigzags



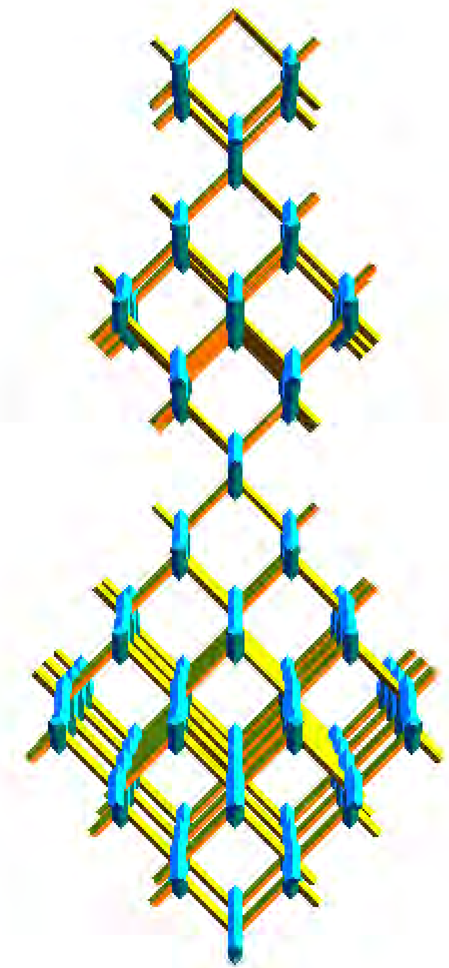
Varying the Lengths of the Zigzags Yields an Octahedron



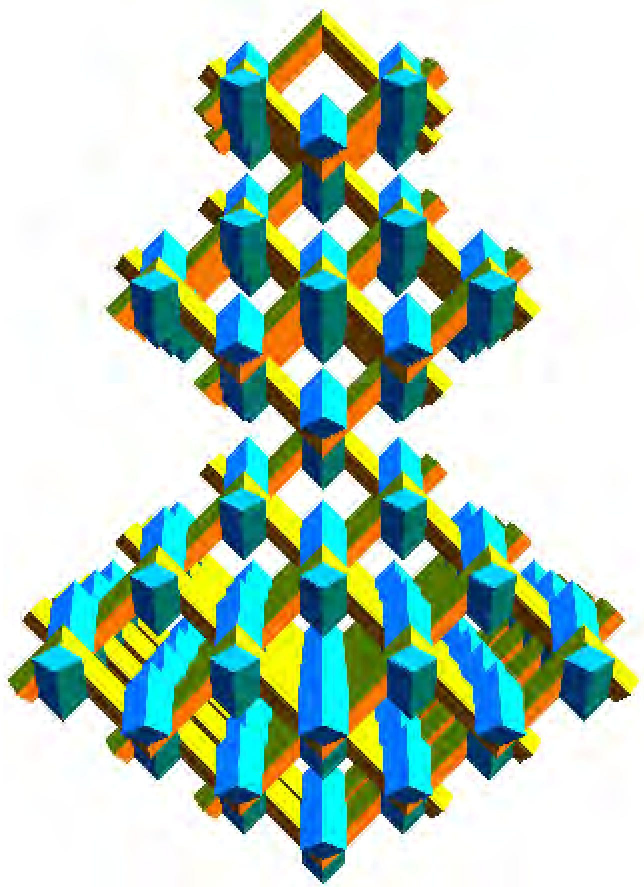
Octahedron Chain



Thickness Variation Affects Alignment



thinner

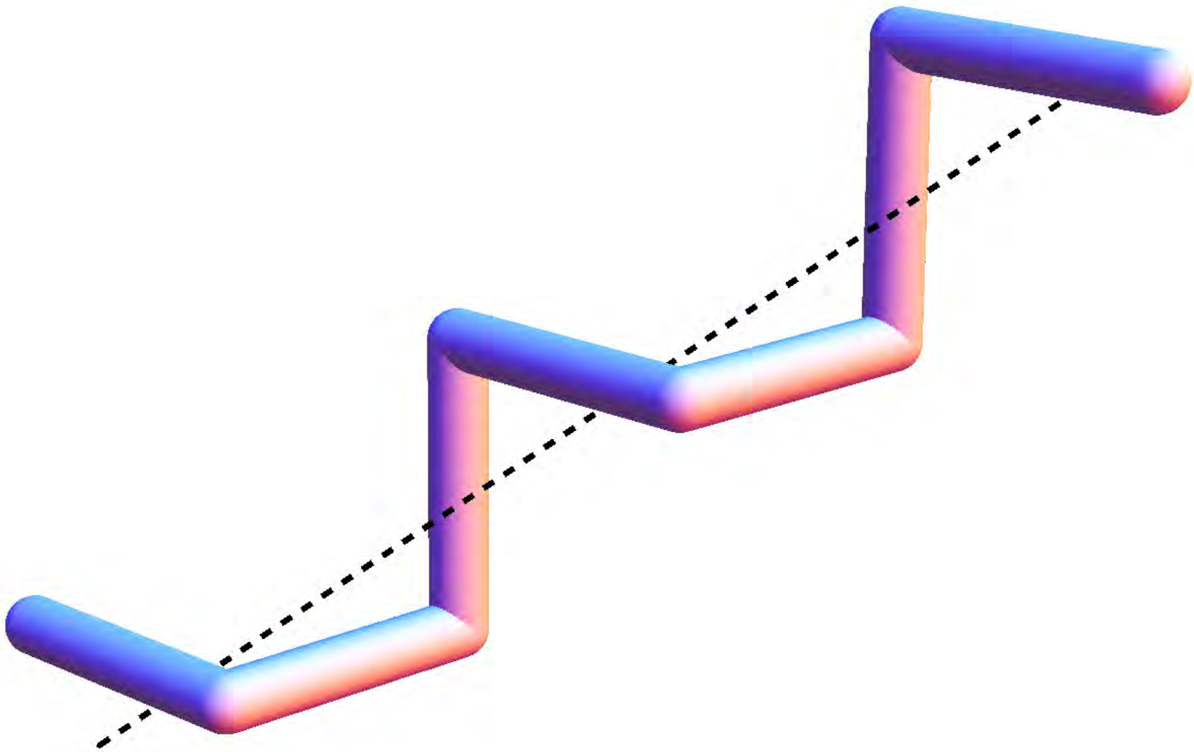


thicker

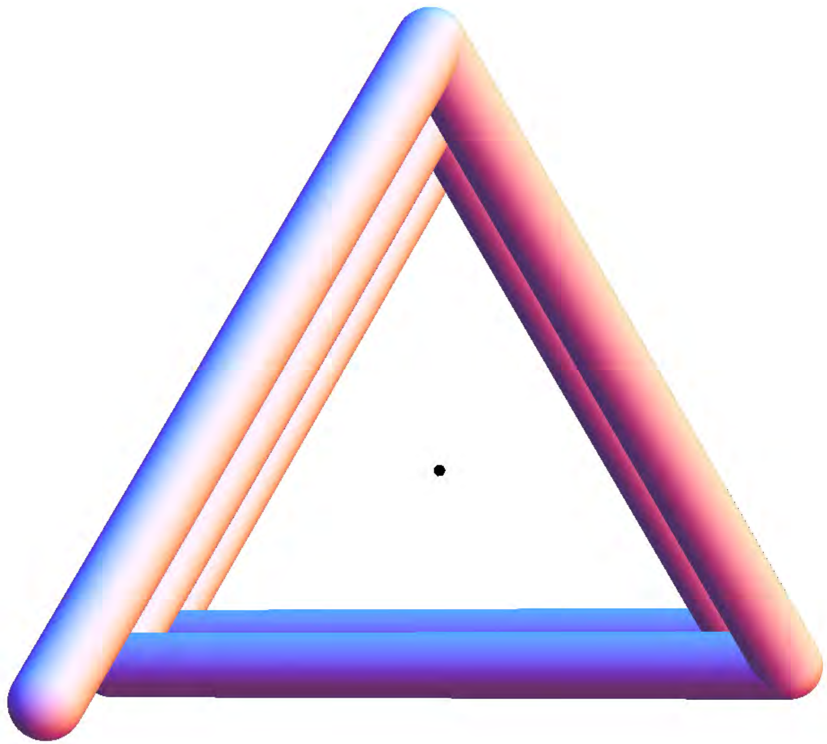
maximal thickness

Generalize Zigzag to 3D: Zigzagzeg

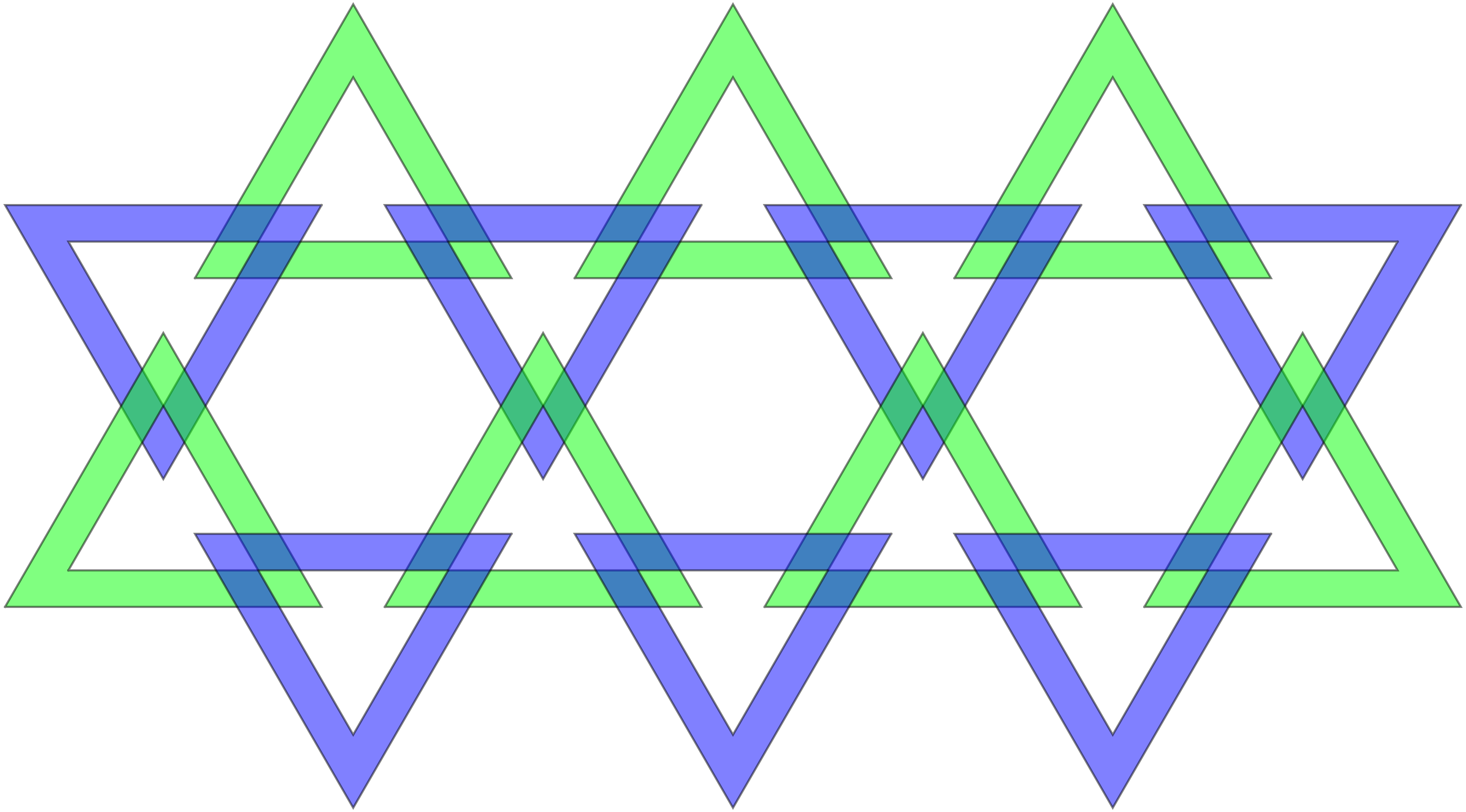
Repeatedly step one unit in x, y, and z direction:



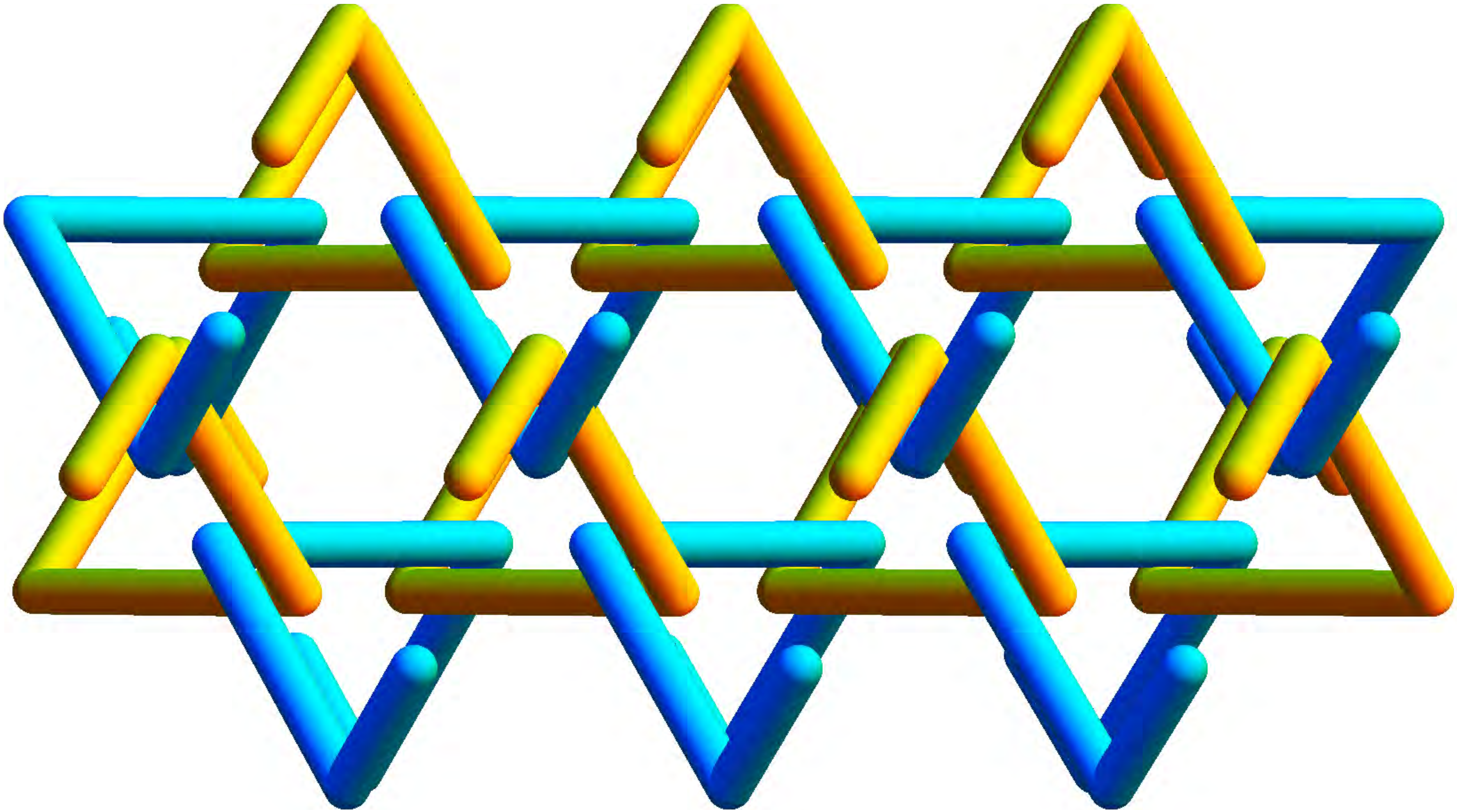
Zigzagzeg Viewed along Axis



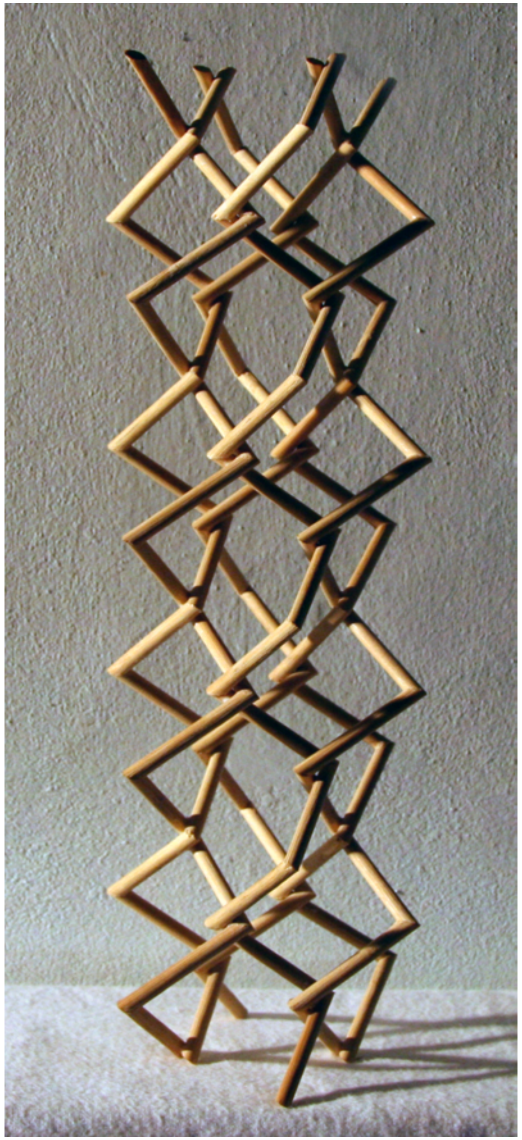
Infinitely Extendible Planar Pattern of Linked Triangles

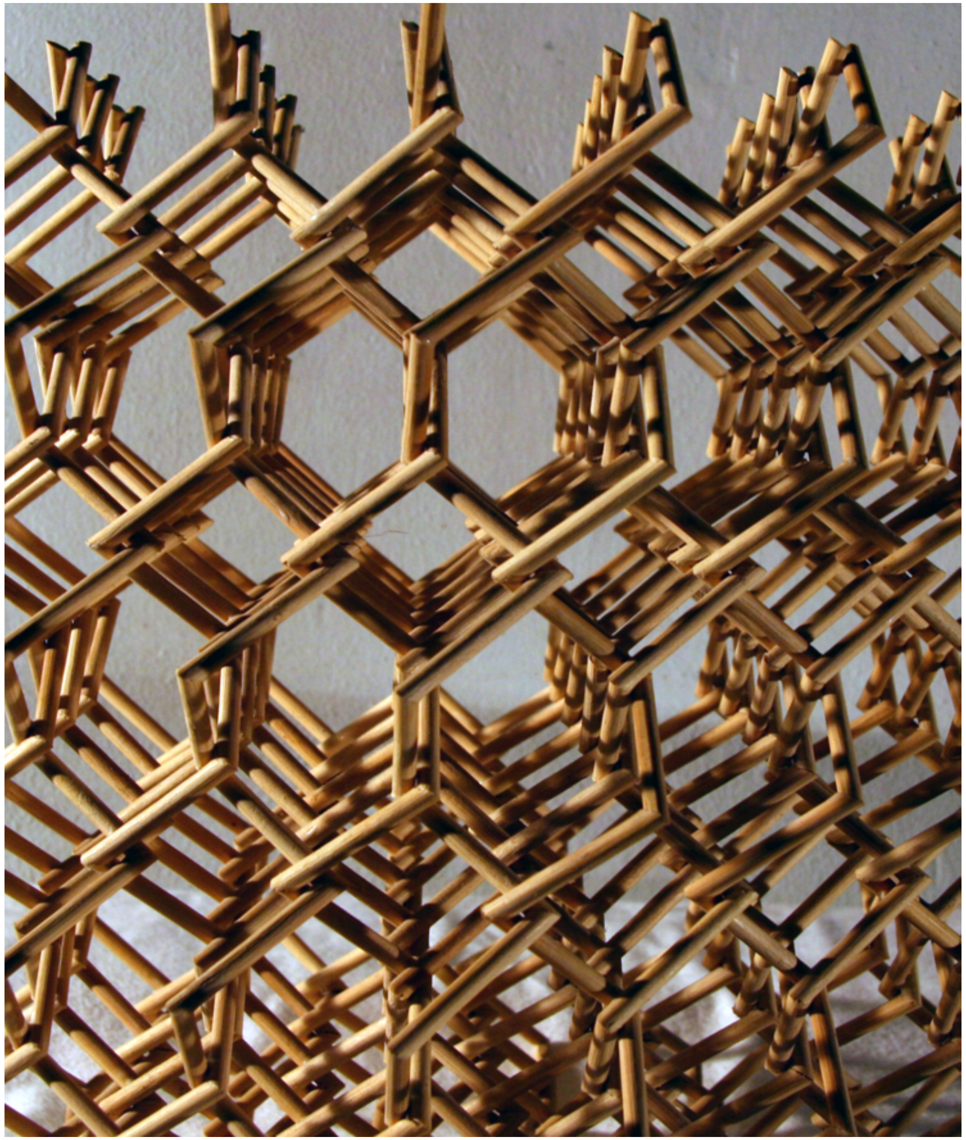


Parallel Woven Zigzagzegs (Top View)

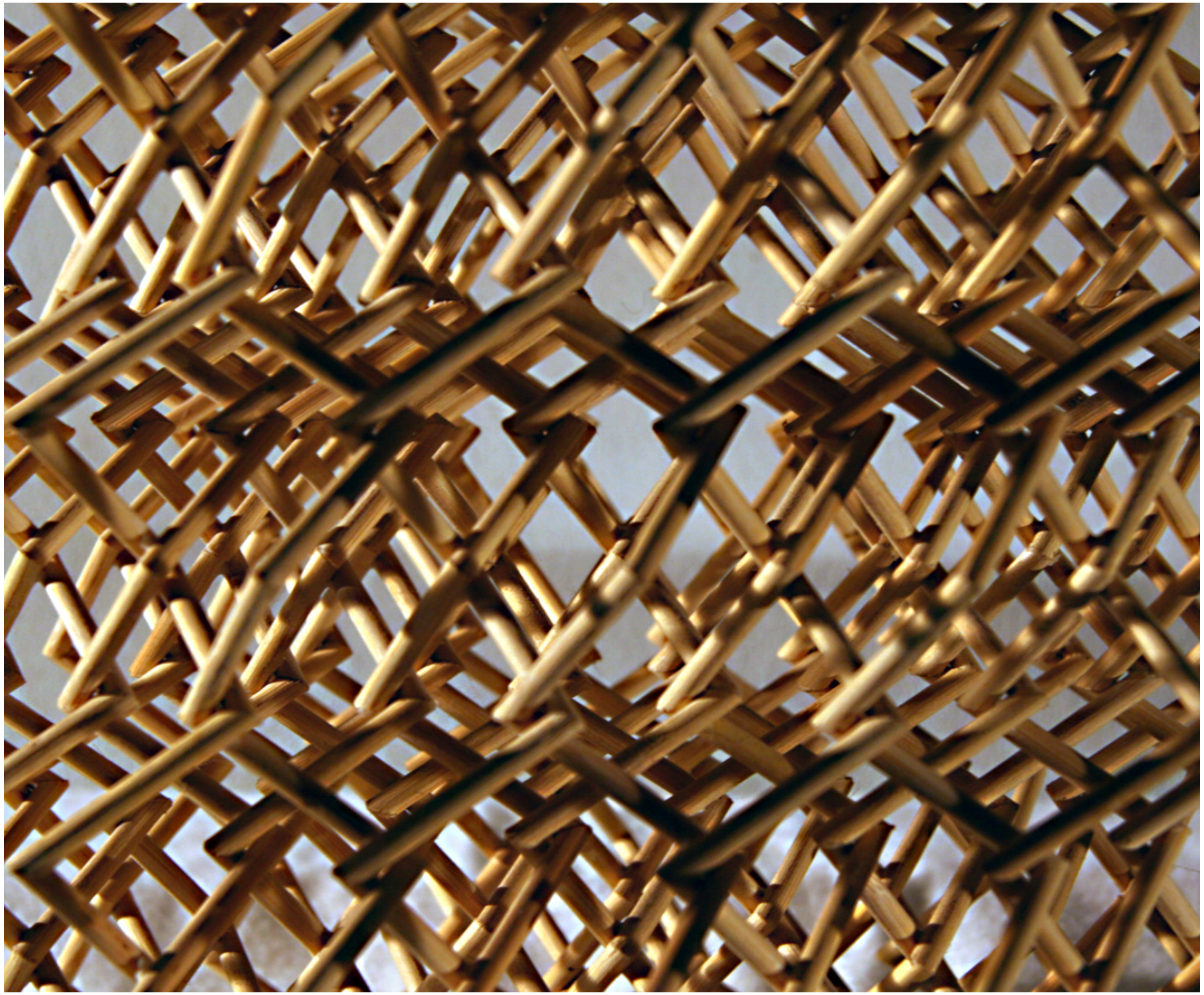


Parallel Woven Zigzagzegs (Side Views)

