Mitered Fractal Trees: Constructions and Properties

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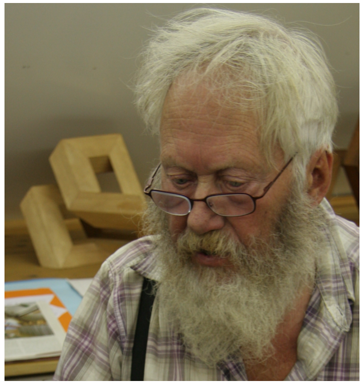

Stichting Wiskunst Koos Verhoeff
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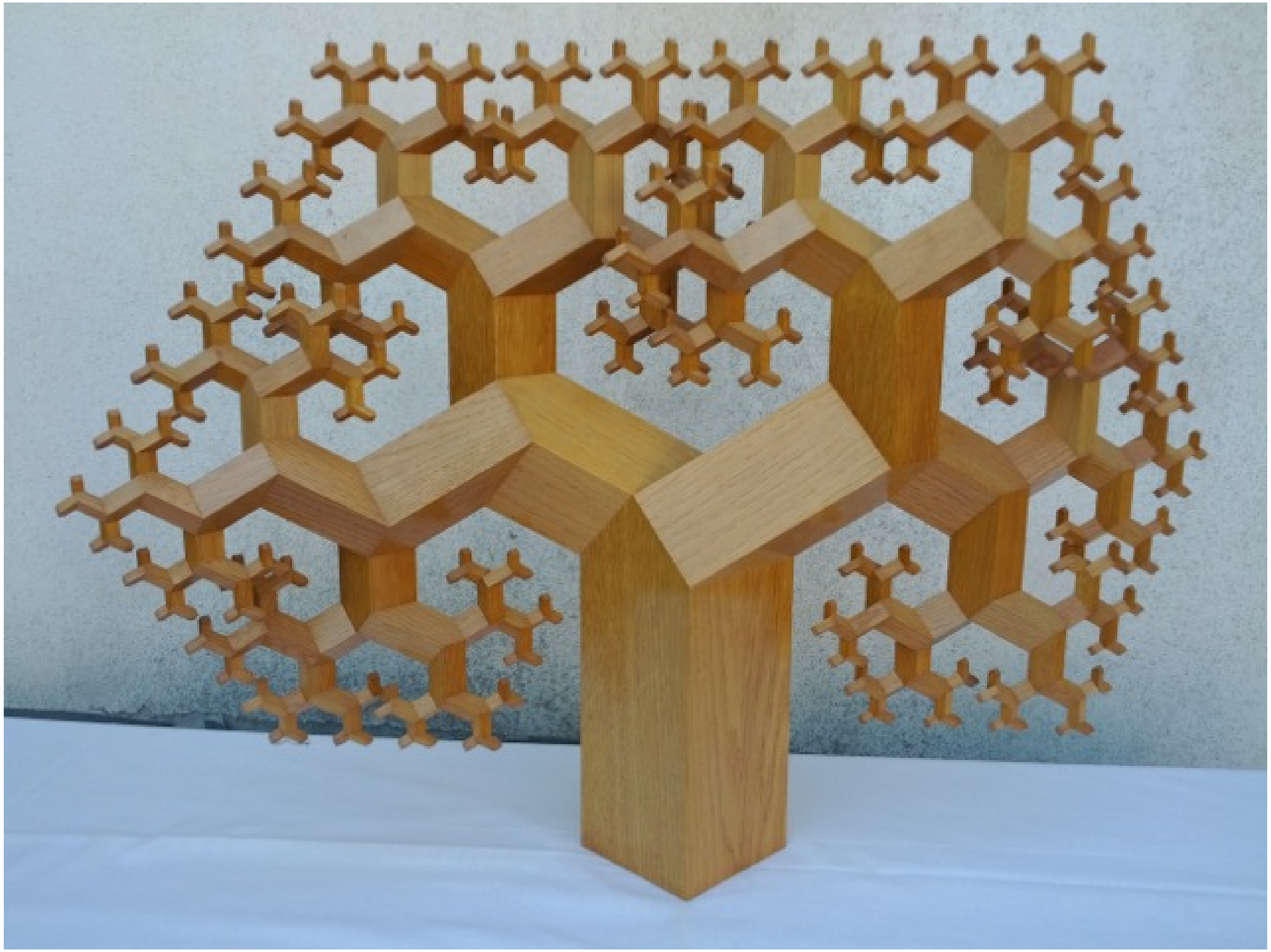
Mitered Fractal Tree Sculpture (late 1980s, bronze)

Mitered Fractal Tree Sculpture (late 1980s, wood)



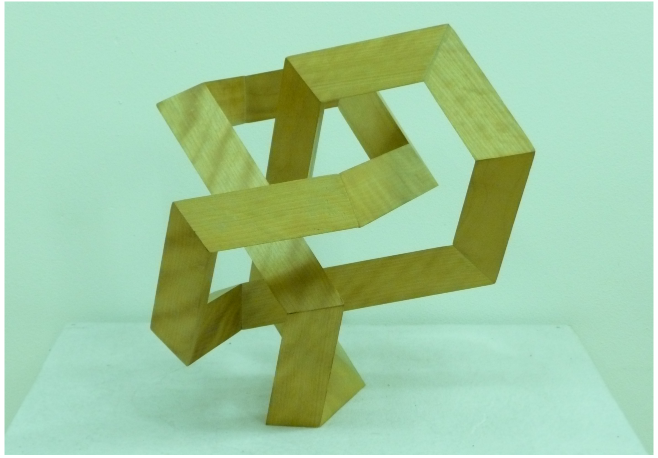
Mitered Fractal Tree I in One-Month Art Exhibition

Mitered Fractal Tree II in Conference Art Exhibition

Earliest Theme: Closed 3D Paths with Miter Joints

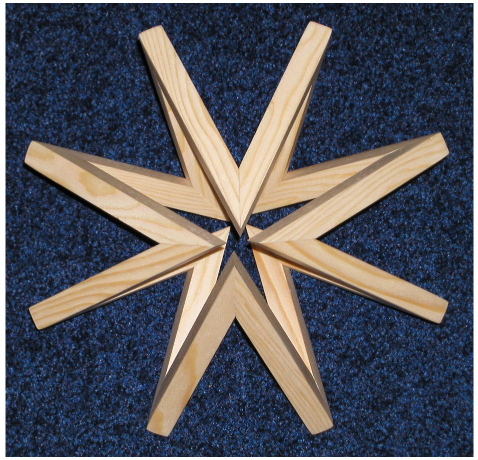
‘Tinkering’

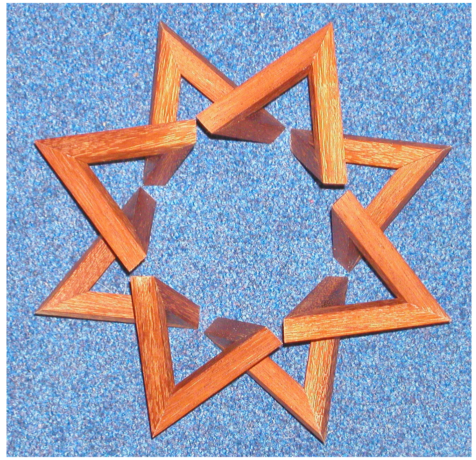
Bridges 2008



Lattice walking

Bridges 2008



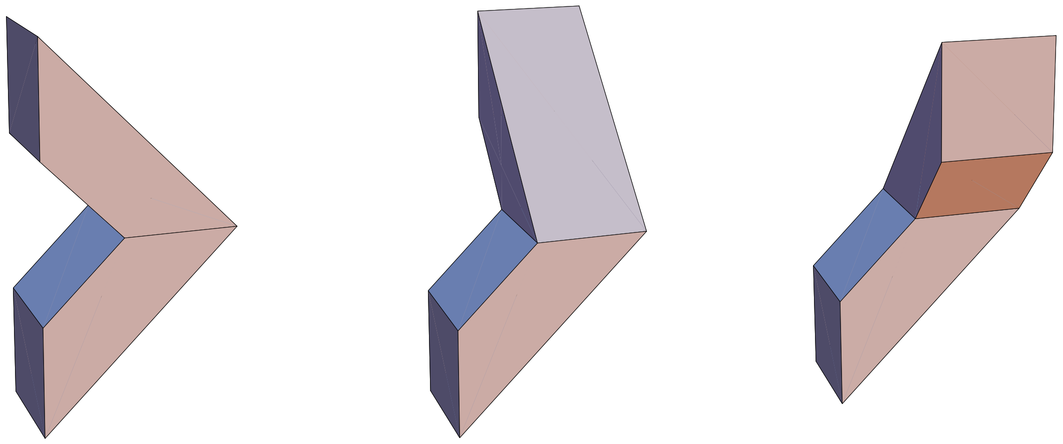


Constant torsion

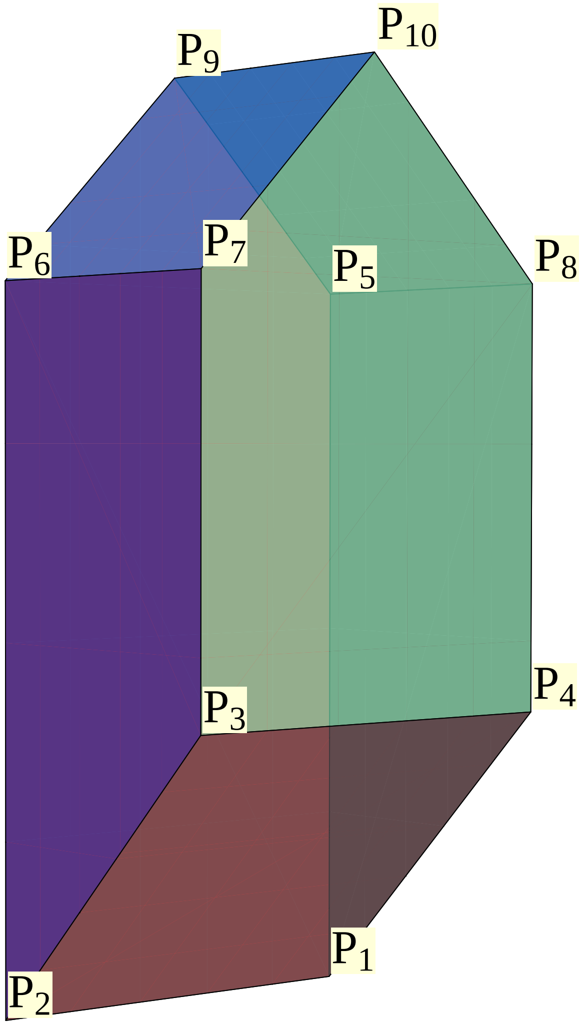
Bridges 2009

Miter Joints with Square Cut Face

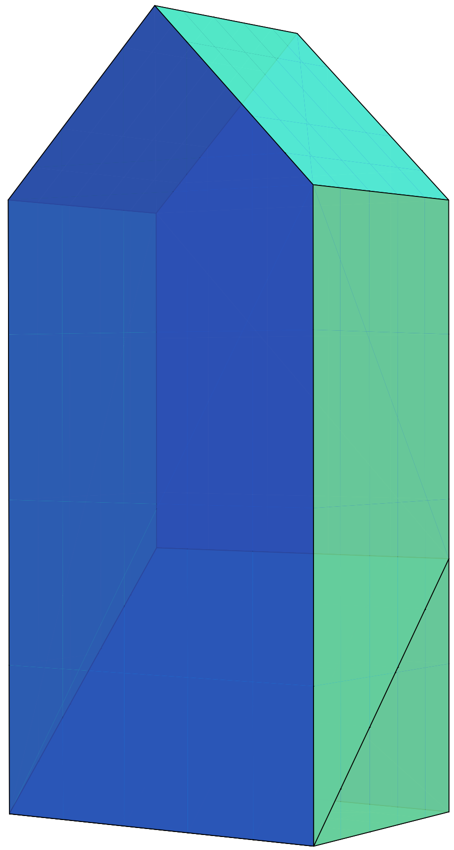
- Beam with $1 : \sqrt{2}$ rectangle as cross section
- Bevel at 45°
- Yields a square cut face