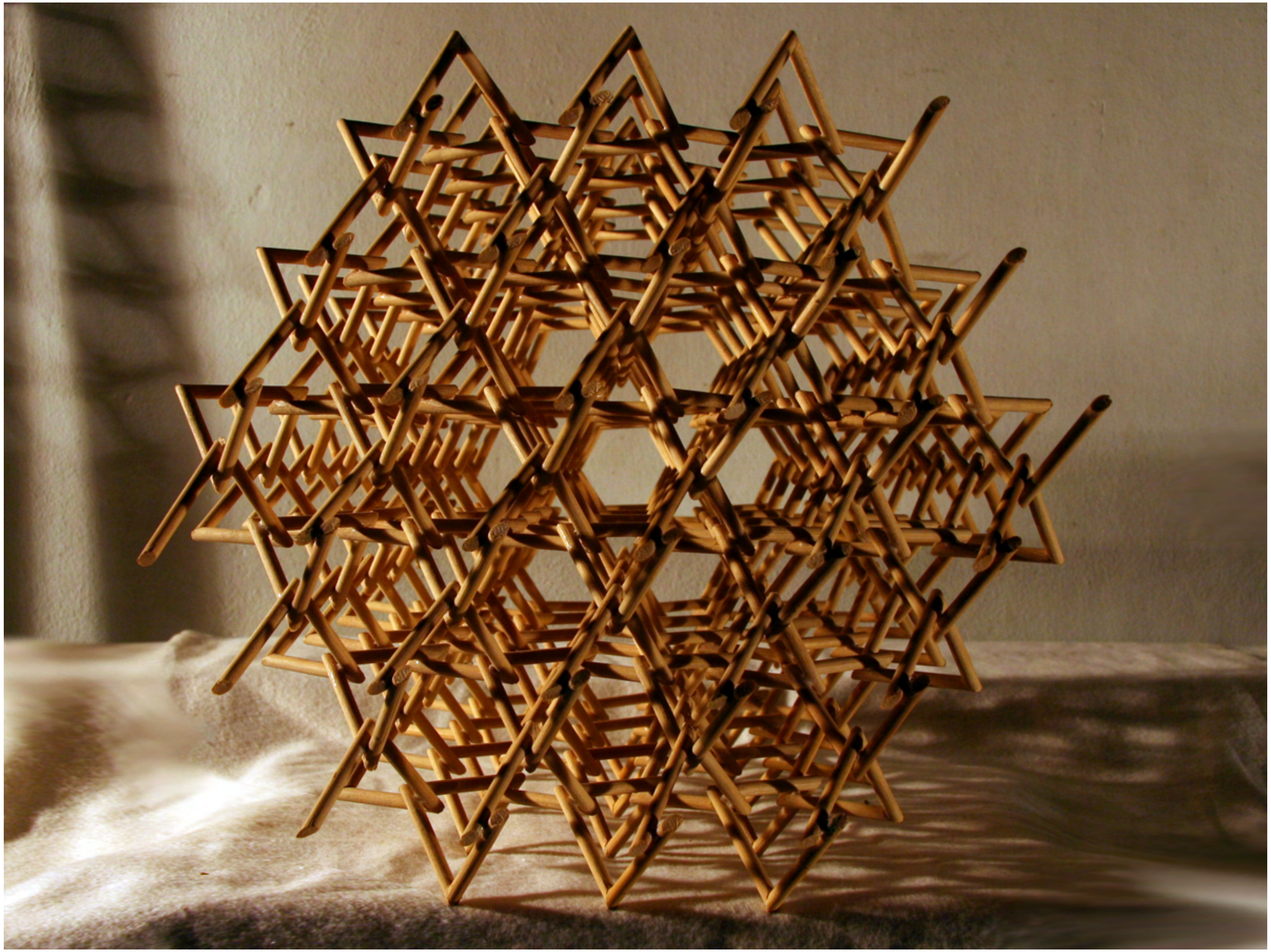




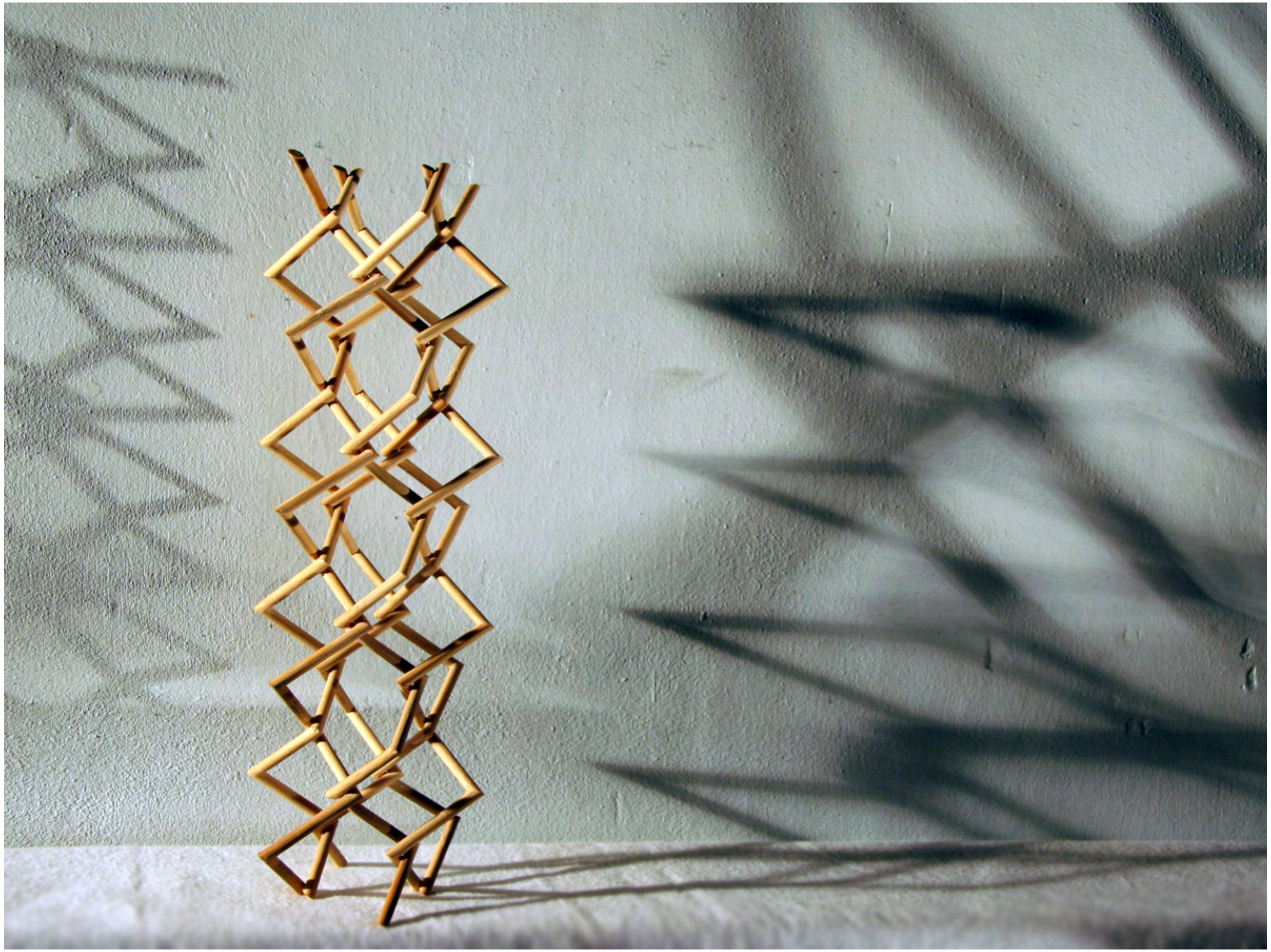
Parallel Woven Zigzagzegs (Side View)



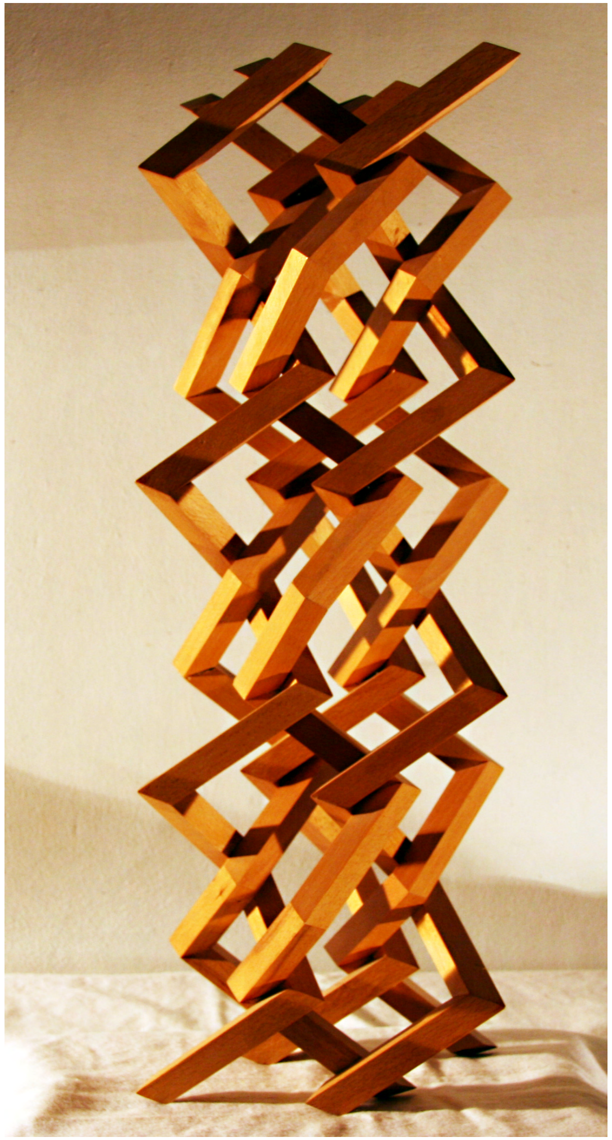
Parallel Woven Zigzagzegs (Top View)

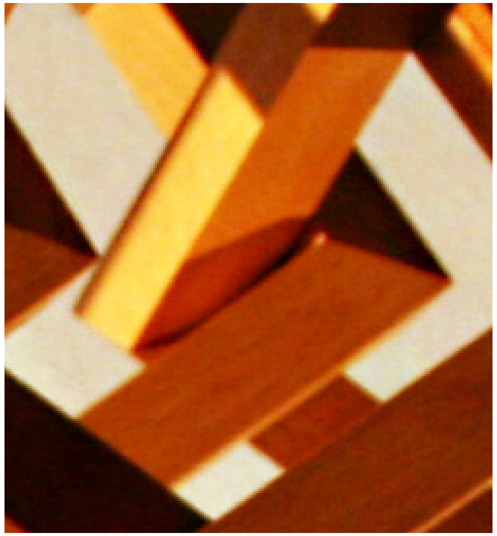


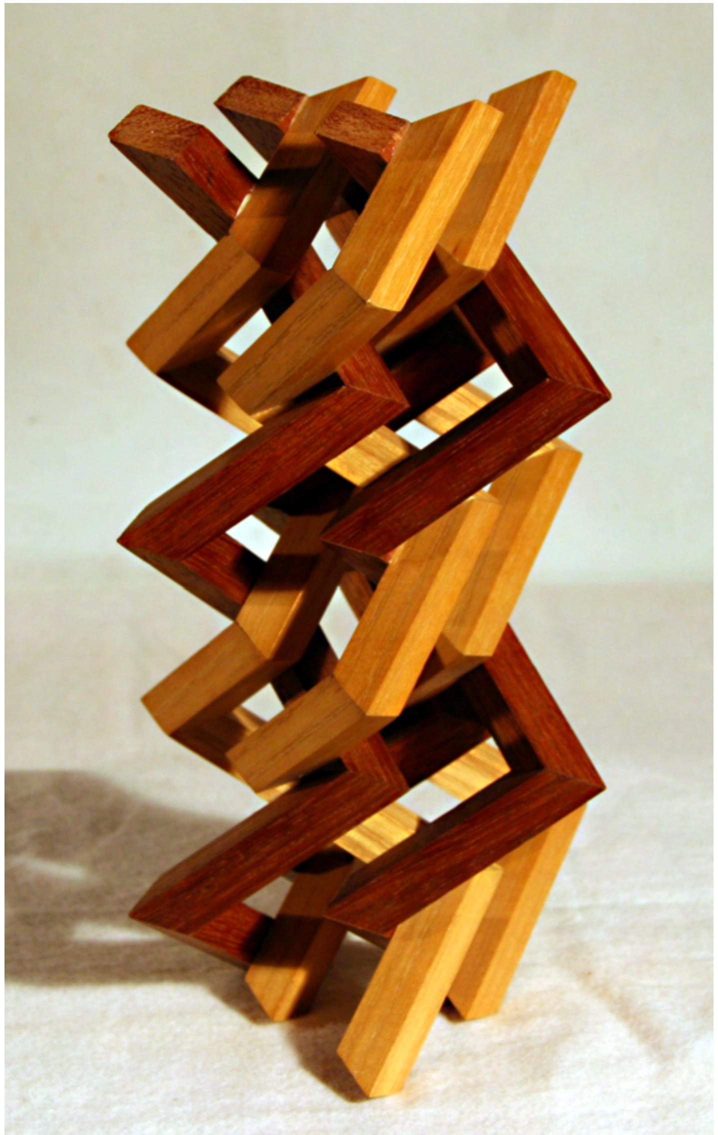
6 Parallel Woven Zigzagzegs (Side View)



6 Parallel Woven Zigzagzegs in Square Beams & Miter Joints

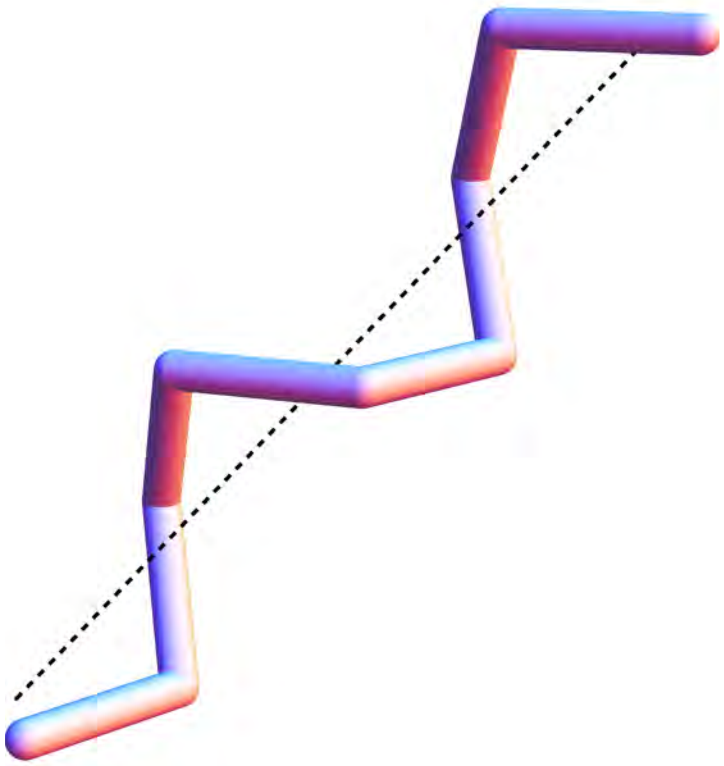




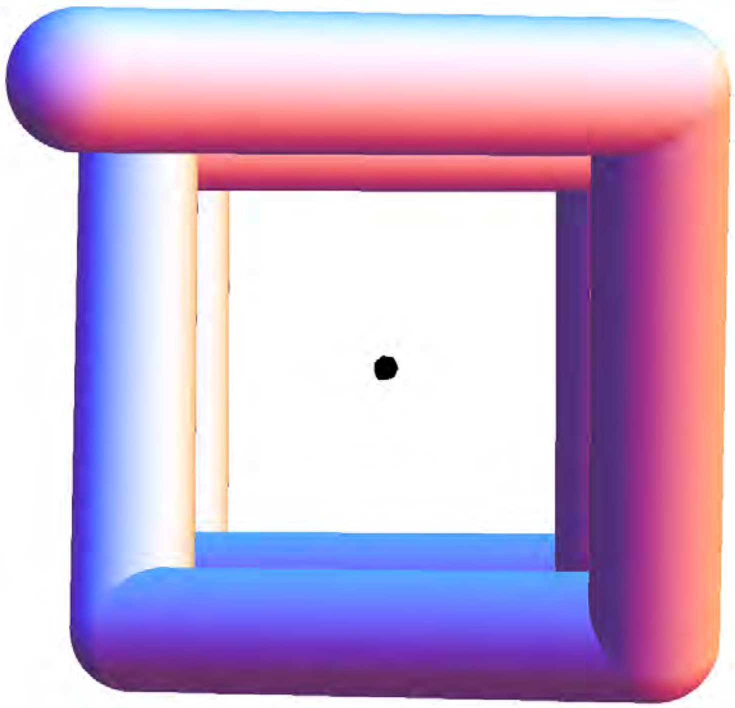


Zigzagzegzug

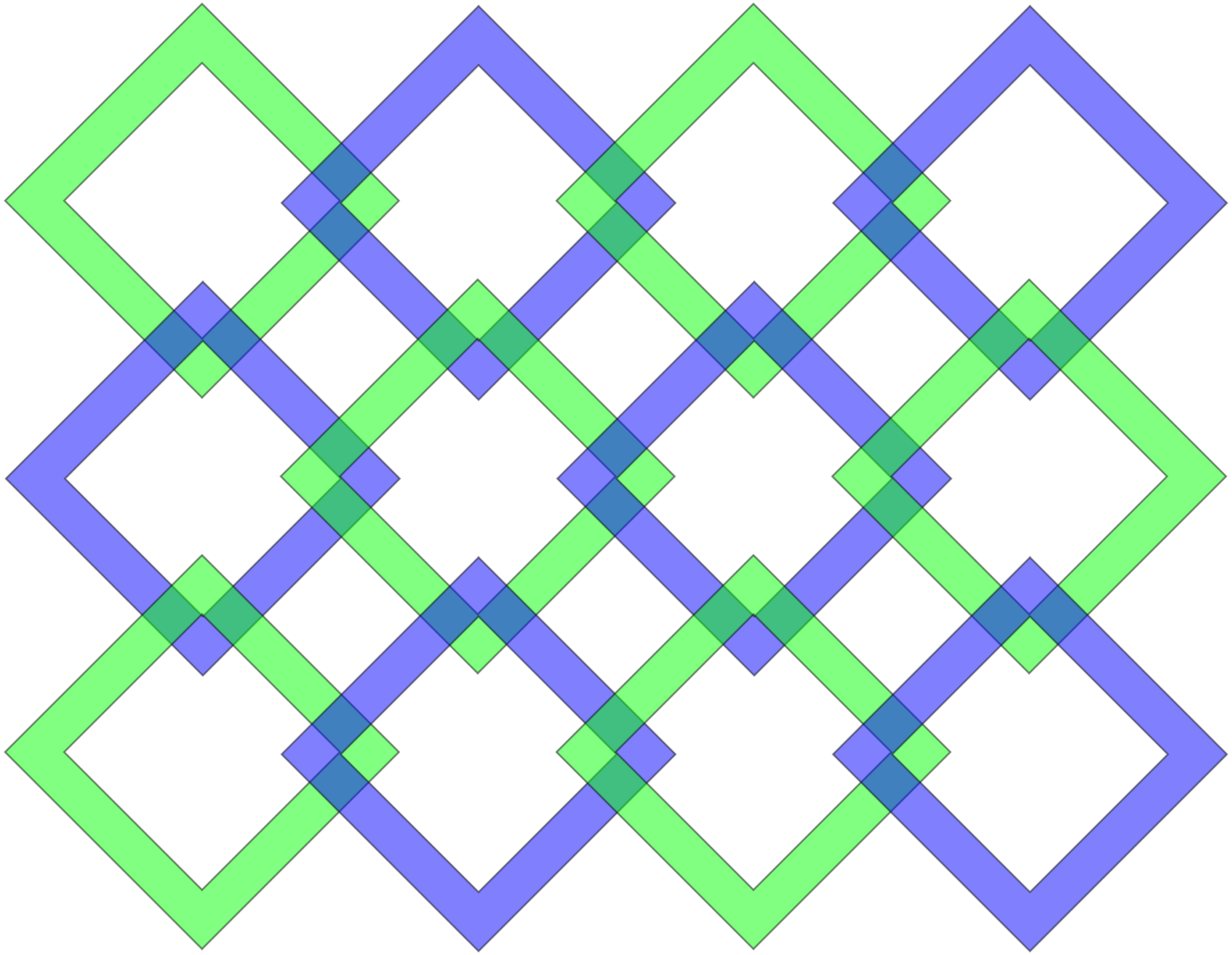
Repeatedly step one unit in direction of 4 main diagonals of the cube:



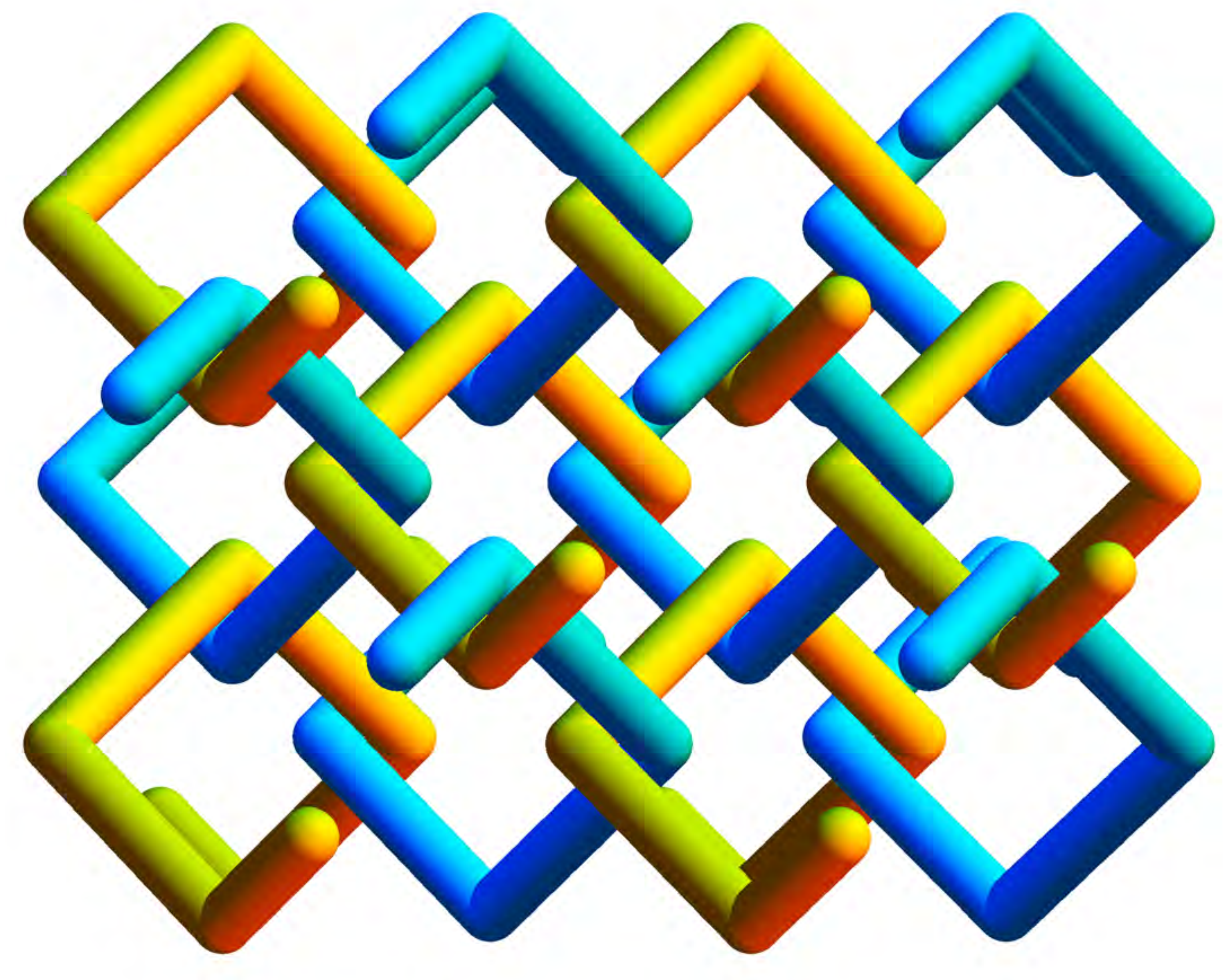
Zigzagzegzug Viewed along Axis



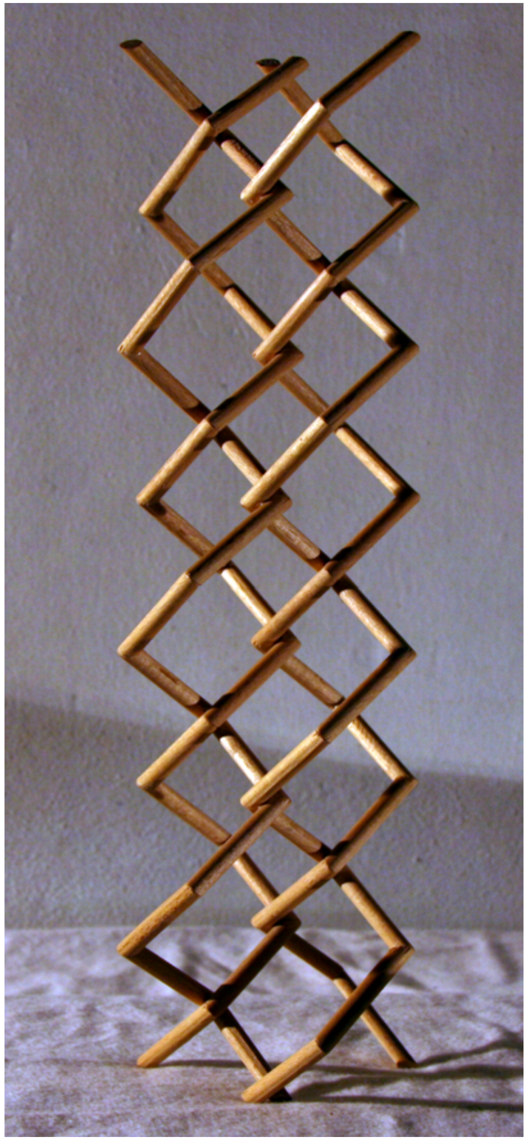
Infinitely Extendible Planar Pattern of Linked Squares

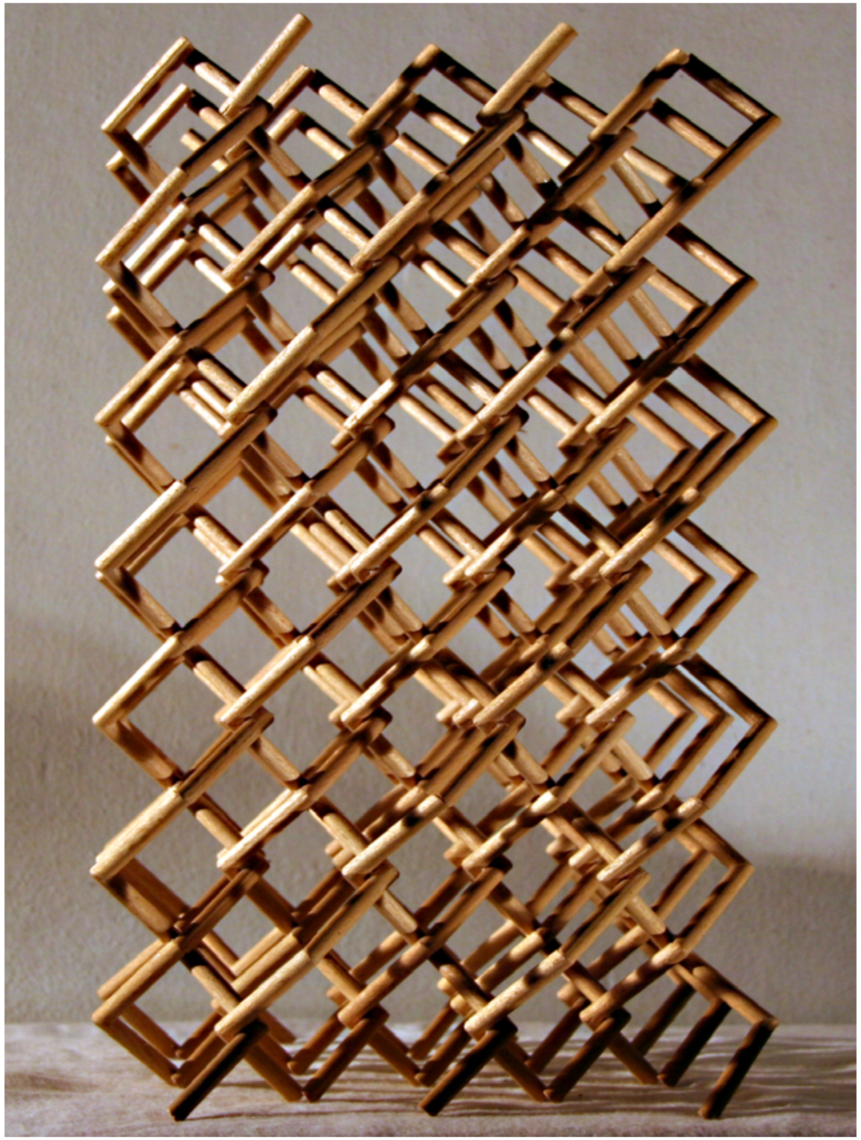


Parallel Woven Zigzagzegzugs (Top View)

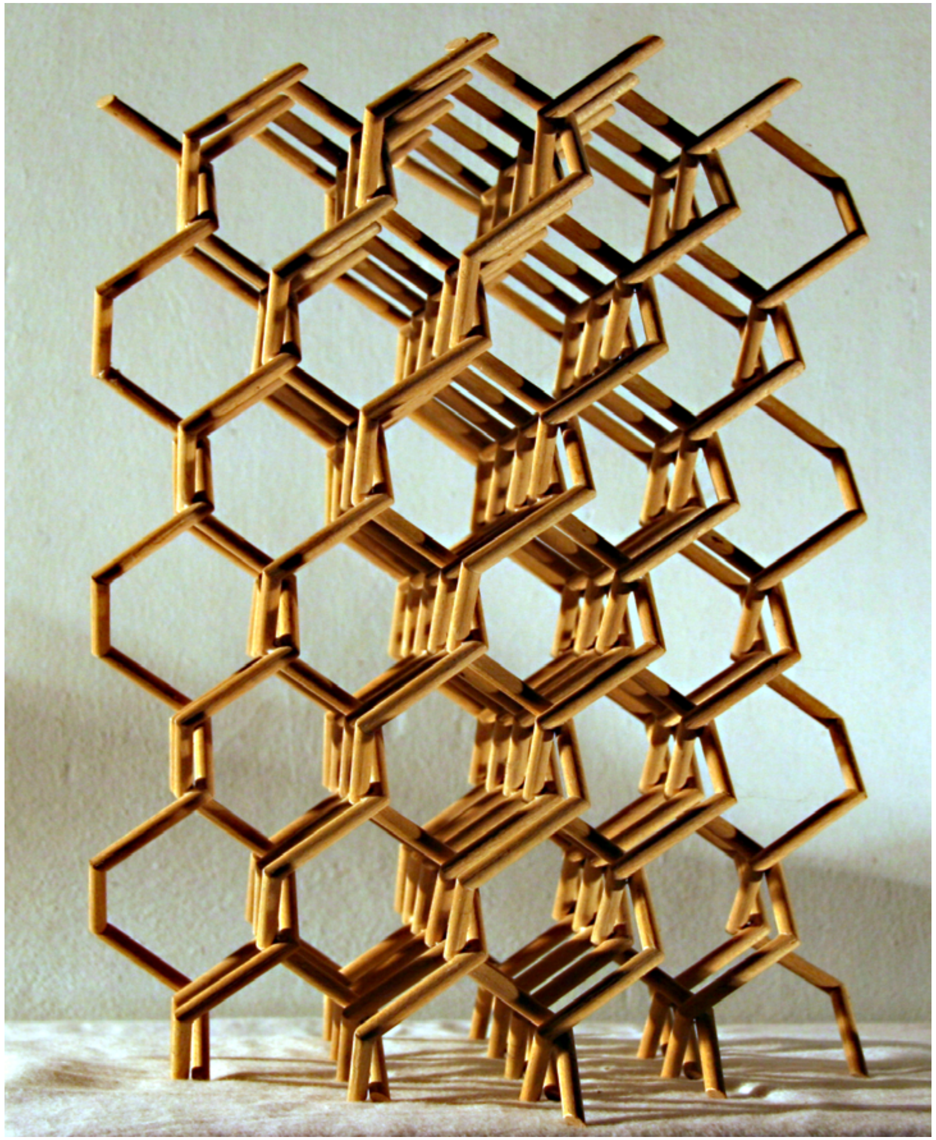


Parallel Woven Zigzagzegzugs (Side Views)