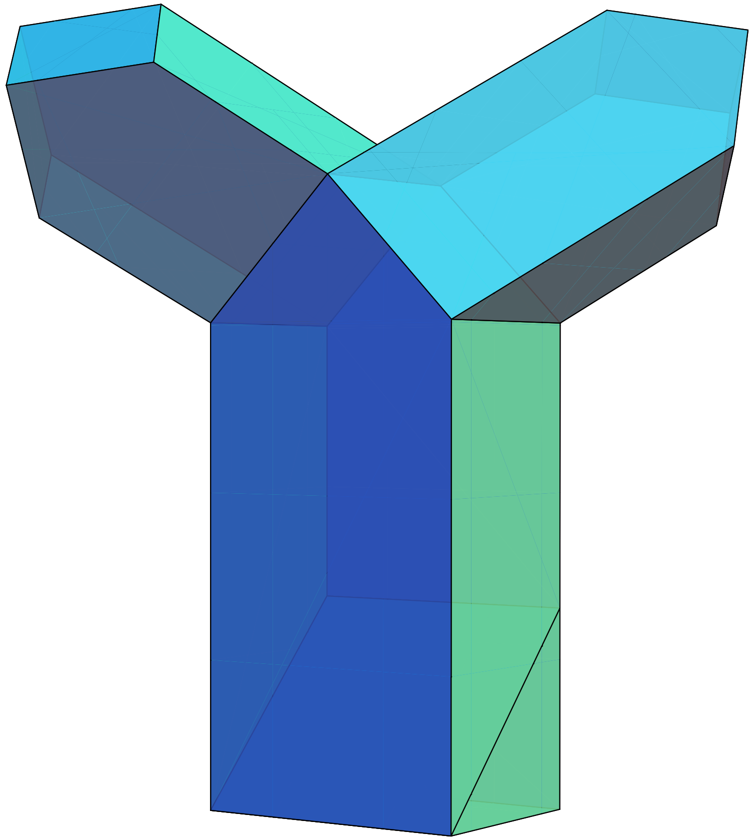
Tree Construction from $1 : \sqrt{2}$ Rectangular Beams: The Piece



Tree Construction from $1 : \sqrt{2}$ Rectangular Beams: Base

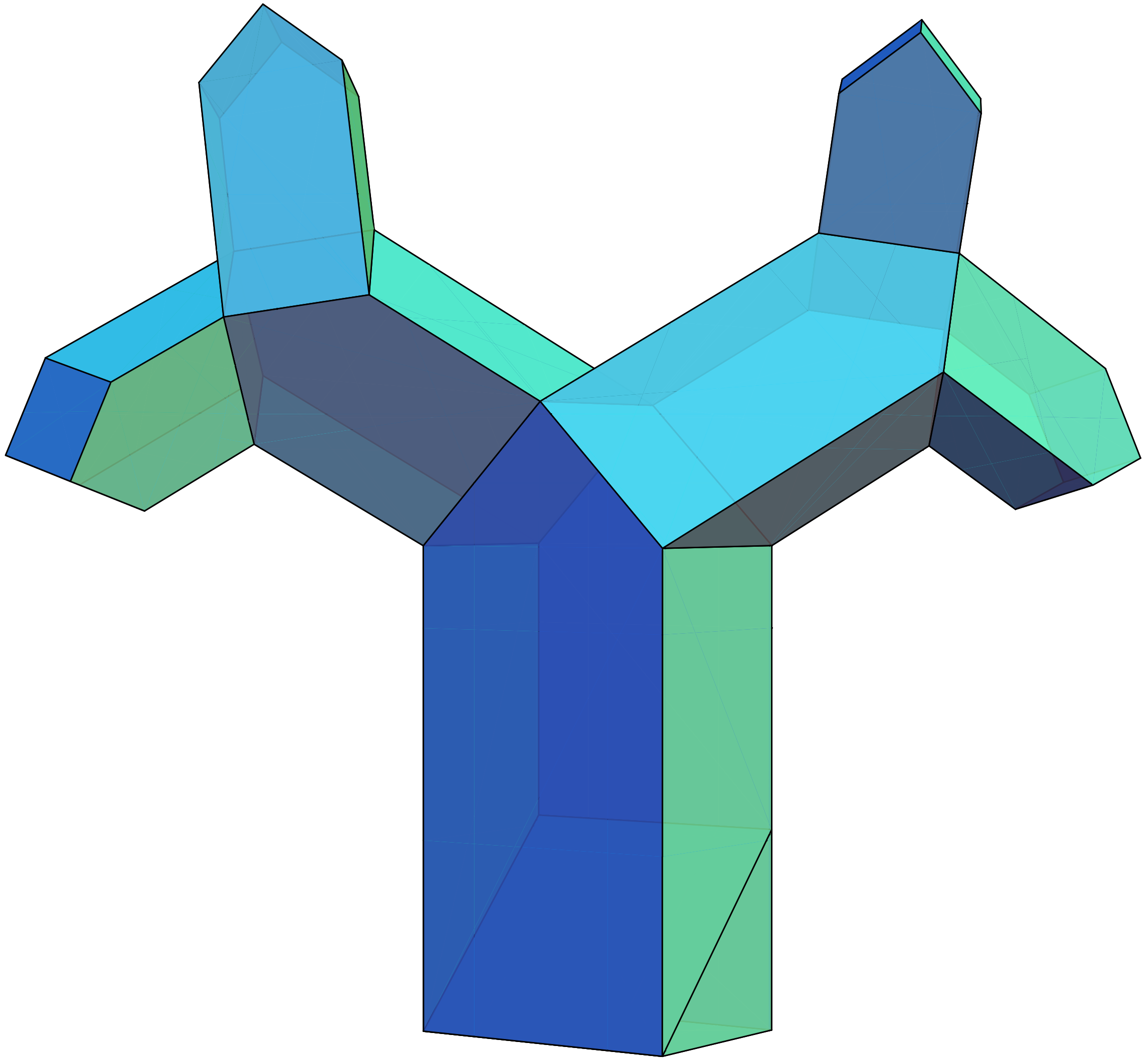


Tree Construction from $1 : \sqrt{2}$ Rectangular Beams: Growth 2

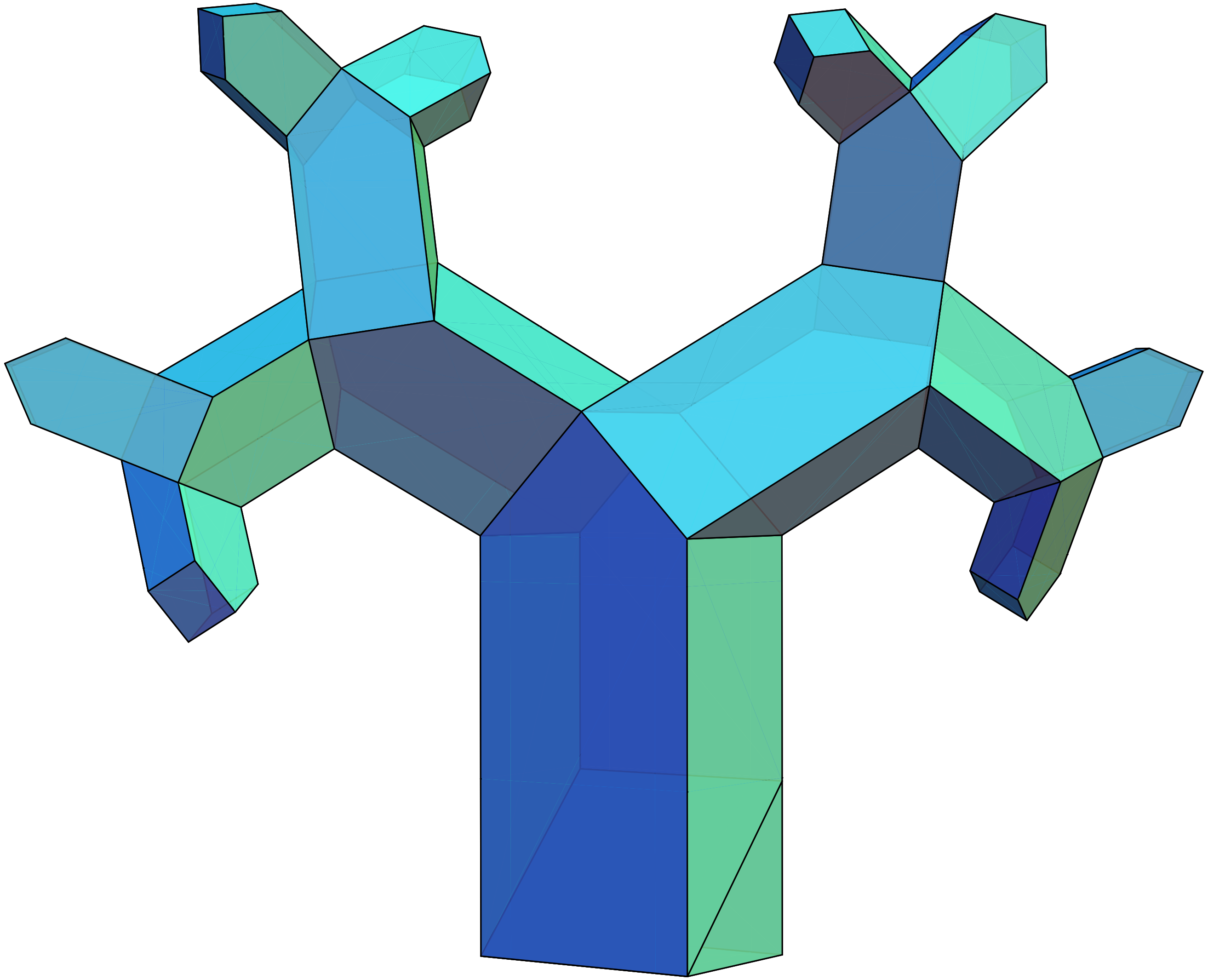


Make two copies of the piece and shrink them by a factor $\sqrt{2}$

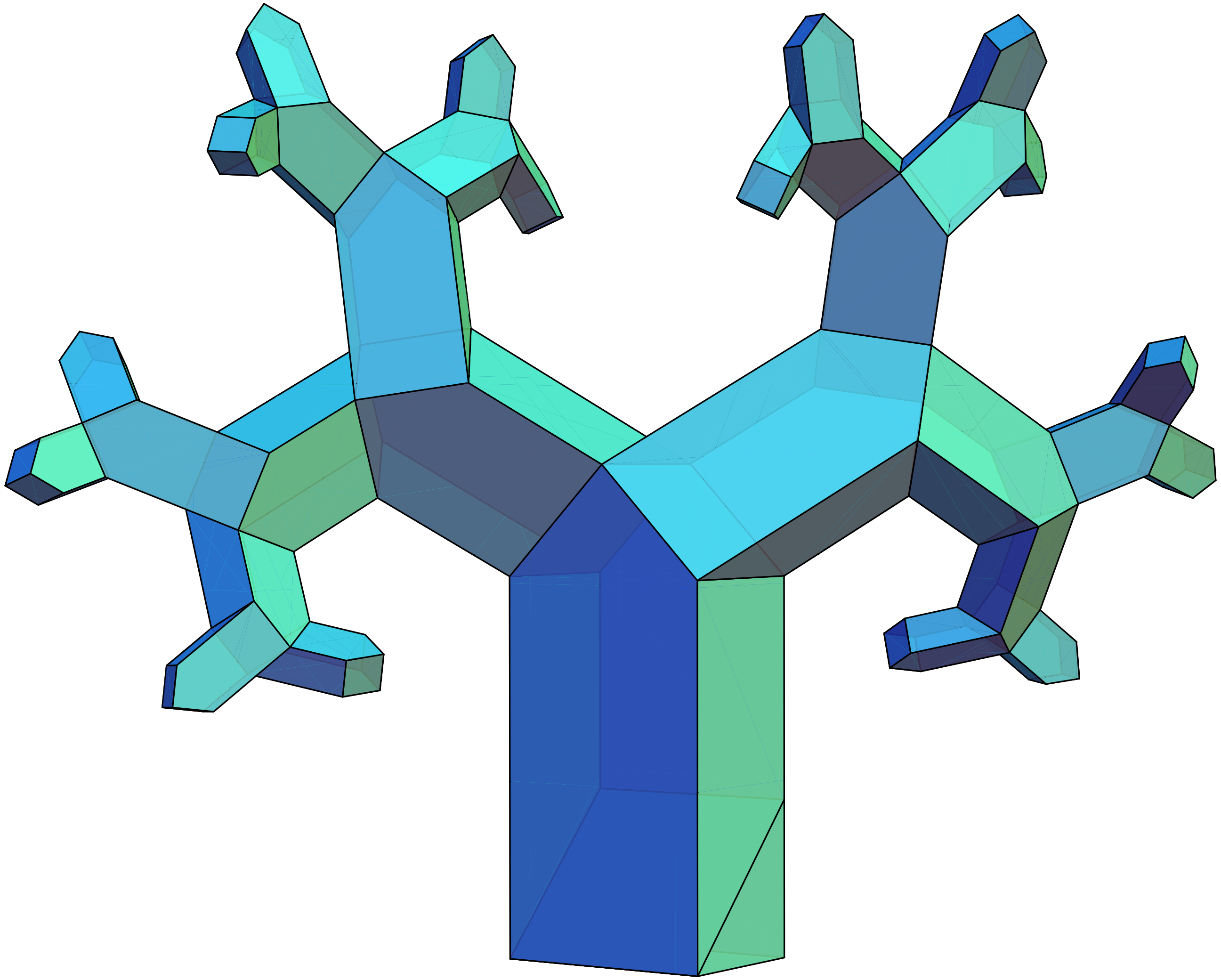
Tree Construction from $1 : \sqrt{2}$ Rectangular Beams: Growth 3



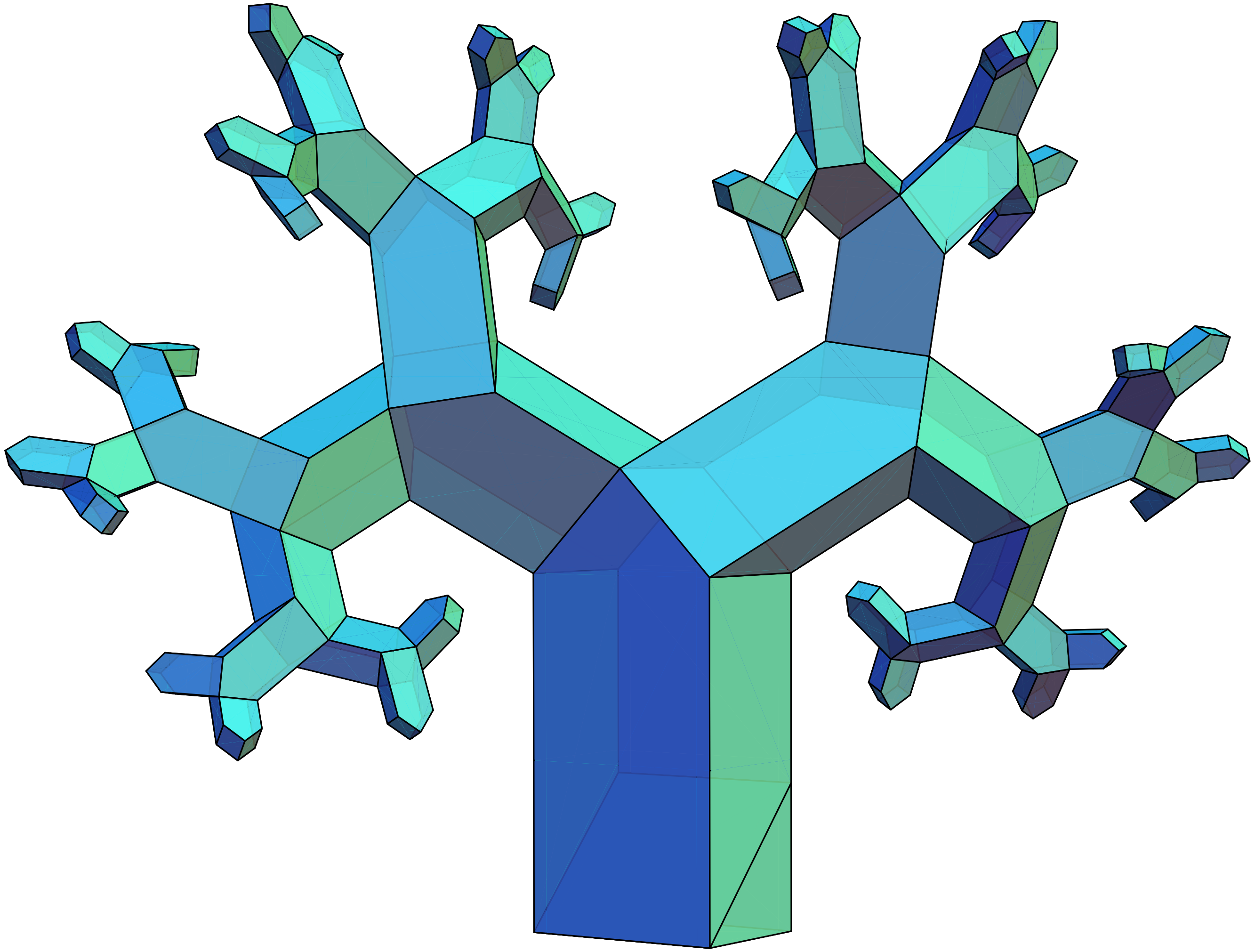
Tree Construction from $1 : \sqrt{2}$ Rectangular Beams: Growth 4



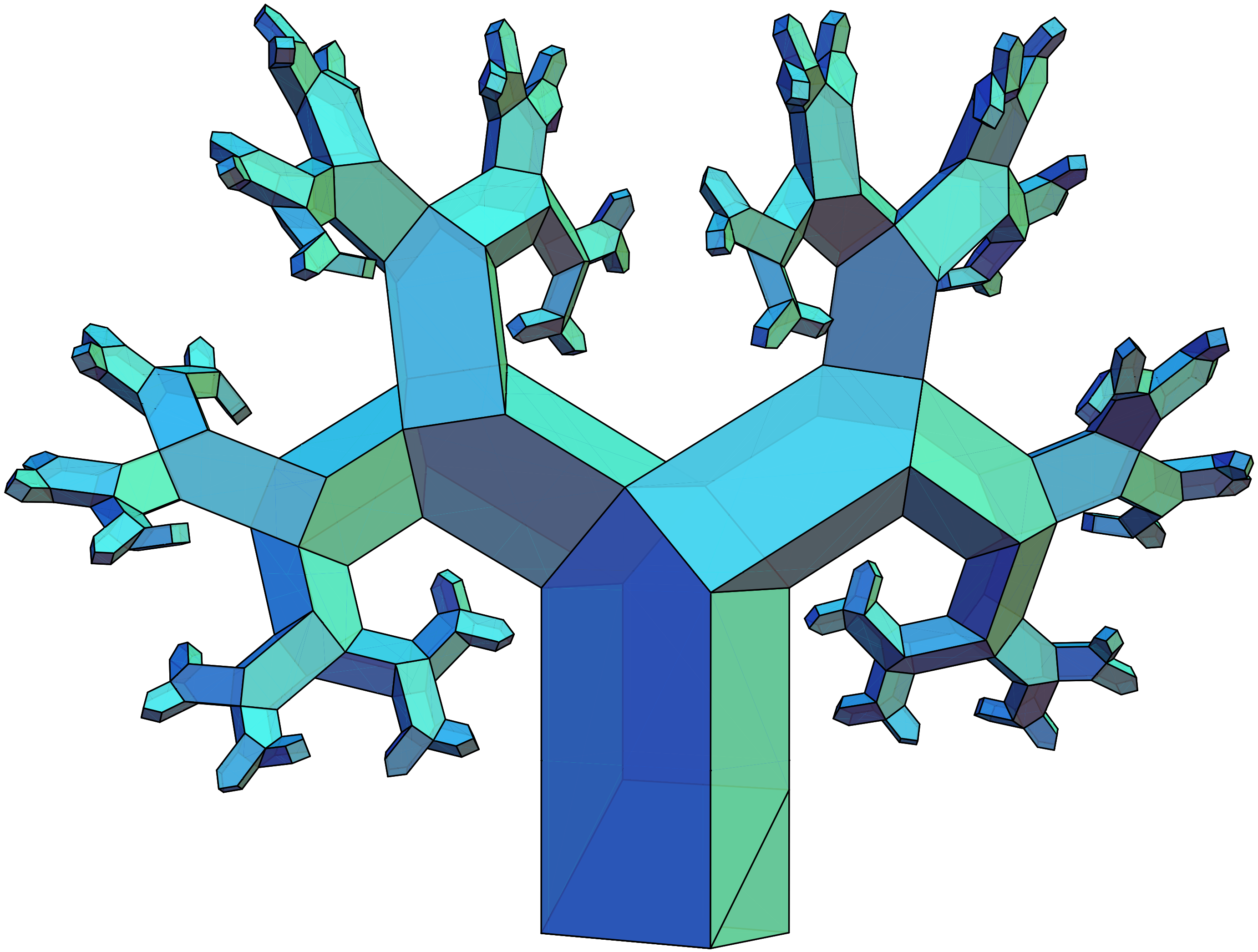
Tree Construction from $1 : \sqrt{2}$ Rectangular Beams: Growth 5



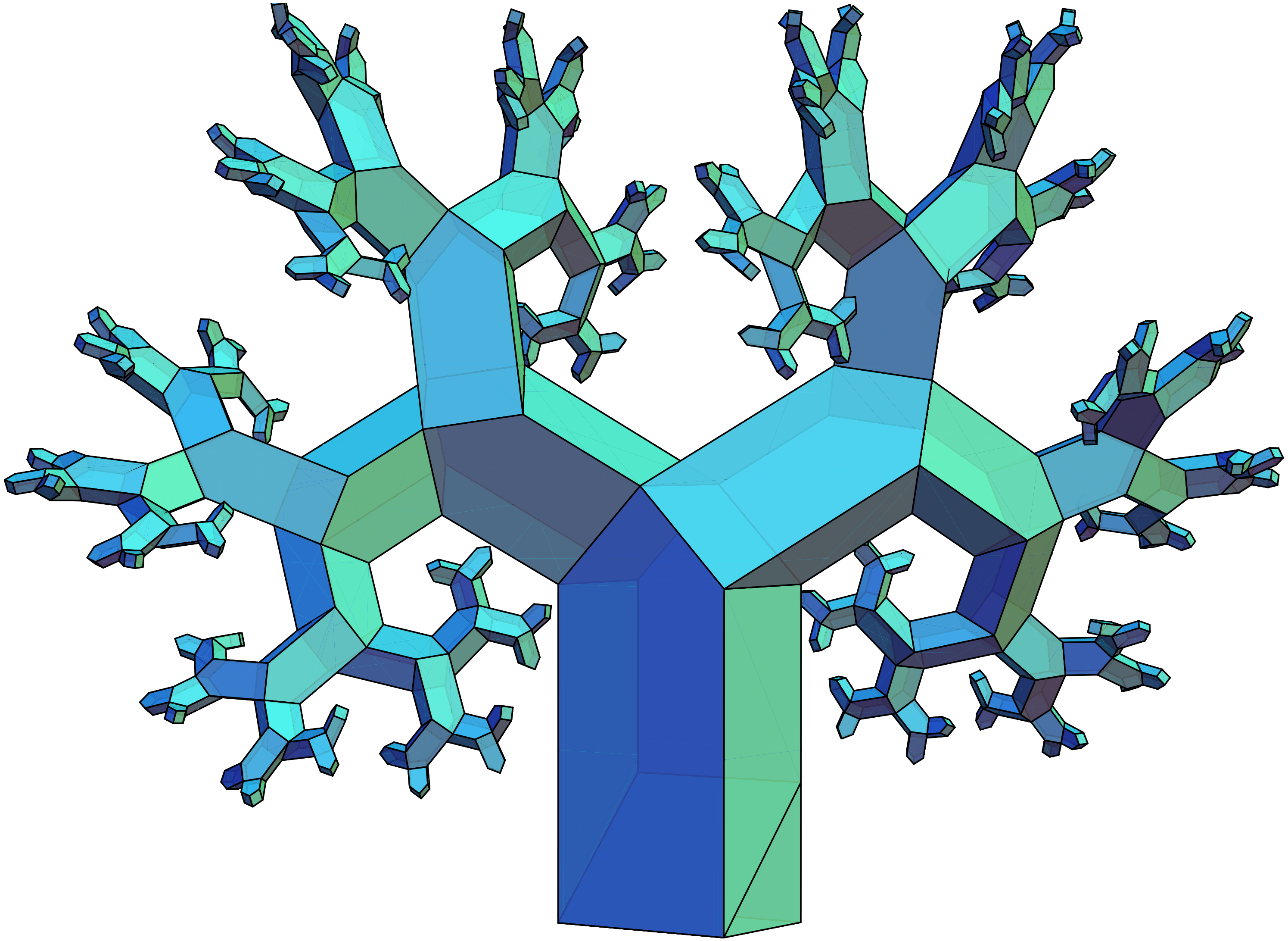
Tree Construction from $1 : \sqrt{2}$ Rectangular Beams: Growth 6



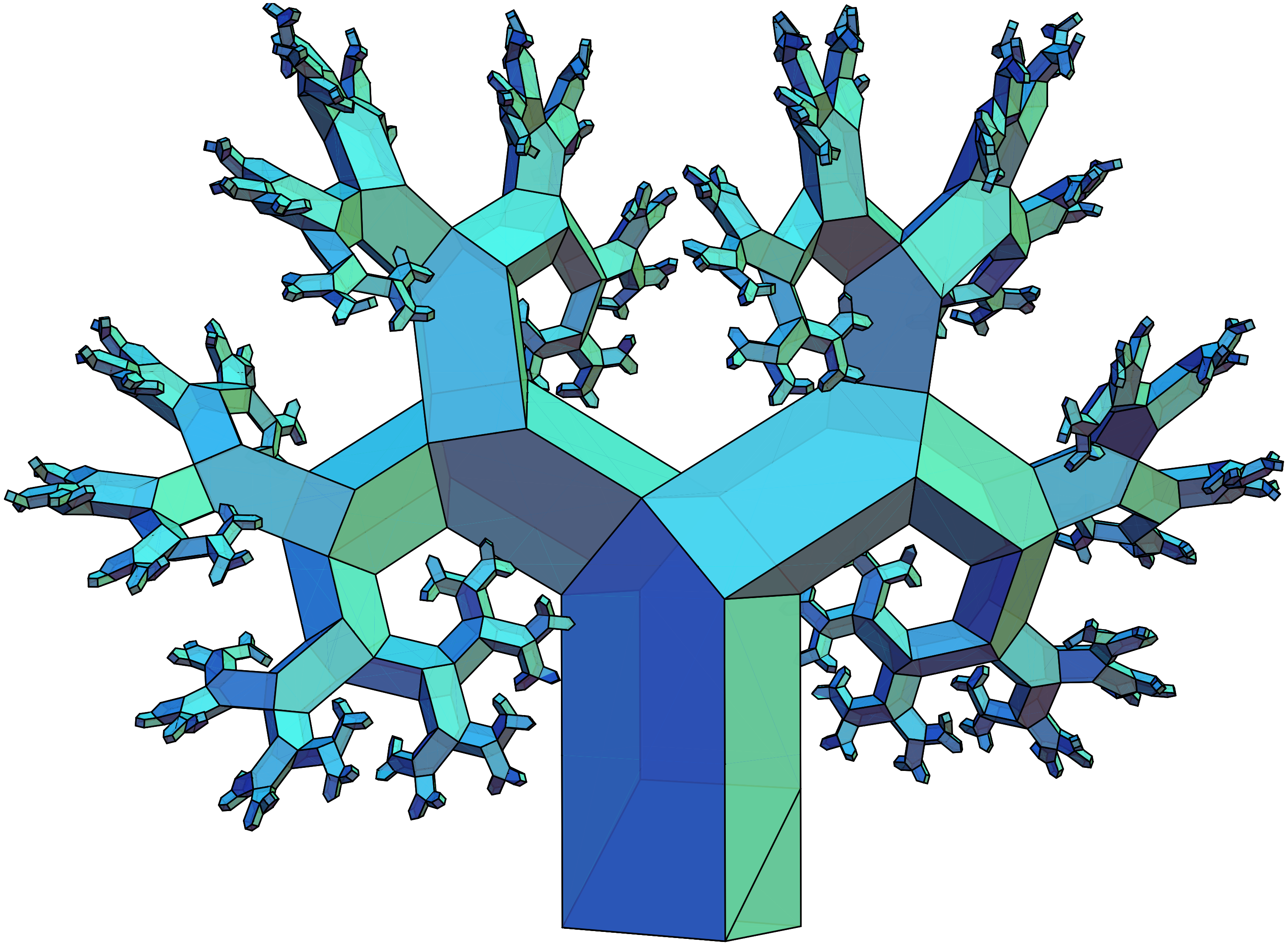
Tree Construction from $1 : \sqrt{2}$ Rectangular Beams: Growth 7