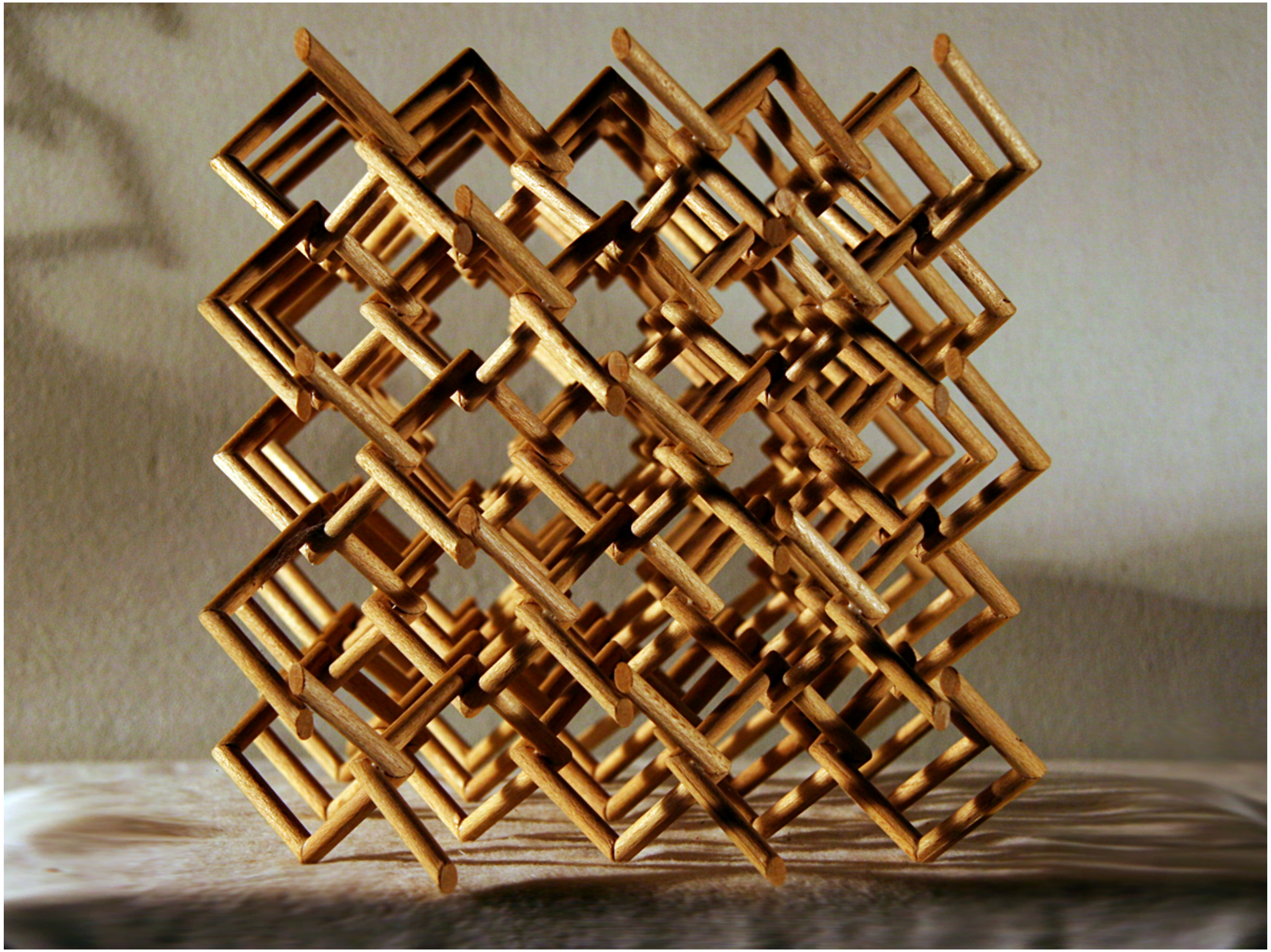




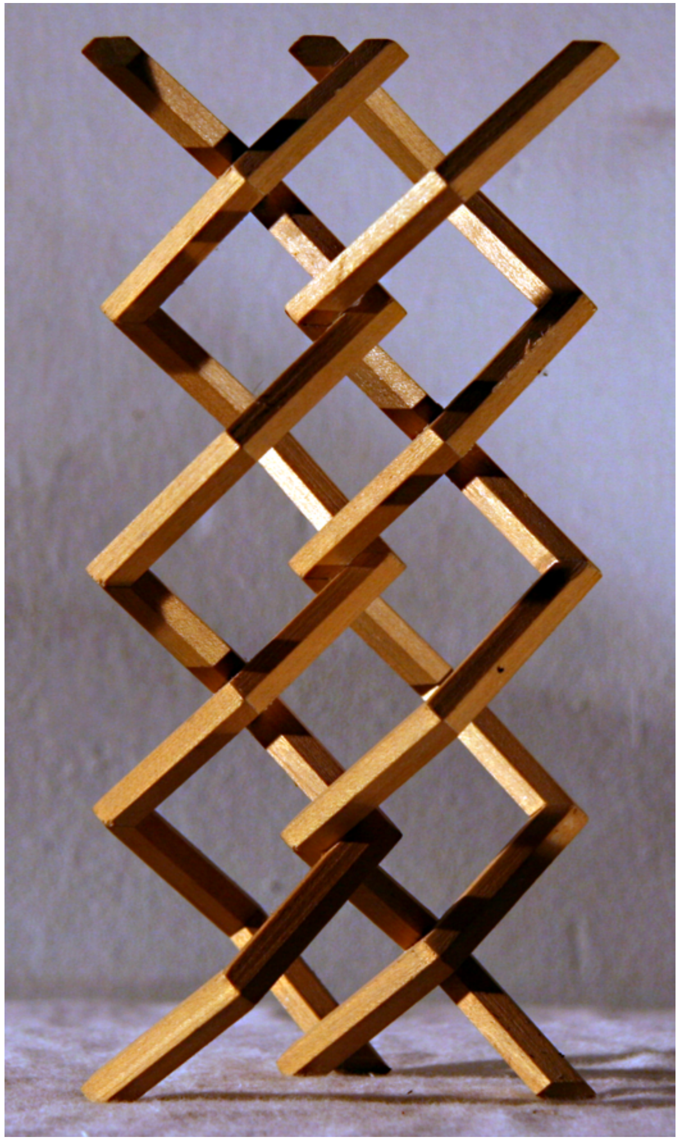
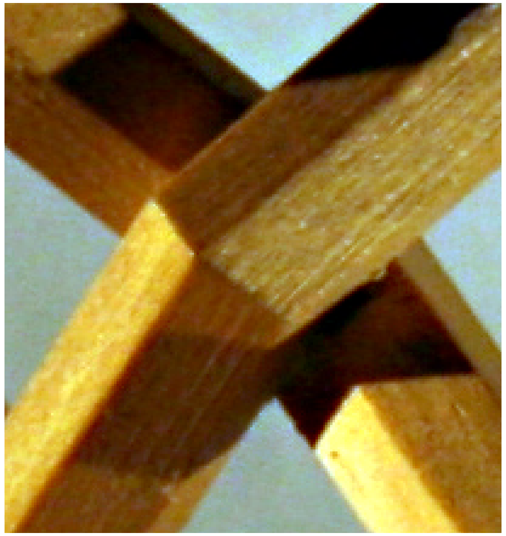
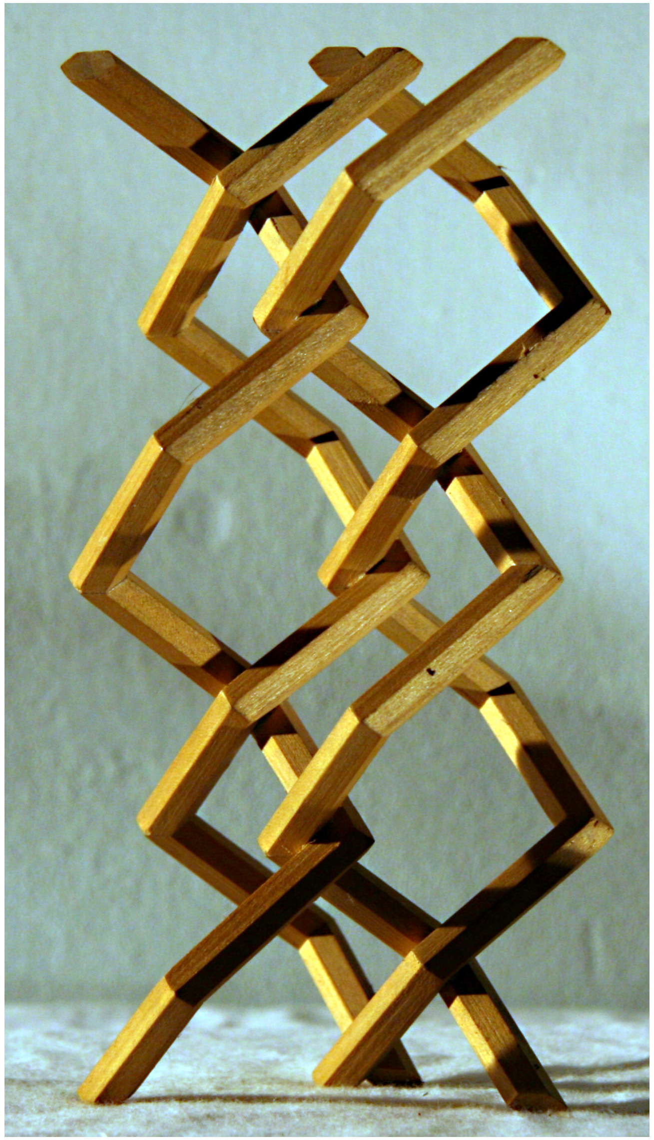
Parallel Woven Zigzagzegzugs (Side View)



Parallel Woven Zigzagzegzugs (Top View)



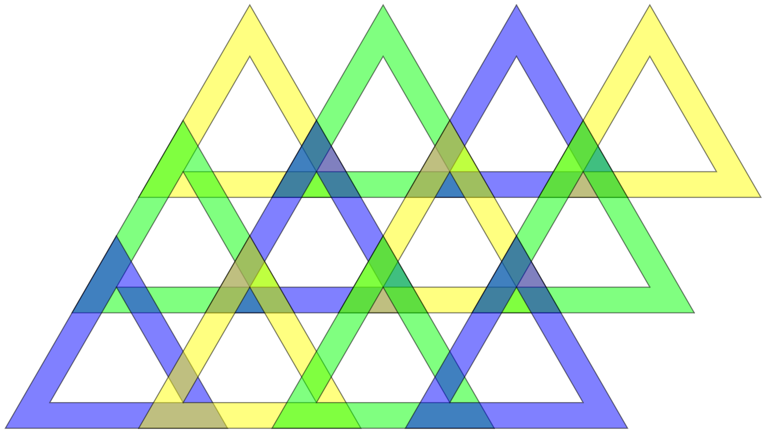
4 Parallel Zigzagzegzugs: Hexagonal Beams & Miter Joints