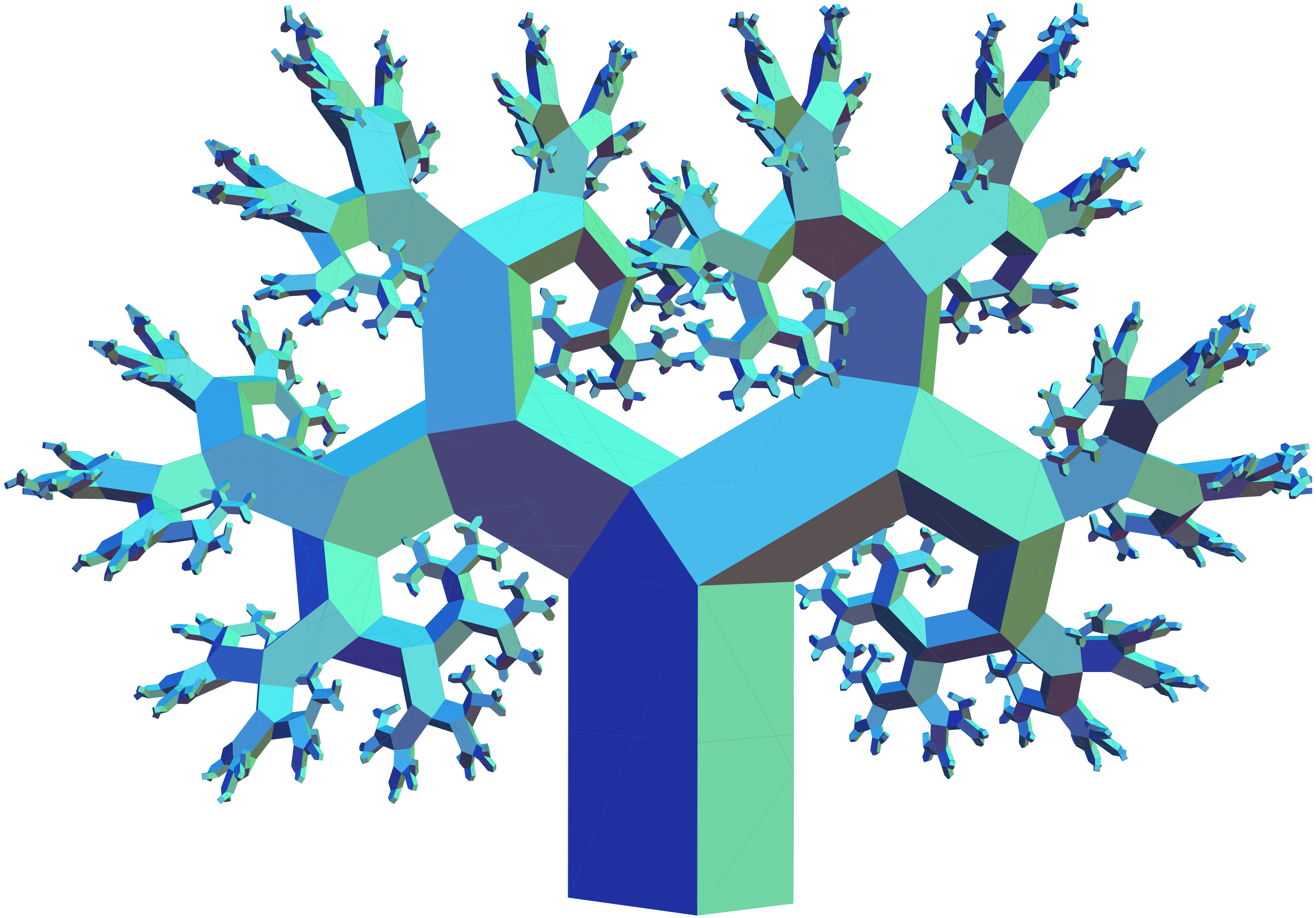
Tree Construction from $1 : \sqrt{2}$ Rectangular Beams: Growth 8



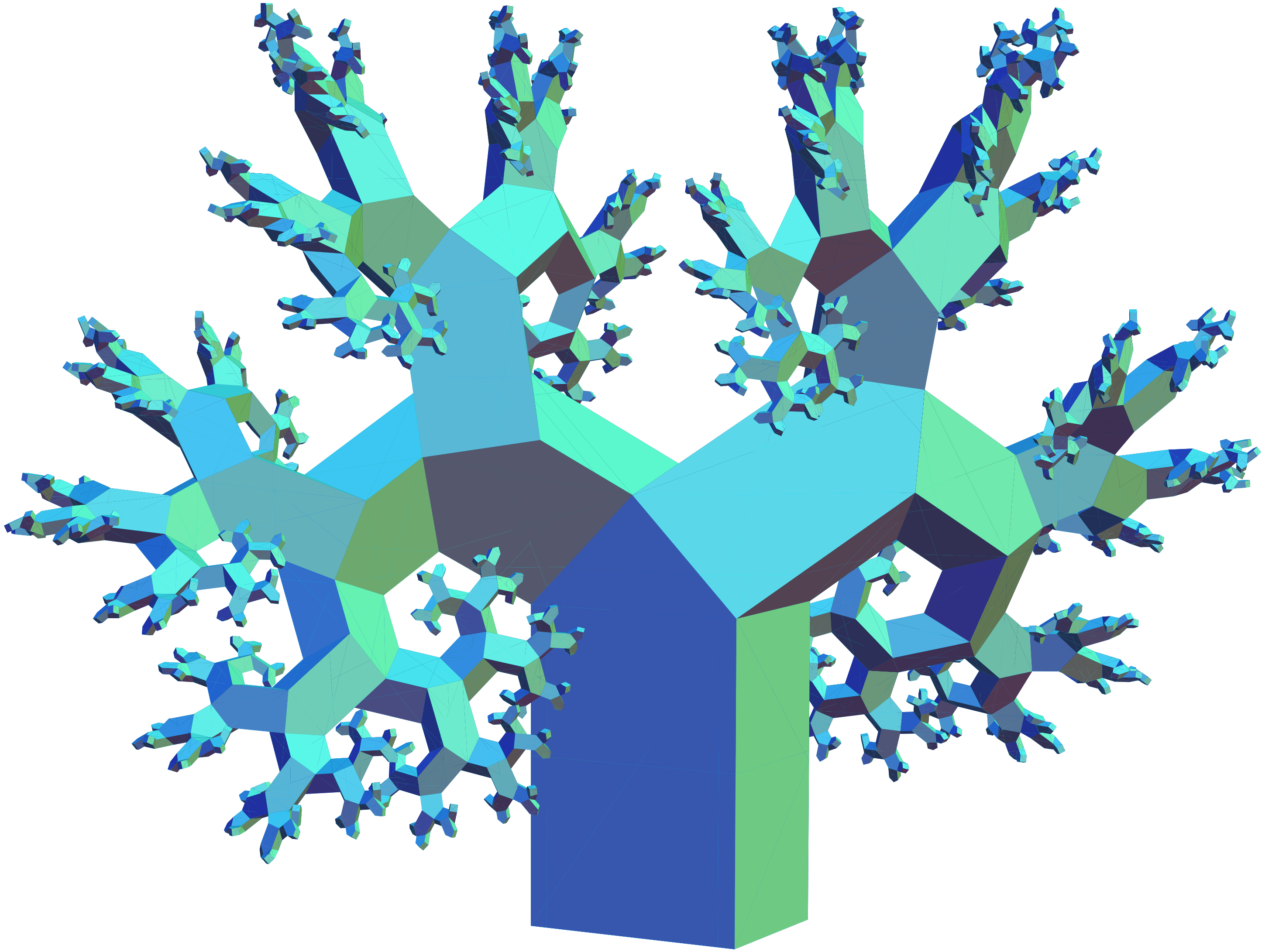
Tree Construction from $1 : \sqrt{2}$ Rectangular Beams: Growth 9



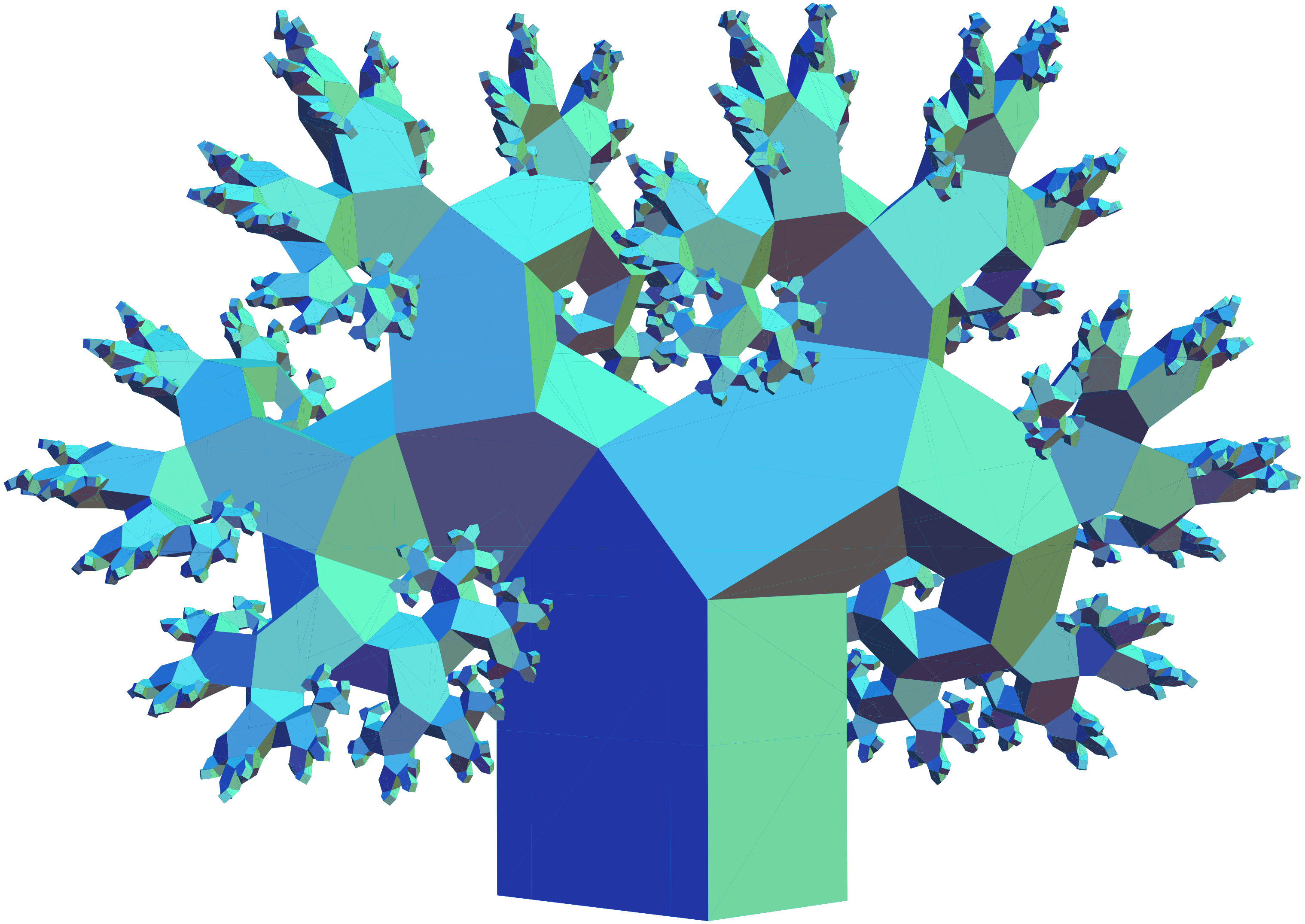
Tree Construction from $1 : \sqrt{2}$ Rectangular Beams: Growth 10



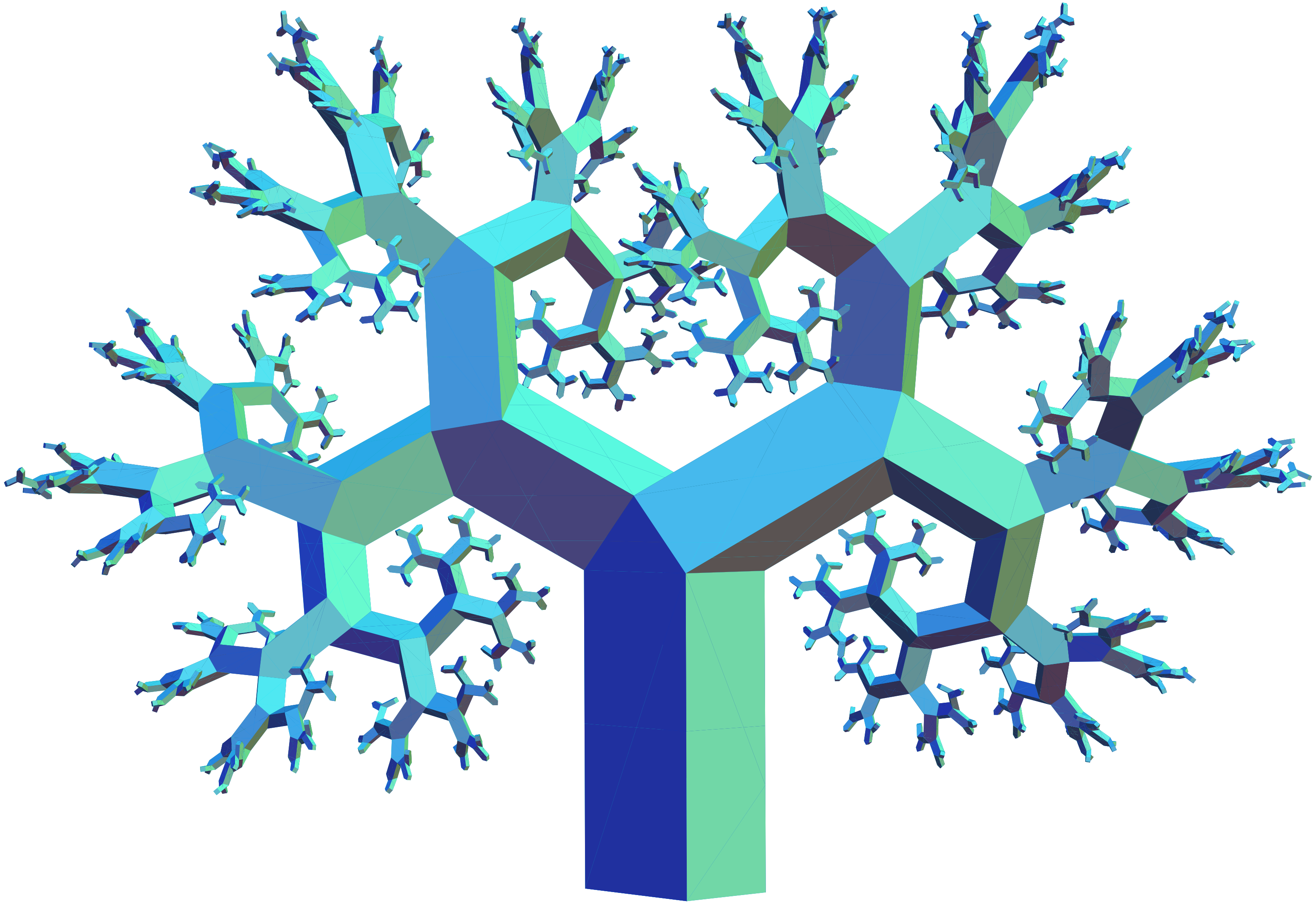
Shorten the Piece Length: $c = 1$



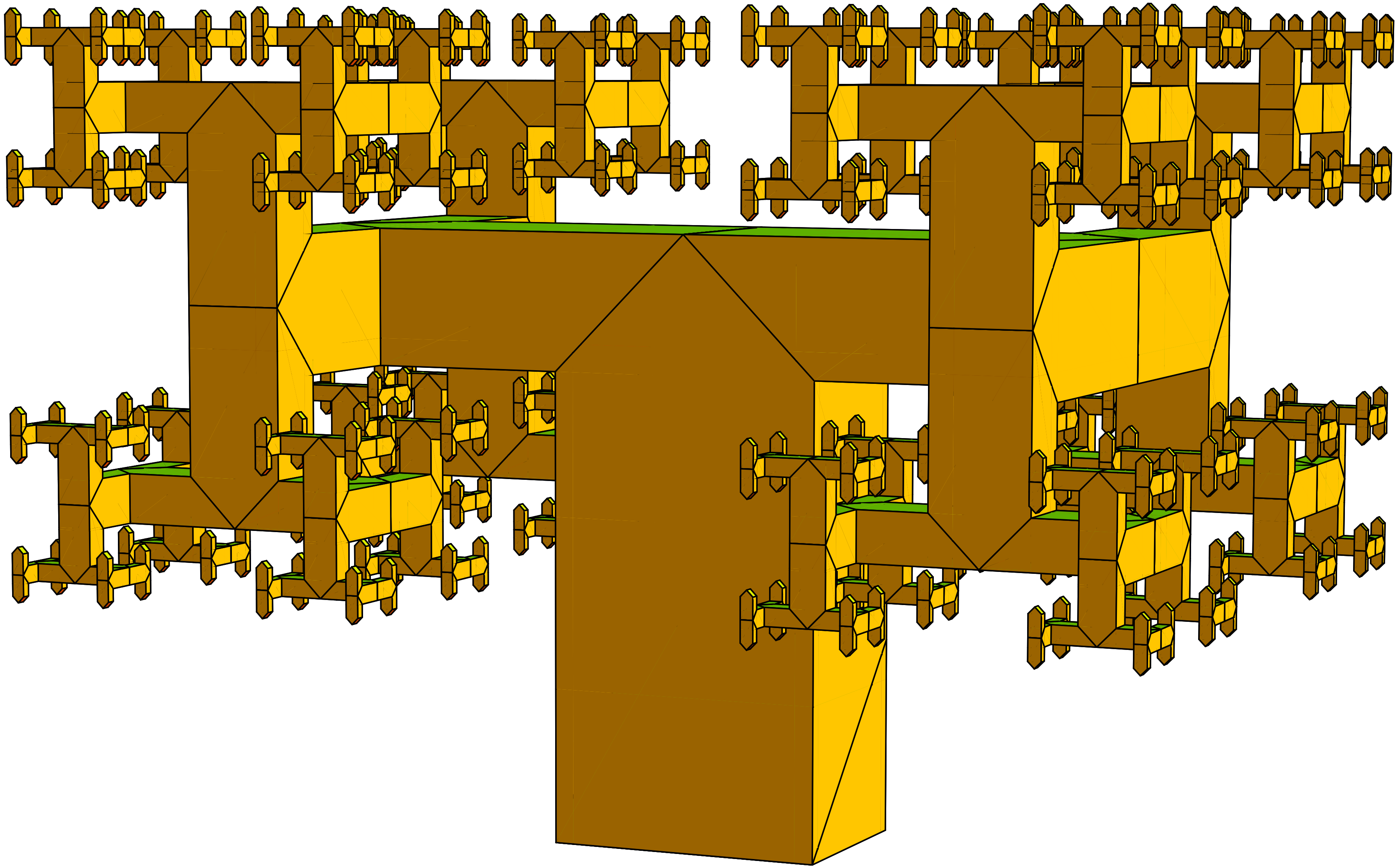
Shorten the Piece Length further: $c = 1/2$