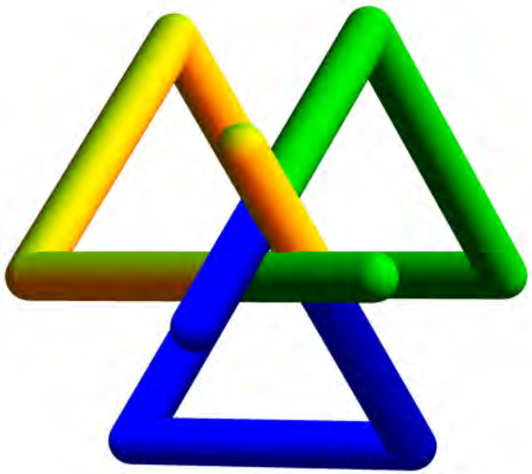




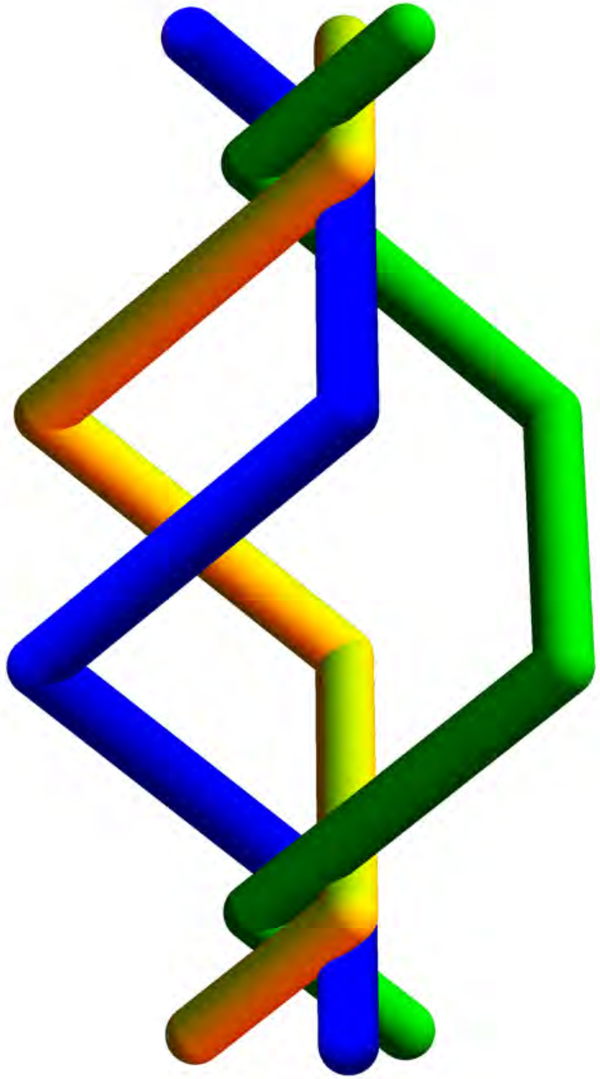
Variations

- More than two helices link
- Not all axes parallel

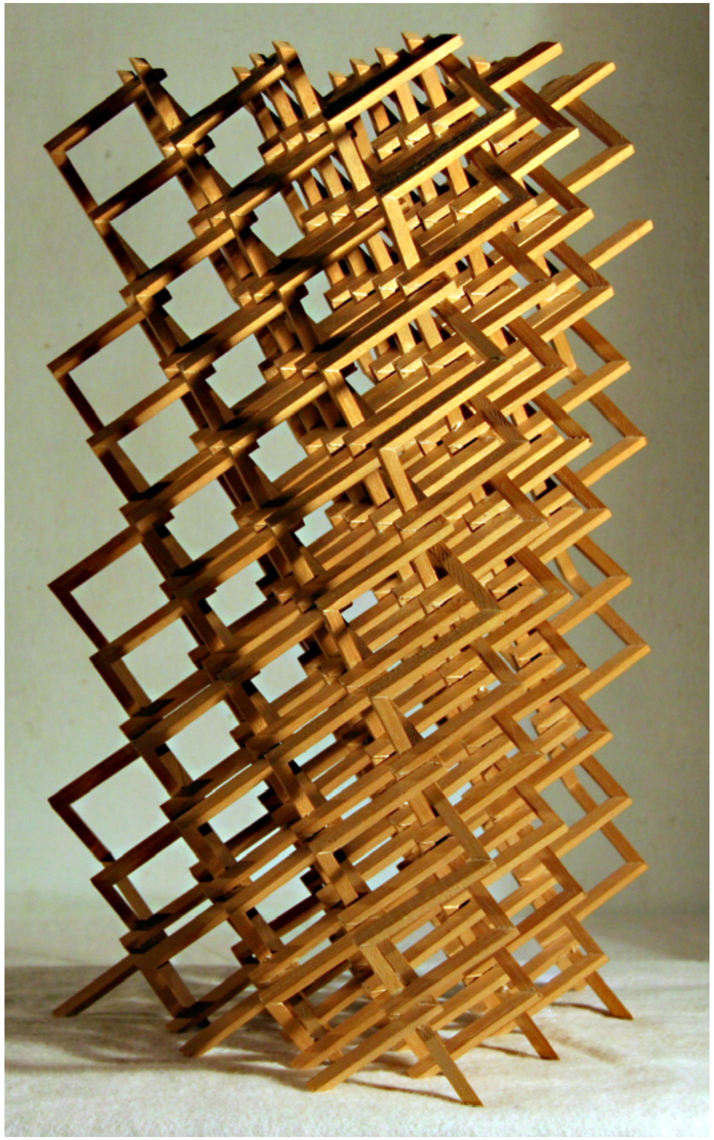
Parallel Zigzagzegs in Ternary Embrace



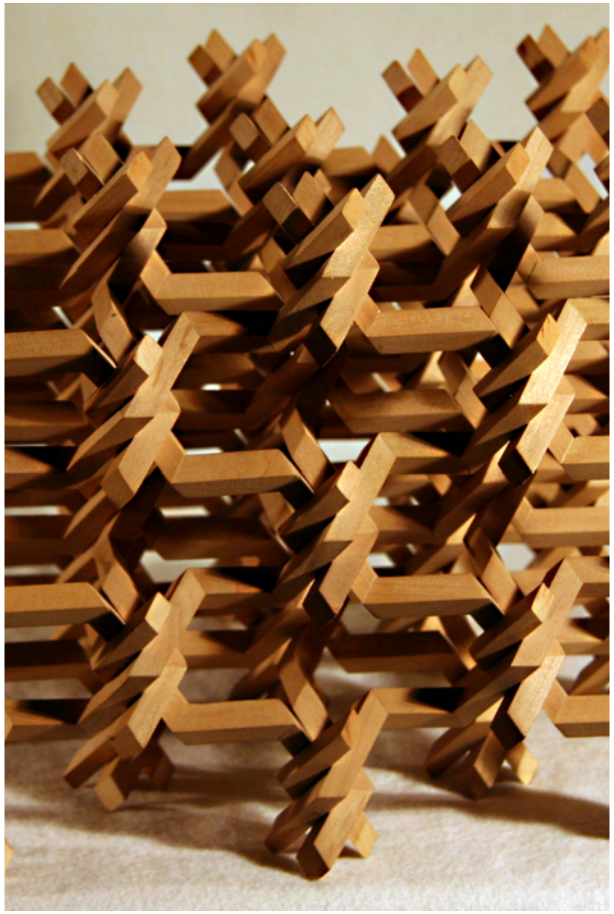


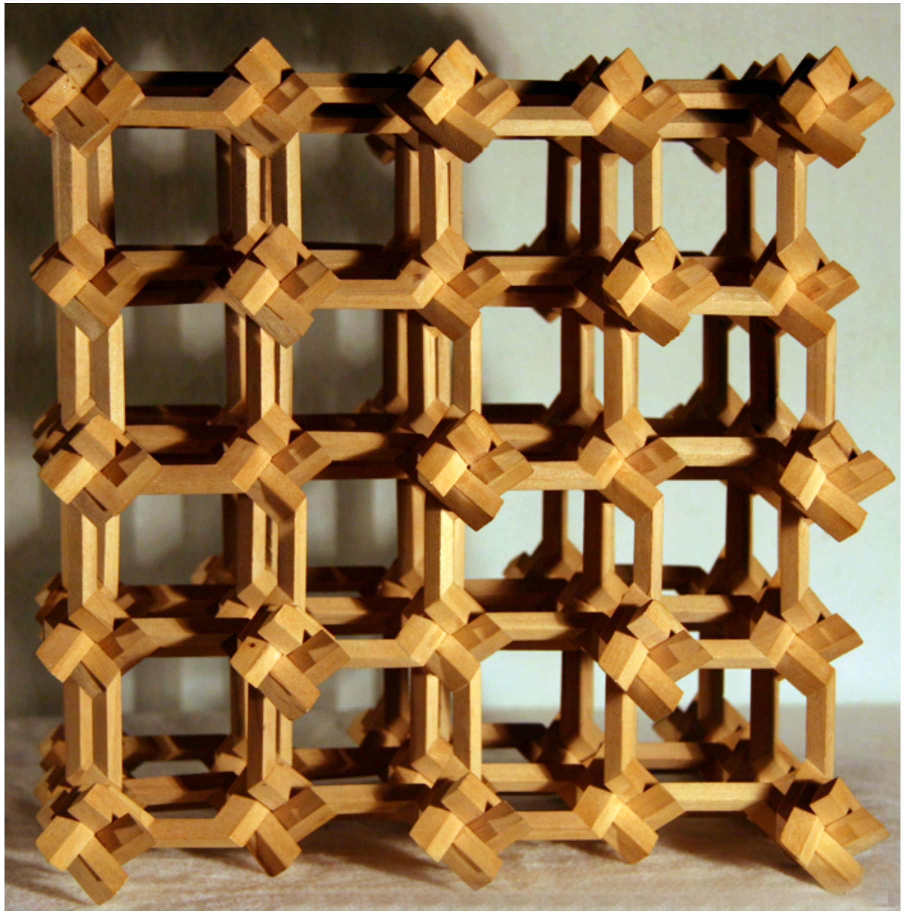


Parallel Zigzagzegs in Ternary Embrace: Square Beams



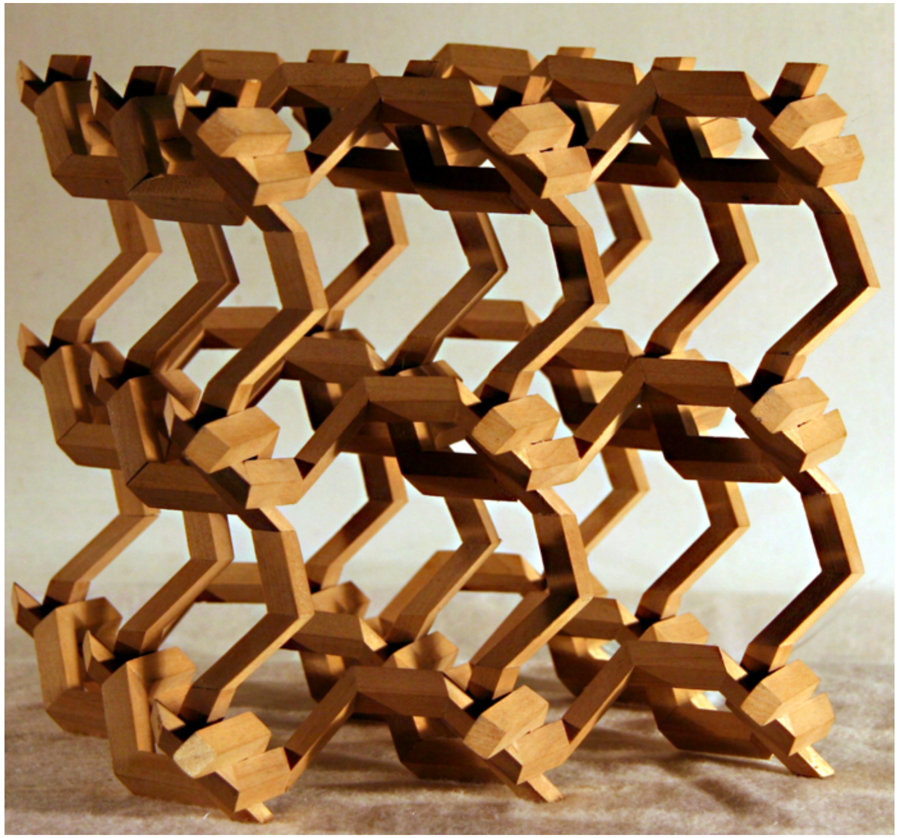
Non-Parallel Zigzagzegs* in Quaternary Embrace

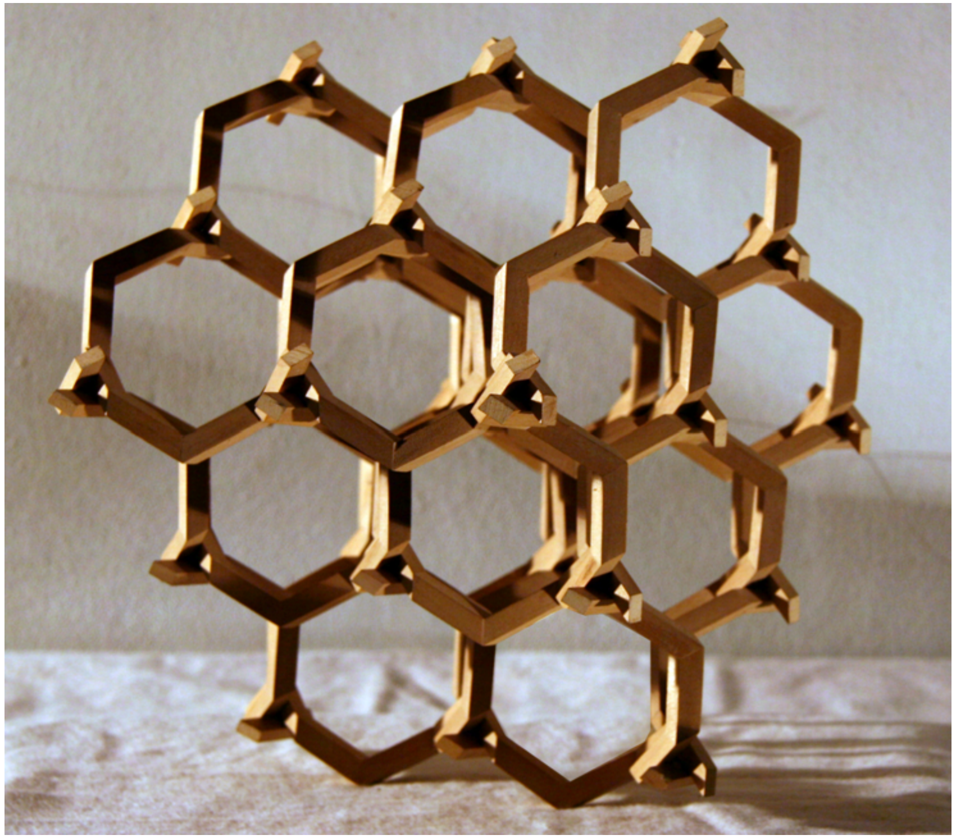




Rhombic cross section, square cut faces, skew miter joints, trapezoid

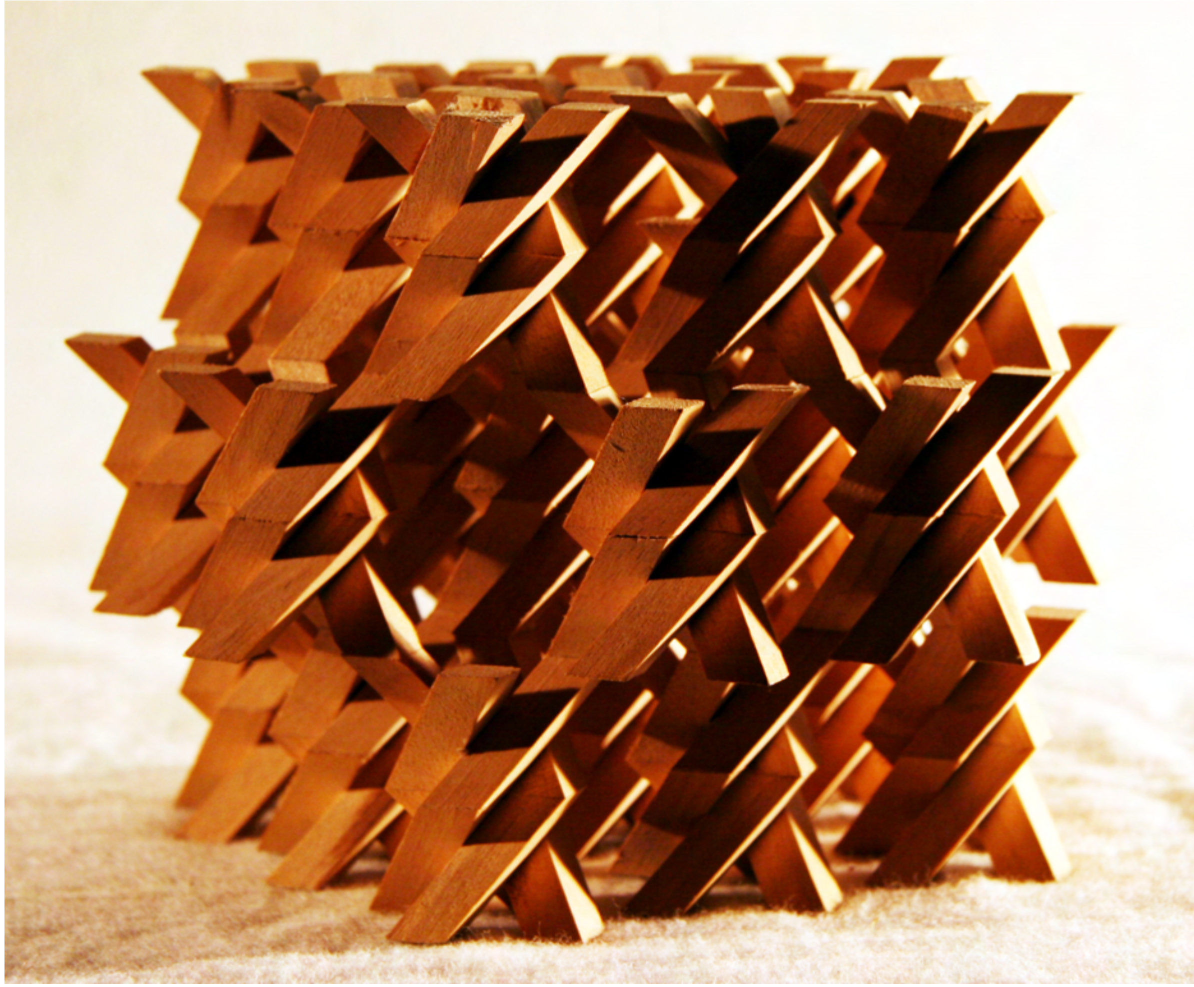
Non-Parallel Zigzagzegzugs† in Ternary Embrace