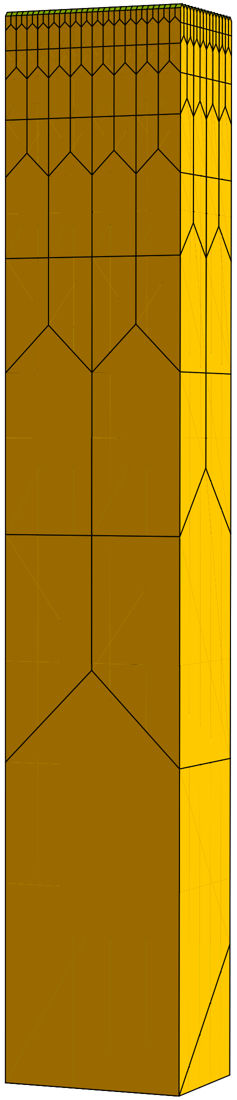
Extend the Piece Length: $c = 2$



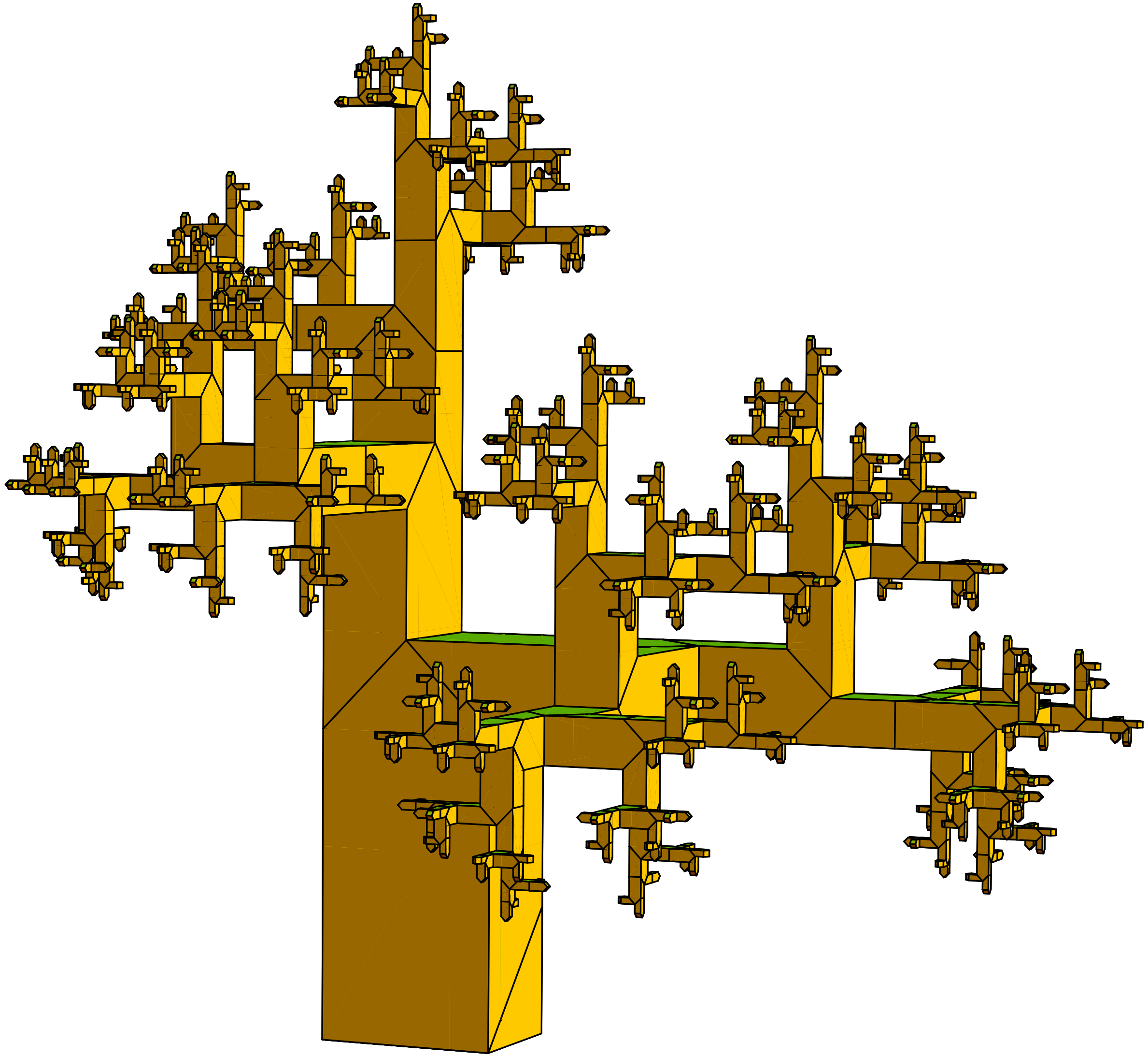
Vary the Growth Pattern: Side – Side



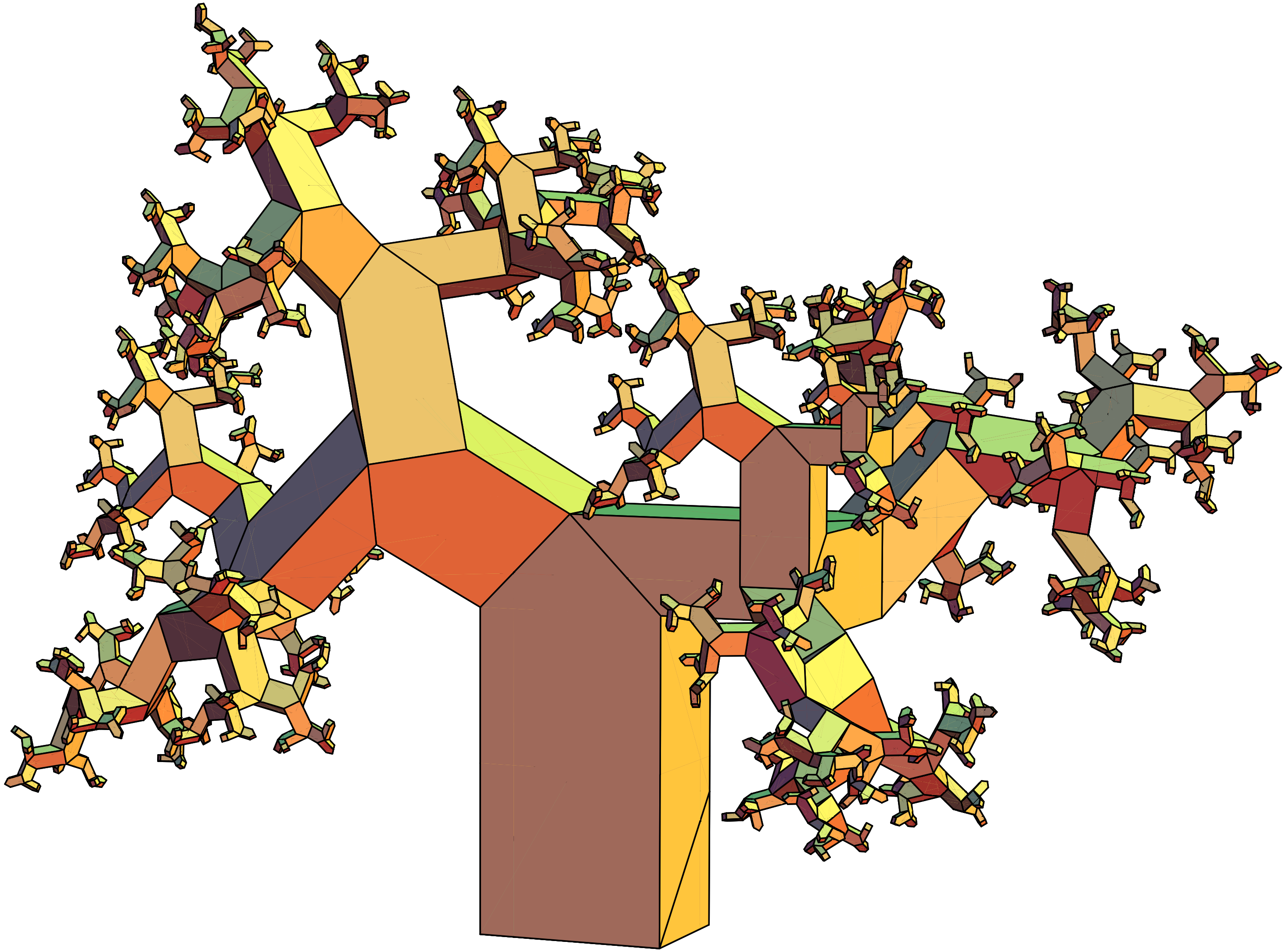
Vary the Growth Pattern: Up – Up



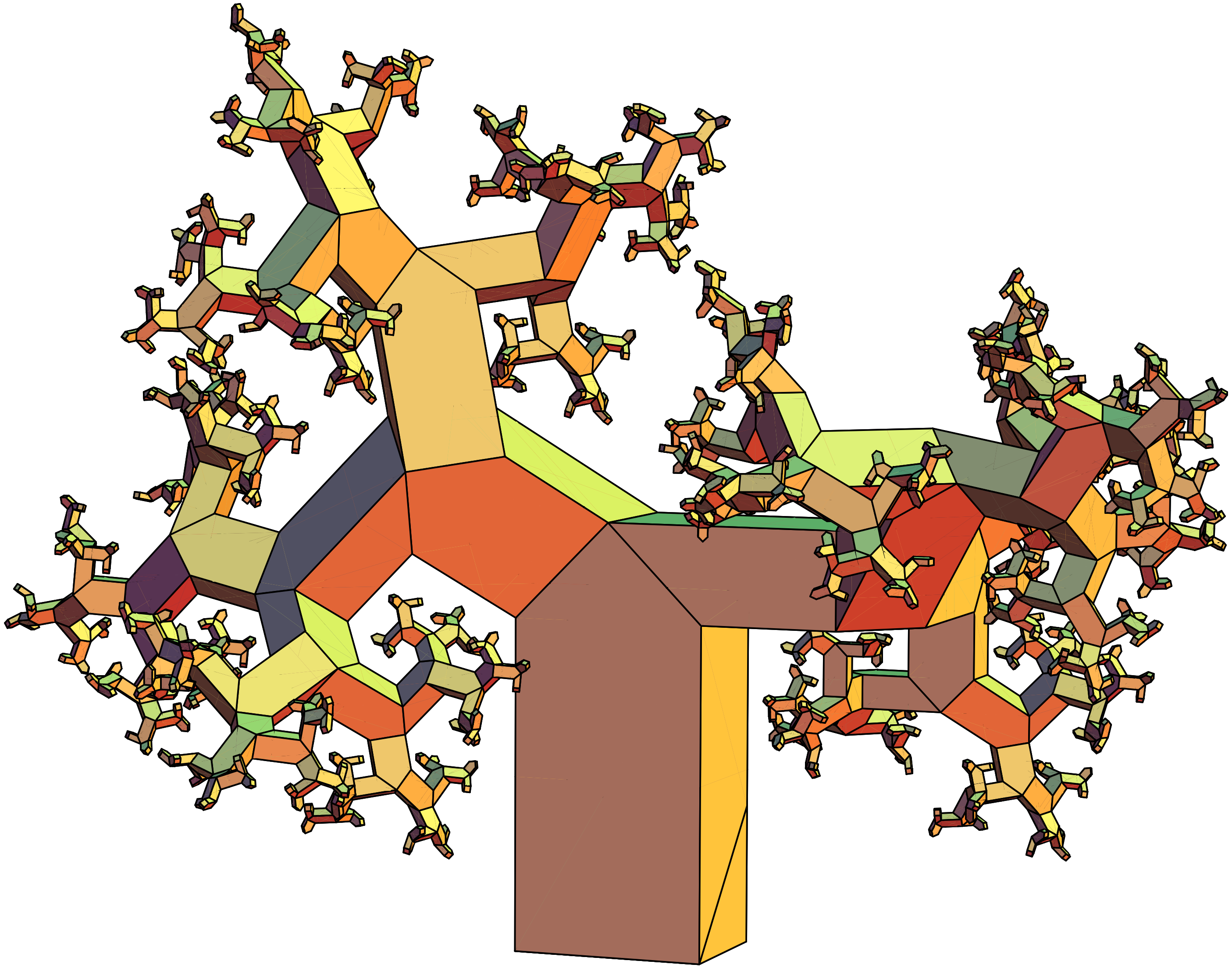
Vary the Growth Pattern: Side – Up



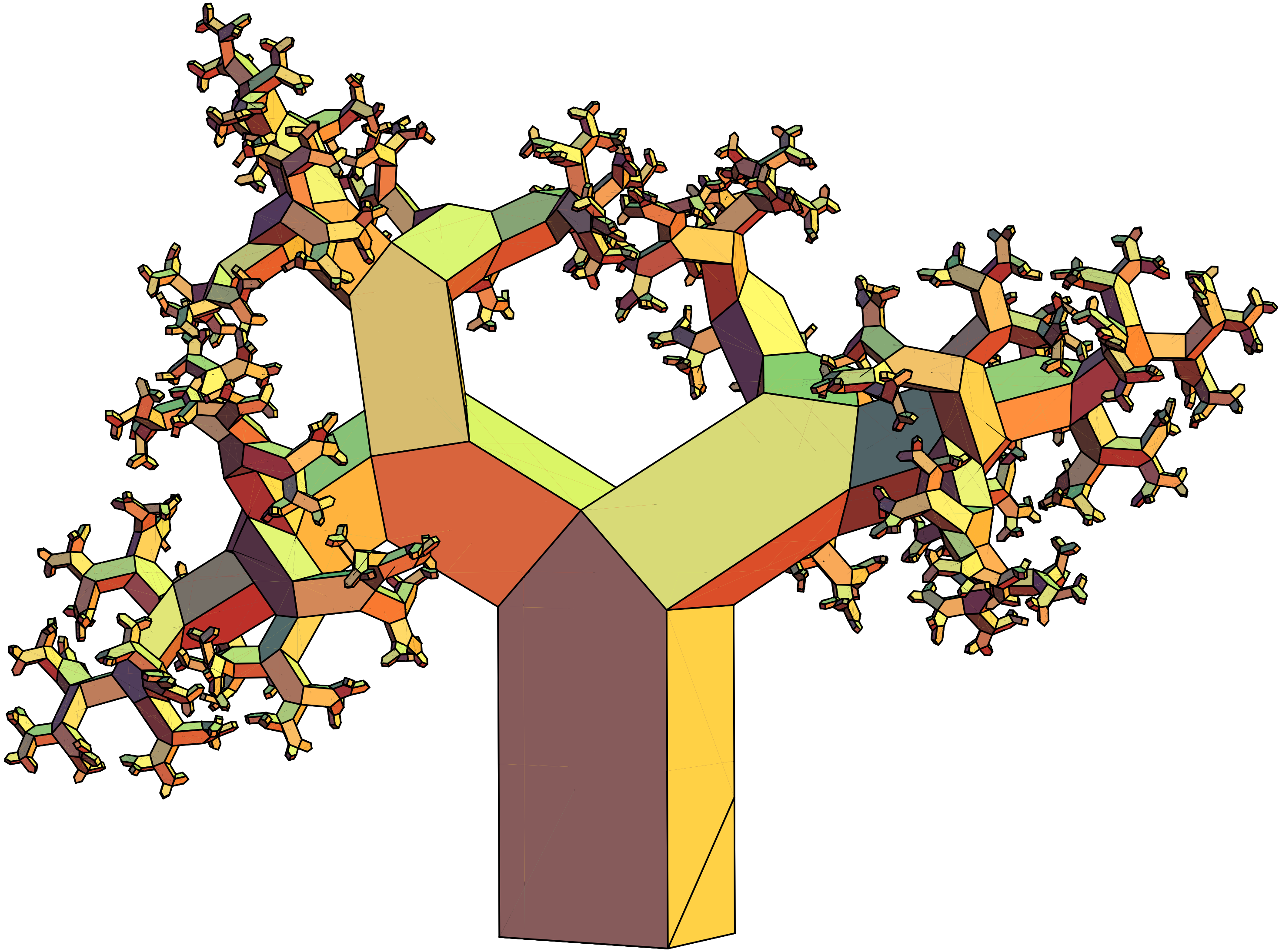
Vary the Growth Pattern: Side – Front



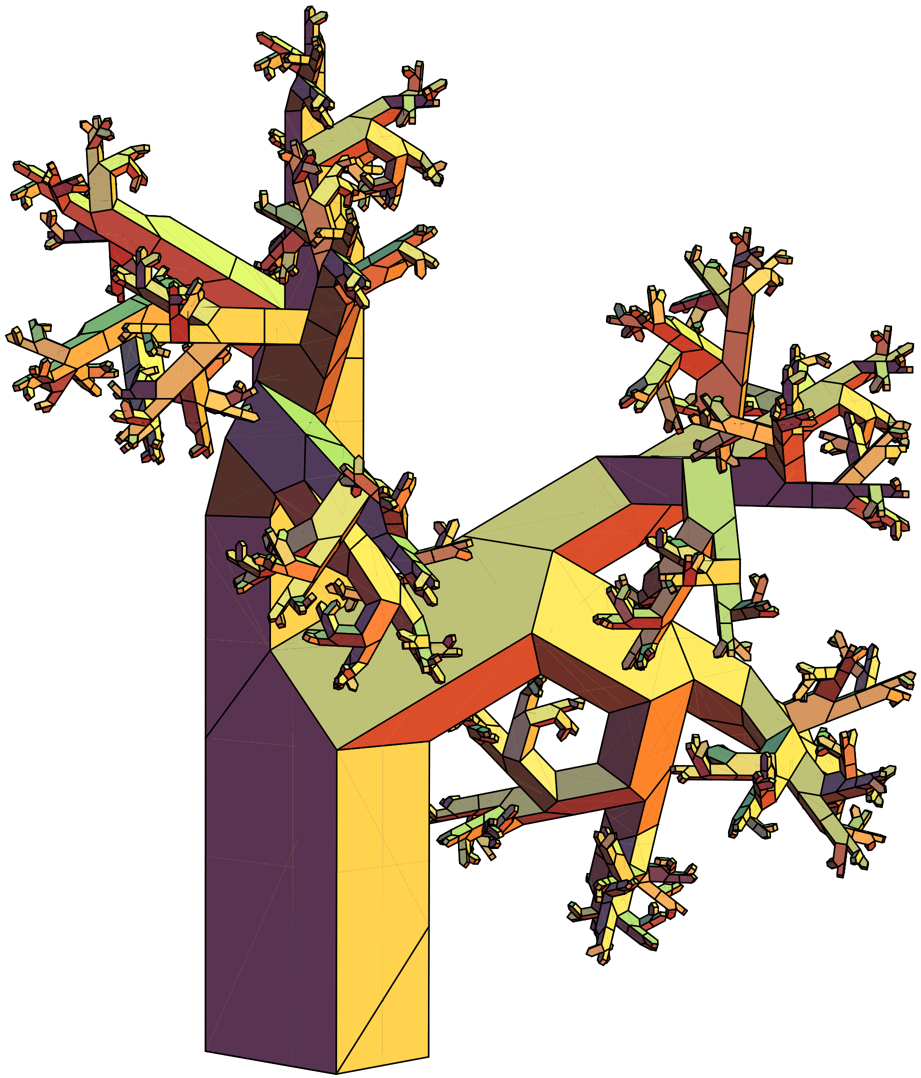
Vary the Growth Pattern: Side-flip – Front