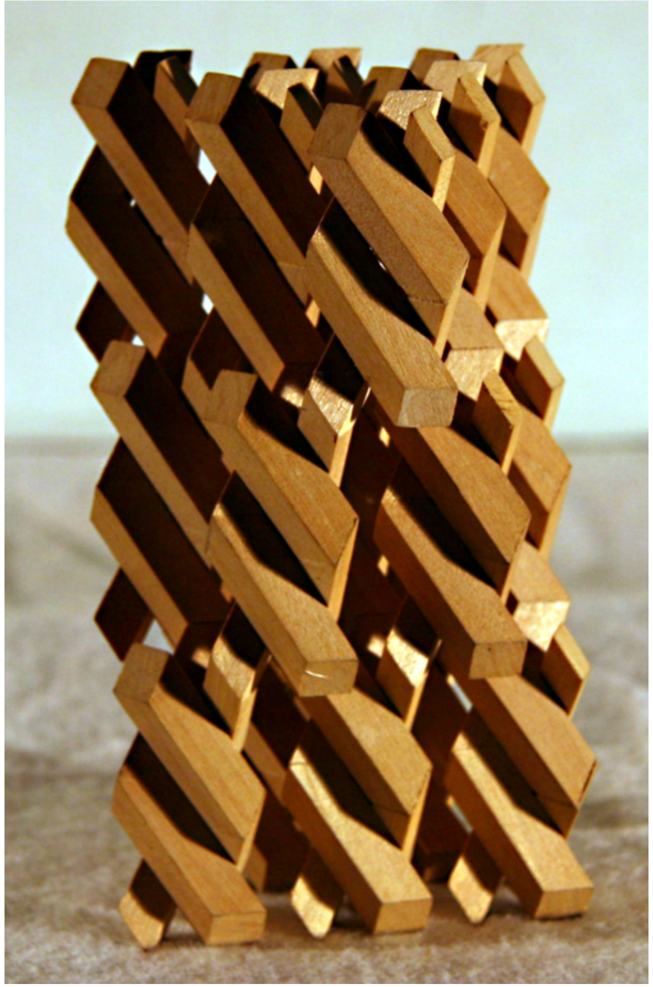




Rhombic beam, square cut faces, skew miter joints, parallelogram

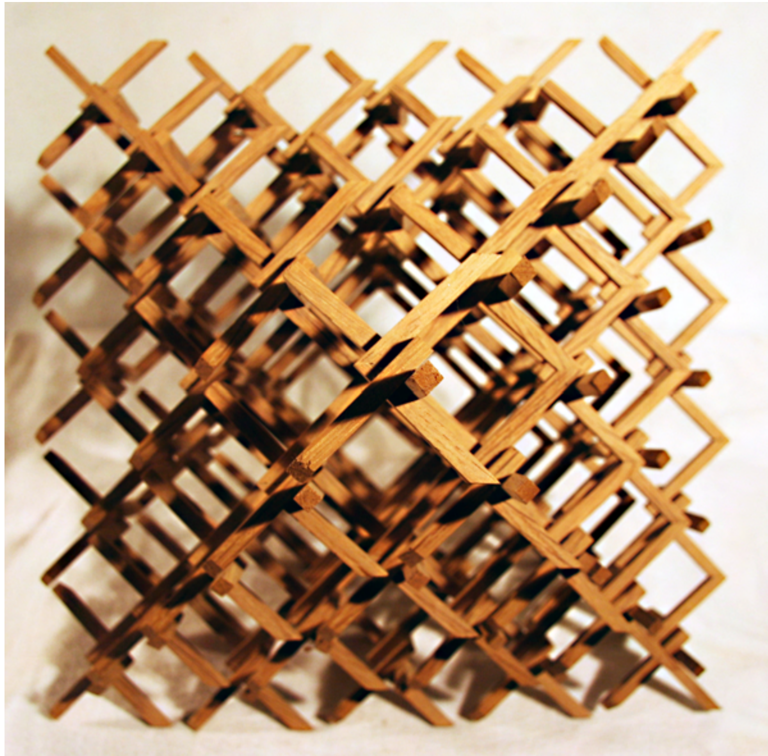
Non-Parallel Zigzags‡ in Quaternary/Ternary Embrace

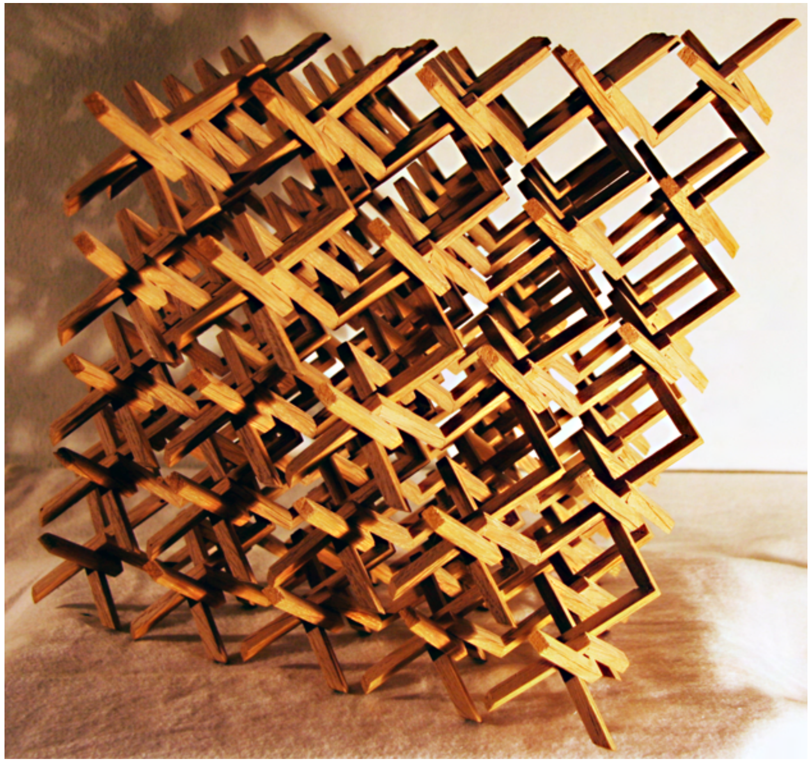




Rhombic beam, square cut faces, skew miter joints, parallelogram

Non-Parallel Zigzags in Ternary Embrace: Square Beam





3D Turtle Geometry Description of Helix

Zigzag:

`Repeat` K `[ Forward` D `Left 90 Forward` D `Right 90 ]`

3D turtle graphics:

`Repeat` K `[ Forward` D `RollLeft 180 TurnLeft 90 ]`

where `RollLeft` rotates the turtle about its *heading* vector, and
`TurnLeft` rotates the turtle about its *normal* vector.

General 3D helix $H(D, \psi, \phi)$:

`Repeat` K `[ Forward` D `RollLeft` ψ `TurnLeft` ϕ `]`

Parameters of Zigzag, Zigzagzeg, Zigzagzegzug

Helix	**Roll** ψ	**Turn** ϕ	**Projection**
Zigzag	180°	90°	—
Zigzagzeg	90°	90°	$\triangle$
Zigzagzegzug	60°	$70.5\ldots^\circ$	$\square$
Zigzagzeg*	$109.5\ldots^\circ$	60°	$\triangle$
Zigzagzegzug†	$70.5\ldots^\circ$	60°	$\square$
Zigzag‡	180°	60°	—

Note: Zigzagzegzug and Zigzagzegzug† have ψ, ϕ swapped

Helix Invariance Theorem

Shape of projection along helix axis is invariant under swapping the helical roll and turn angles ψ and ϕ.

Consider the *parallel projection* of $H(D, \psi, \phi)$ along its axis.

This is a regular polygon (possibly infinite).

Denote the *exterior angle* of this polygon by $\theta(\psi, \phi)$.

Helix Invariance Theorem:

$$\theta(\psi, \phi) \;=\; \theta(\phi, \psi)$$

Discovered by “accident”

Conclusion 1

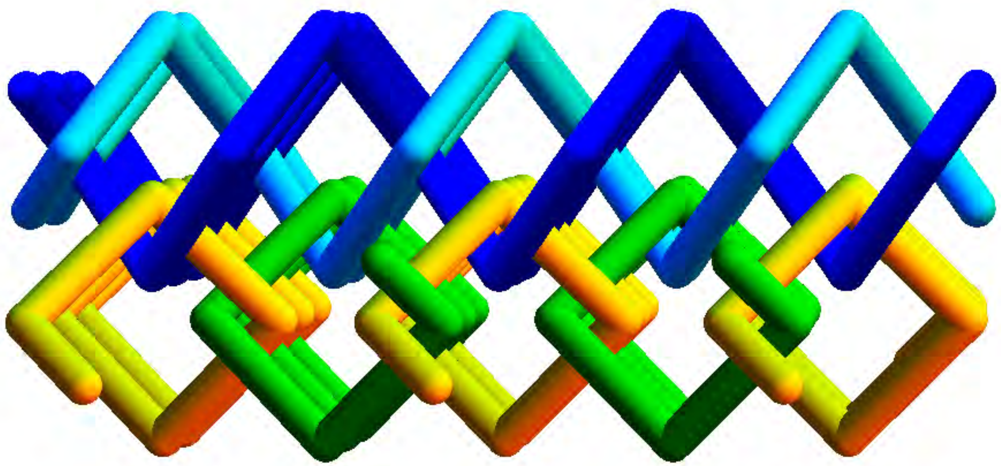
- Generalize planar zigzag to discrete (turtle-geometry) helices
- Describe discrete helix by roll angle ψ and turn angle ϕ
- Weave helices into space-spanning structures in various ways
- Use regular/skew miter joints with appropriate beam cross section
- New Theorem:

  Shape of projection is invariant under swapping roll and turn angle

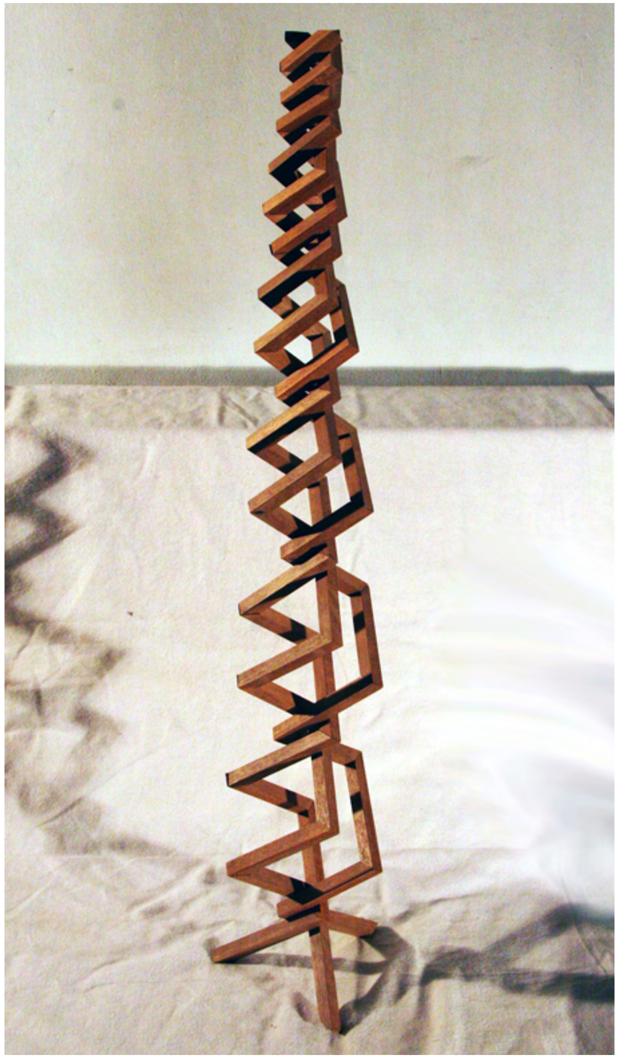
Conclusion 2

Move Variations:

- Helices with varying step size
- Mix right-handed and left-handed helices



Photographs by *Tom Goris*



Related Work

- Tom Verhoeff & Koos Verhoeff.
  "The Mathematics of Mitering and Its Artful Application",
  Bridges 2008, Leeuwarden, Netherlands, pp.225–234.

- Tom Verhoeff & Koos Verhoeff.
  "Regular 3D Polygonal Circuits of Constant Torsion",
  Bridges 2009, Banff, Canada, pp.223–230.