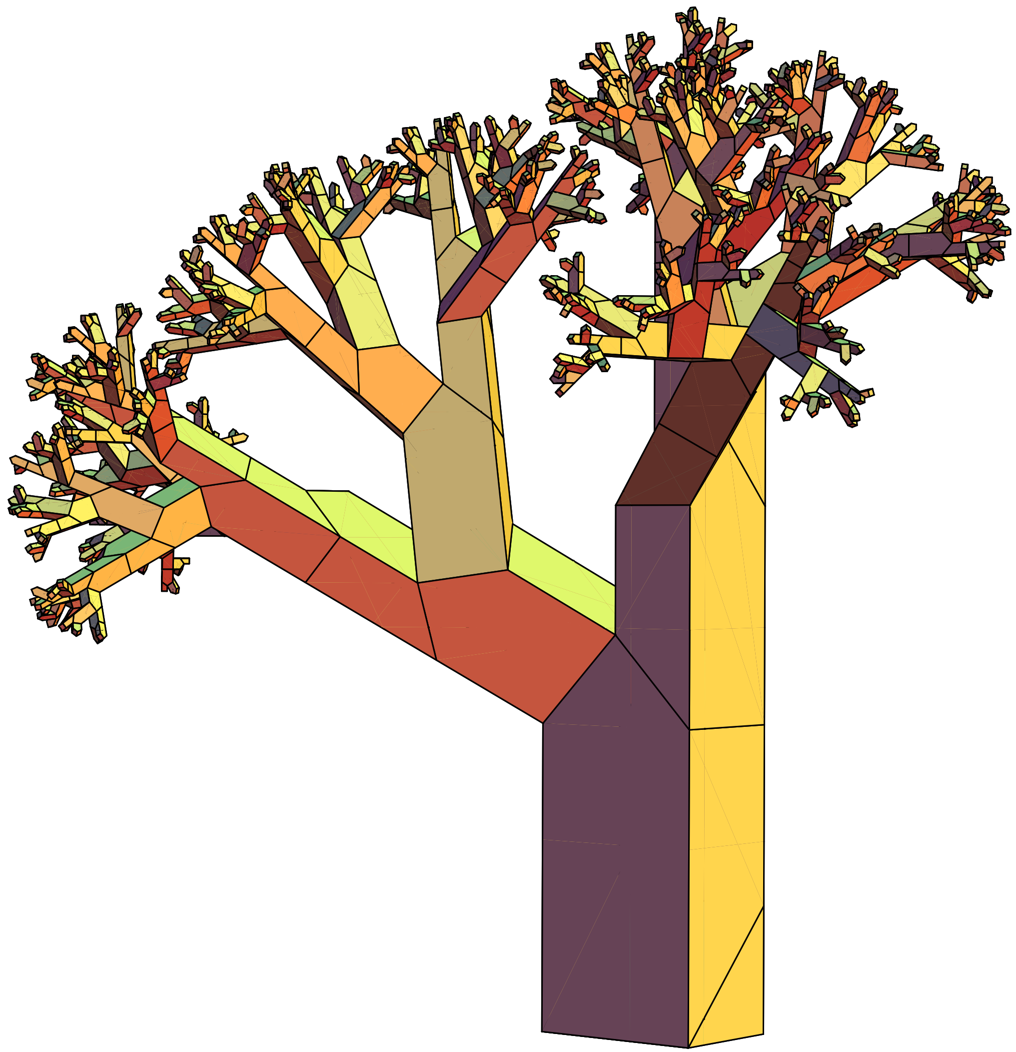
Vary the Growth Pattern: Back-flip – Front