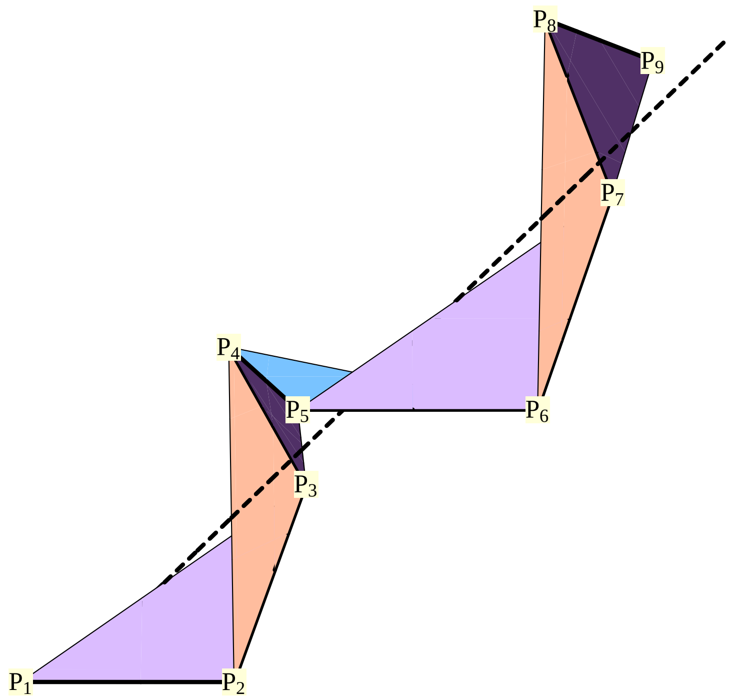
- Tom Verhoeff.
  "3D Turtle Geometry: Artwork, Theory, Program Equivalence and Symmetry".
  Int. Journal of Arts and Technology, **3**(2/3):288-319 (2010).

Also see: `http://www.win.tue.nl/~wstomv/publications/`

Proof (Sketch) of Helix Invariance Theorem

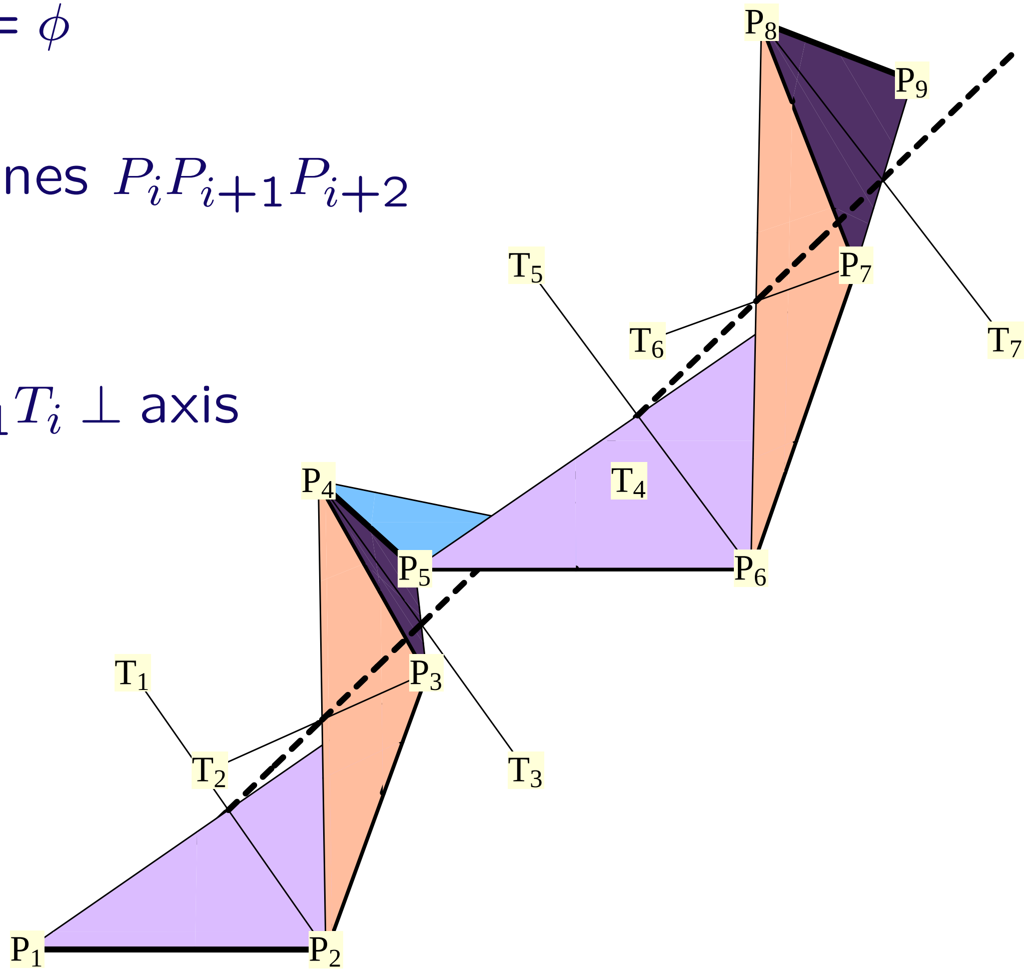
Shape of projection along helix axis is invariant under swapping the helical roll and turn angles ψ and ϕ.

Let P_i be the i-th vertex of helix $H(1, \psi, \phi)$, and let $\theta = \theta(\psi, \phi)$.



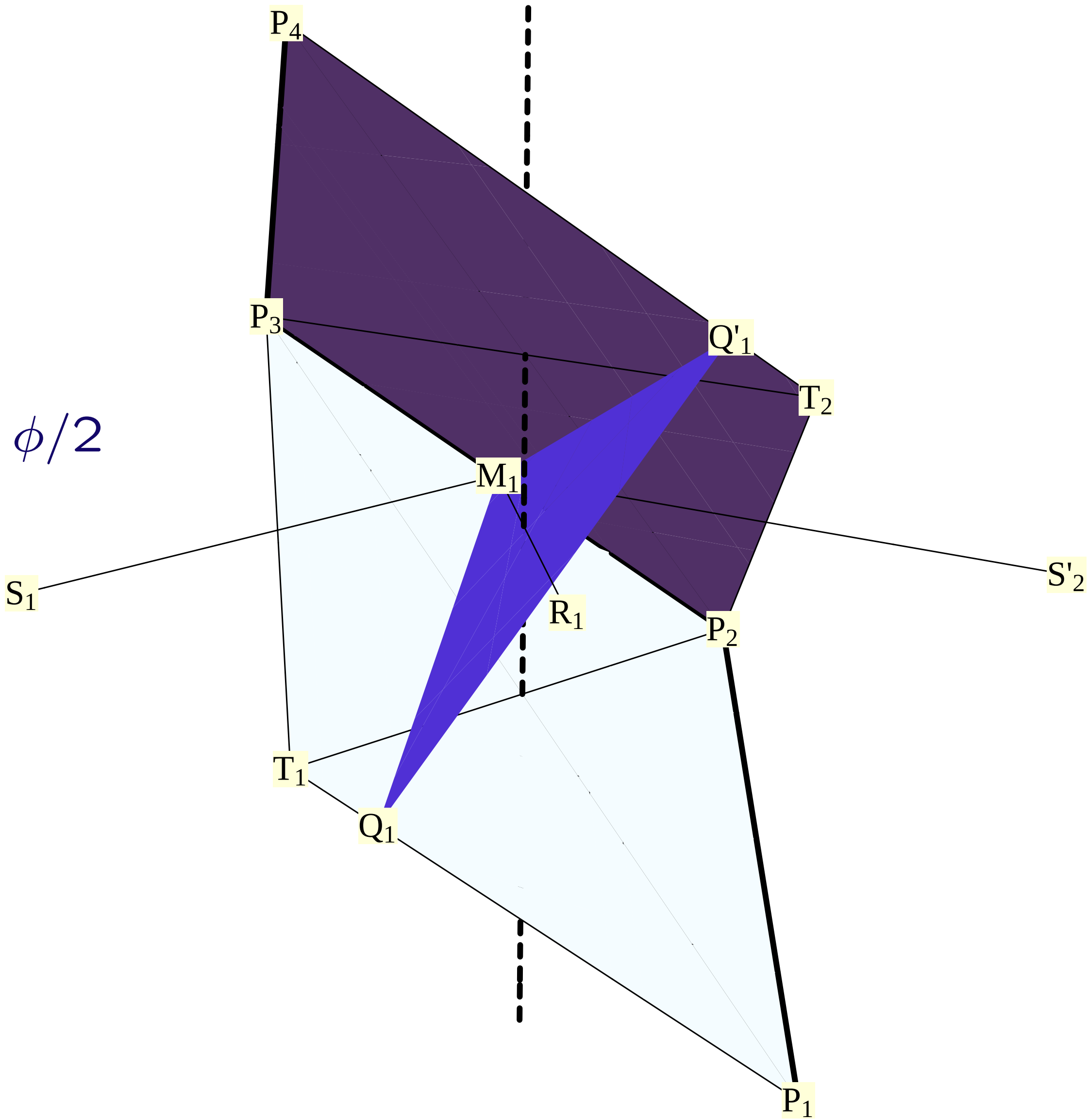
Some Observations

- Exterior angle $P_i P_{i+1} P_{i+2} = \phi$
- Angle between adjacent planes $P_i P_{i+1} P_{i+2}$ and $P_{i+1} P_{i+2} P_{i+3} = \psi$
- Interior angle bisectors $P_{i+1} T_i \perp$ axis
- Angle between adjacent angle bisectors $= \theta$



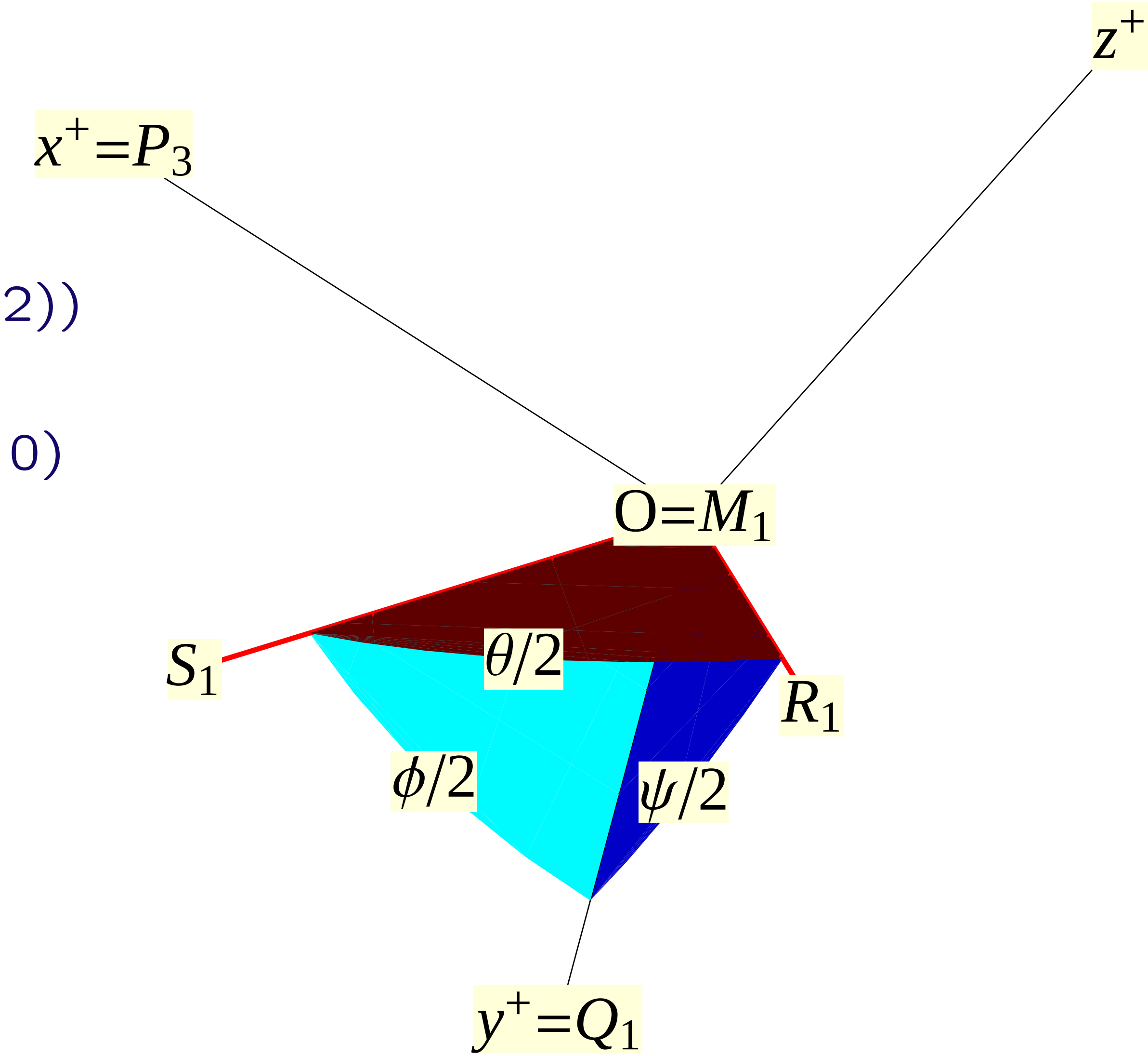
Further Observations

- $\angle P_2P_1T_1 = \phi$, hence
  $\angle P_1P_3M_1 = \phi/2$
- $M_1S_1 \perp P_1P_3$
  $\angle P_3M_1Q_1 = 90°$, hence
  $\angle S_1M_1Q_1 = \angle P_1P_3M_1 = \phi/2$
- $\angle Q_1M_1Q'_1 = \psi$, hence
  $\angle Q_1M_1R_1 = \psi/2$
- $\angle S_1M_1S'_2 = \theta$, hence
  $\angle S_1M_1R_1 = \theta/2$



Final Observations

- $R_1 = (0, \cos(\psi/2), \sin(\psi/2))$
- $S_1 = (\sin(\phi/2), \cos(\phi/2), 0)$
- $\cos(\theta/2) = R_1 \cdot S_1$



Proof of Helix Invariance Theorem

In general, 3D rotations do not commute.

But here we have

$$\cos(\theta/2) \;=\; \cos(\psi/2)\cos(\phi/2)$$

Observe that the right-hand side is symmetric in ψ and ϕ.

Hence,

$$\theta(\psi,\phi) \;=\; \theta(\phi,\psi)$$

Q.E.D.

Special Property

Helix with roll angle $\psi = 60^\circ$ and turn angle $\phi = \arccos(1/3) \approx 70.5\ldots^\circ$ is special:

> Its axis passes through the centers of the angle spanning rhombi.