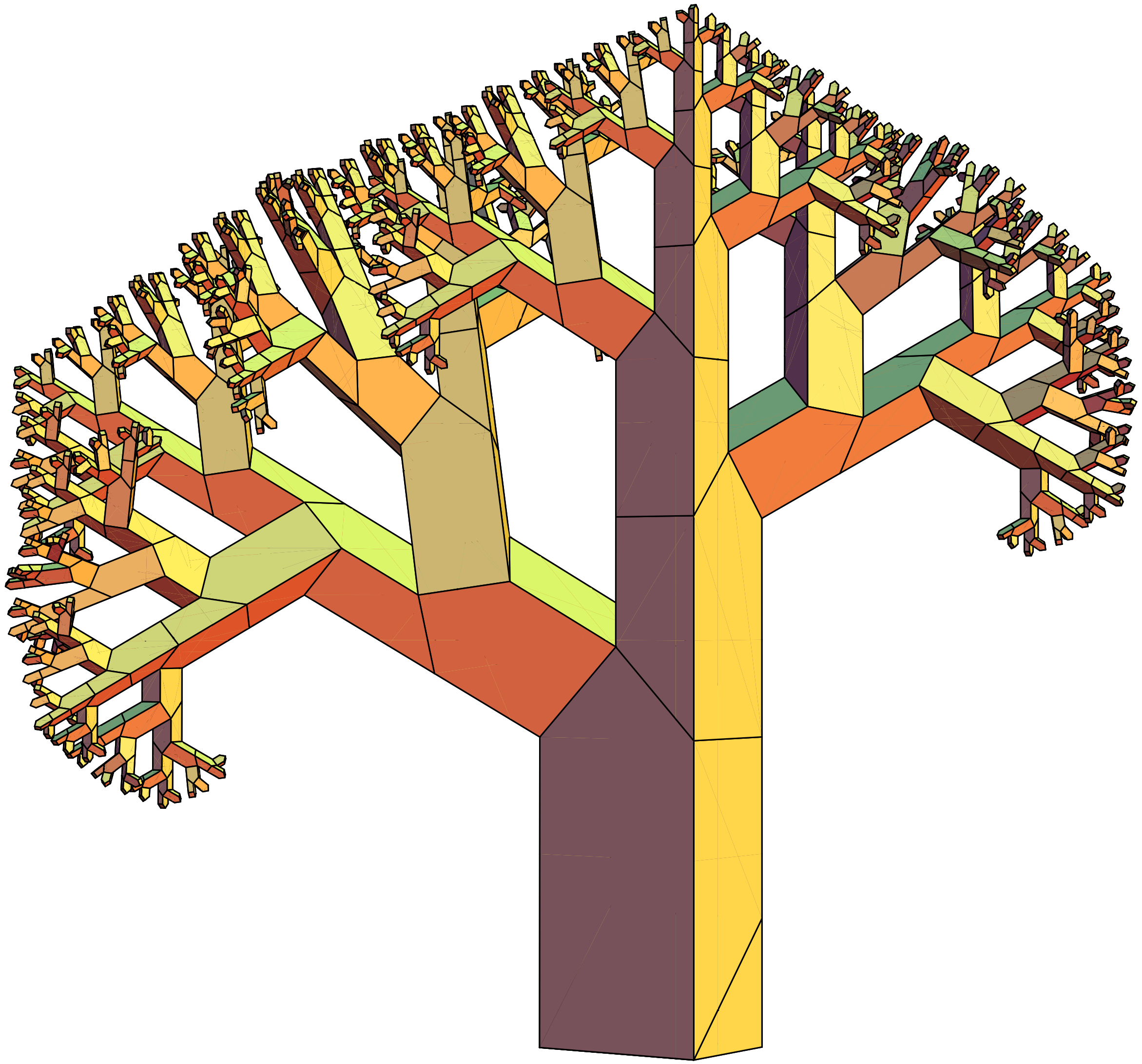
Vary the Growth Pattern: Back – Up



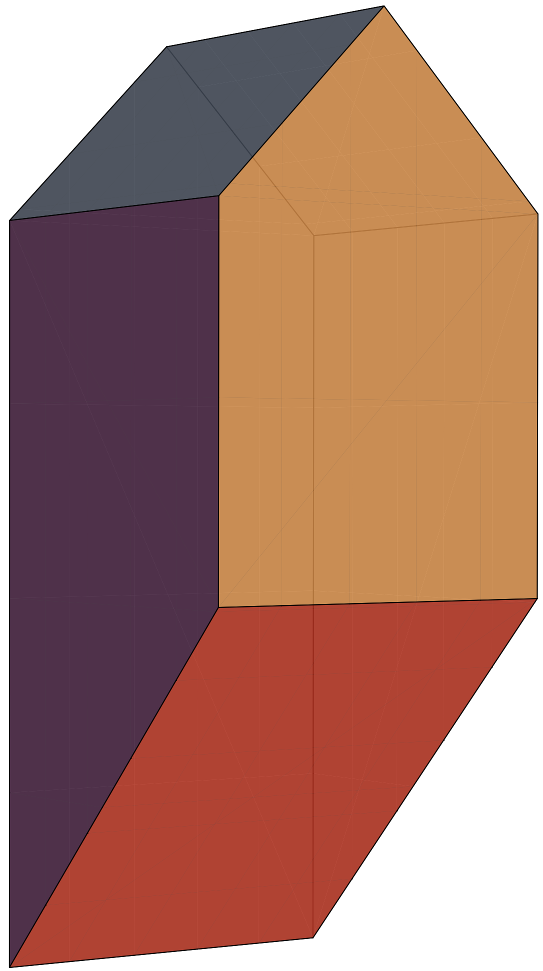
Vary the Growth Pattern: Up – Front



Vary the Growth Pattern: Up-flip – Front

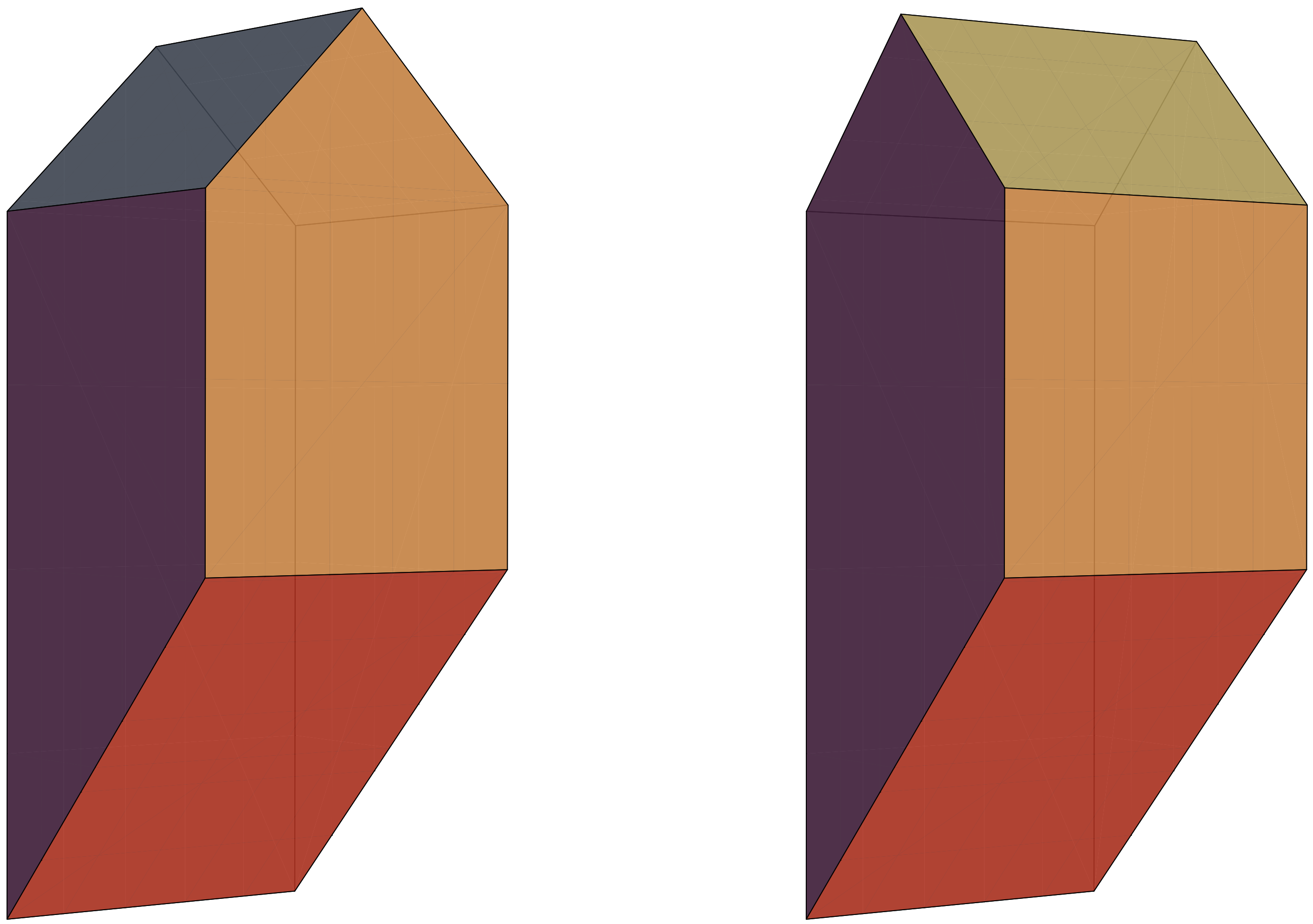


Tree Construction from Square Beams: The Piece

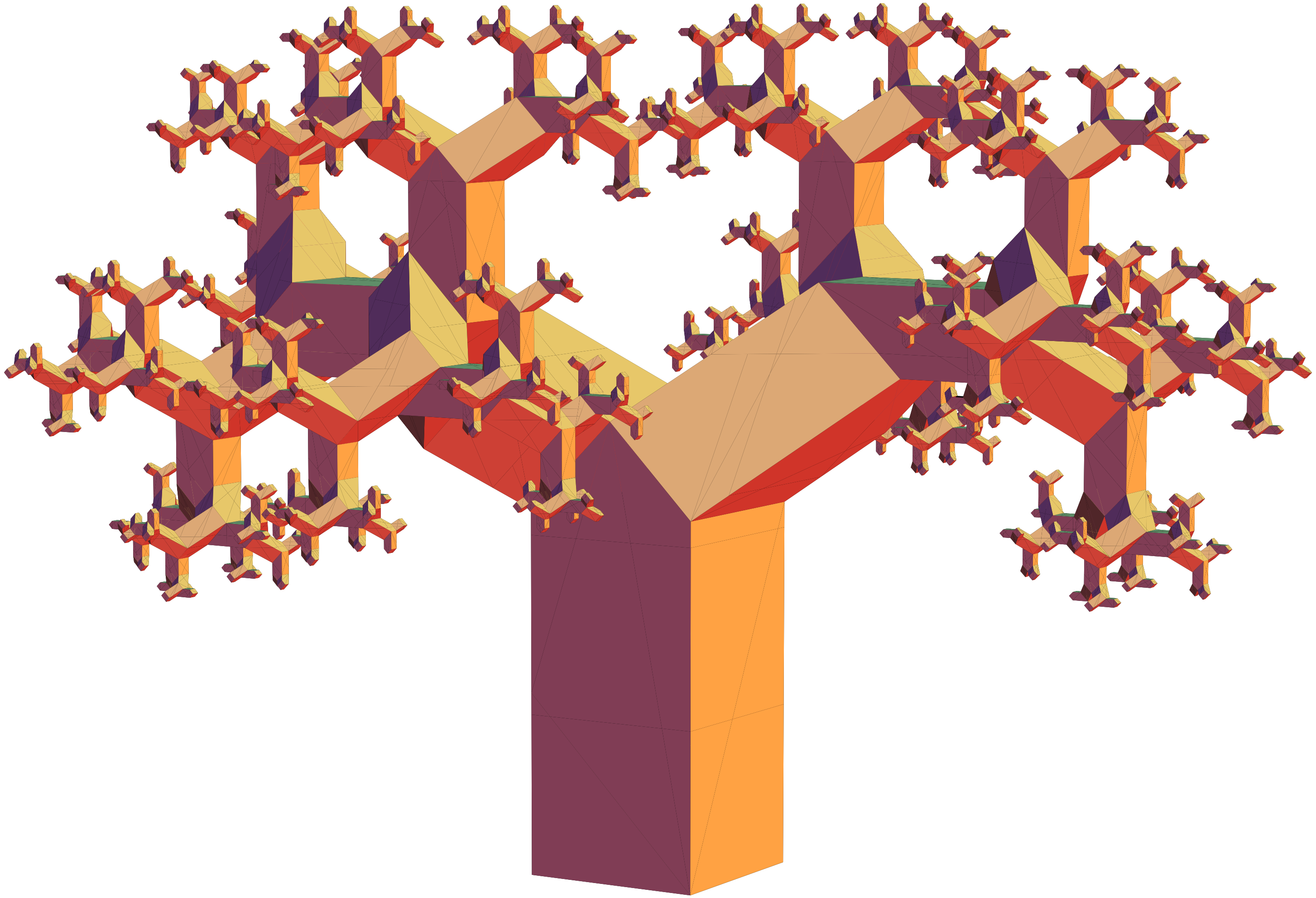


Tree Construction from Square Beams: Back – Front

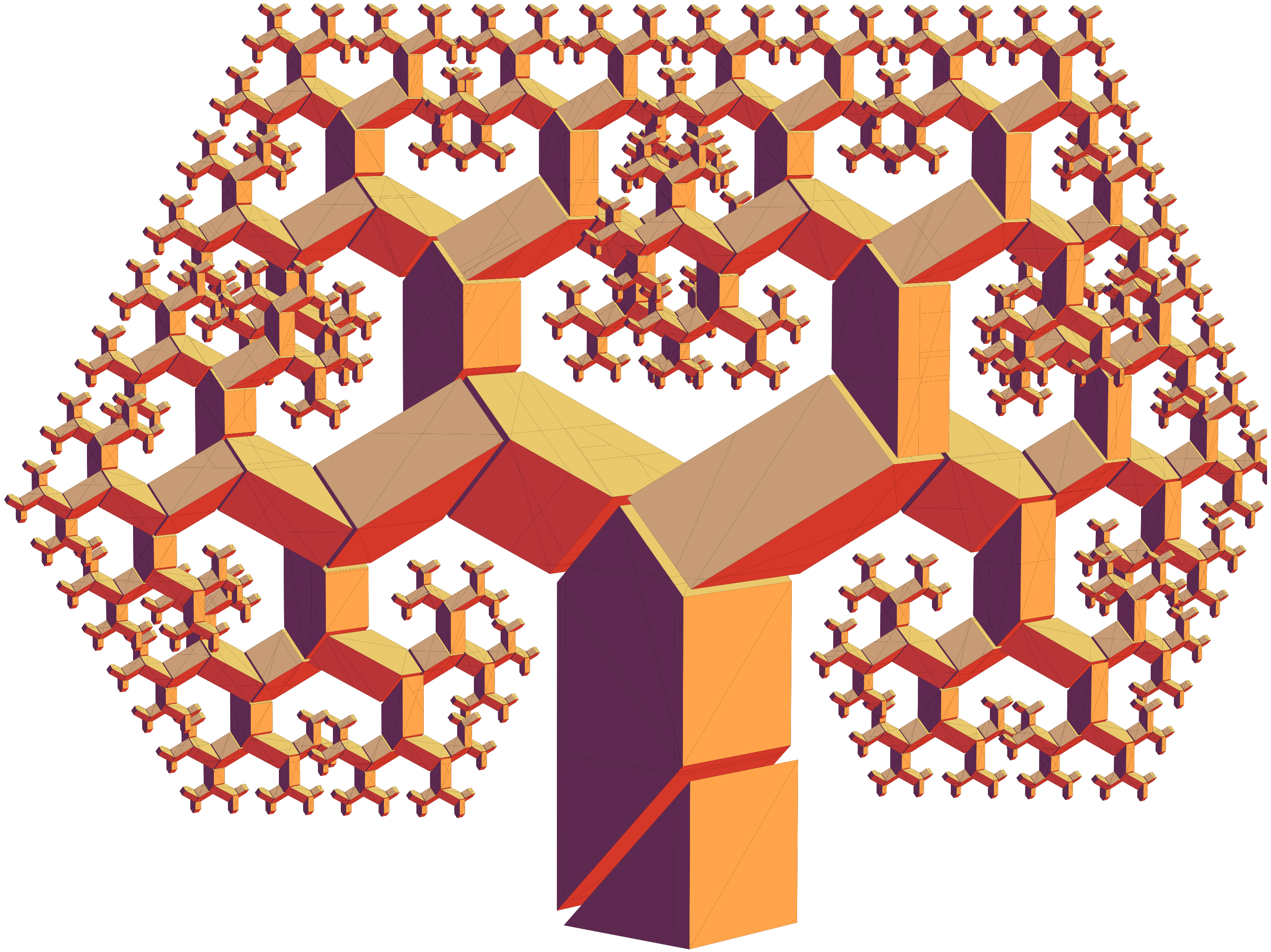
Tree Construction from Square Beams: Rotated-Roof Piece



Tree Construction from Square Beams: Back – Front



Tree Construction from Square Beams: Back-flip – Front-flip



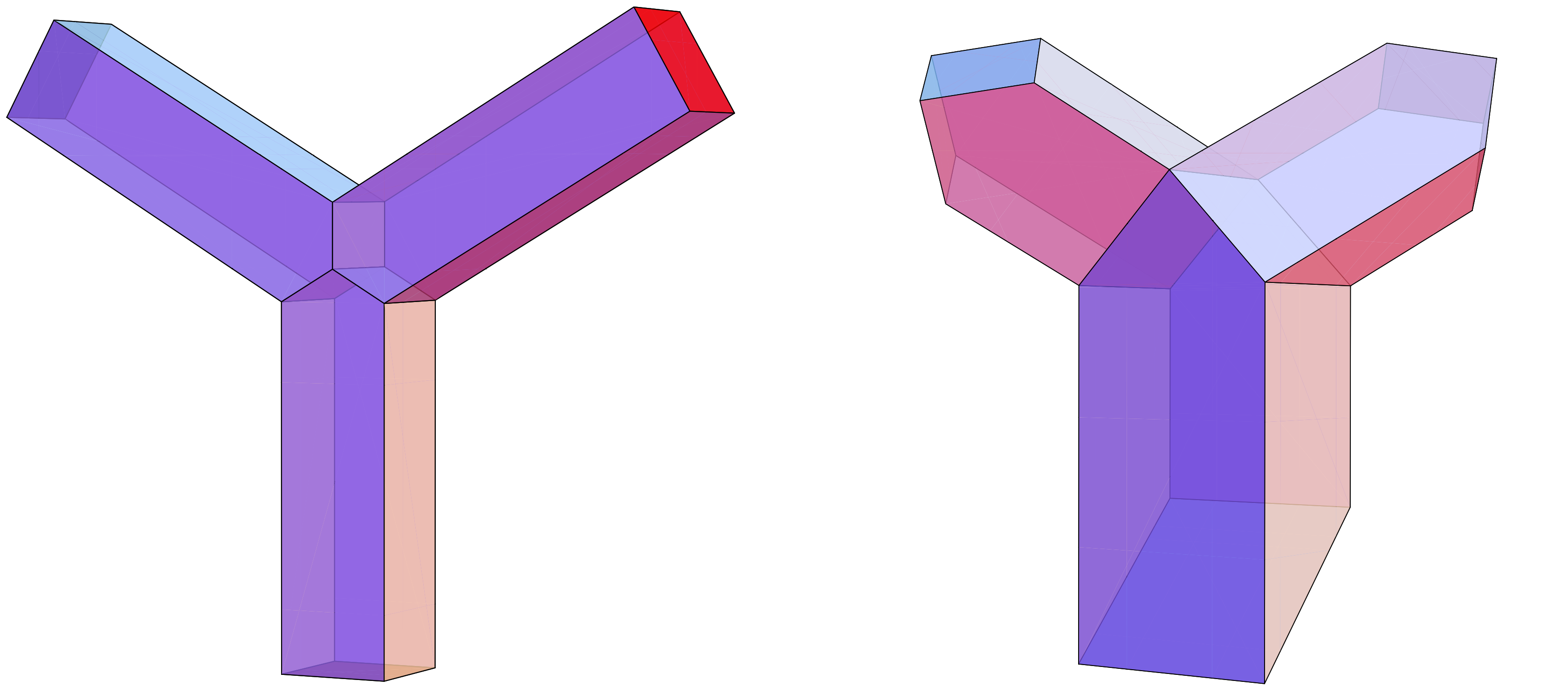
Constraints for General Binary Mitered Fractal Trees

- The trunk has a *polygonal* cross section.
- Each subtree is a *scaled-down* copy of the whole tree, possibly reflected.

  All branches have a *similar* polygonal cross section as the trunk.
- The longitudinal edges of the beams properly meet up at the three-way joints.
- Sibling branches share the roof's ridge,

  rather than a surface as in a *ternary miter joint* (cf. Bridges 2010)
- Three dimensional

Ternary Miter Joint versus Binary Tree Miter Joint



Variation Points

- Cross section: must be a *strip* (2-gon), *triangle*, or *quadrangle*
- Cut face: must be similar to roof panels

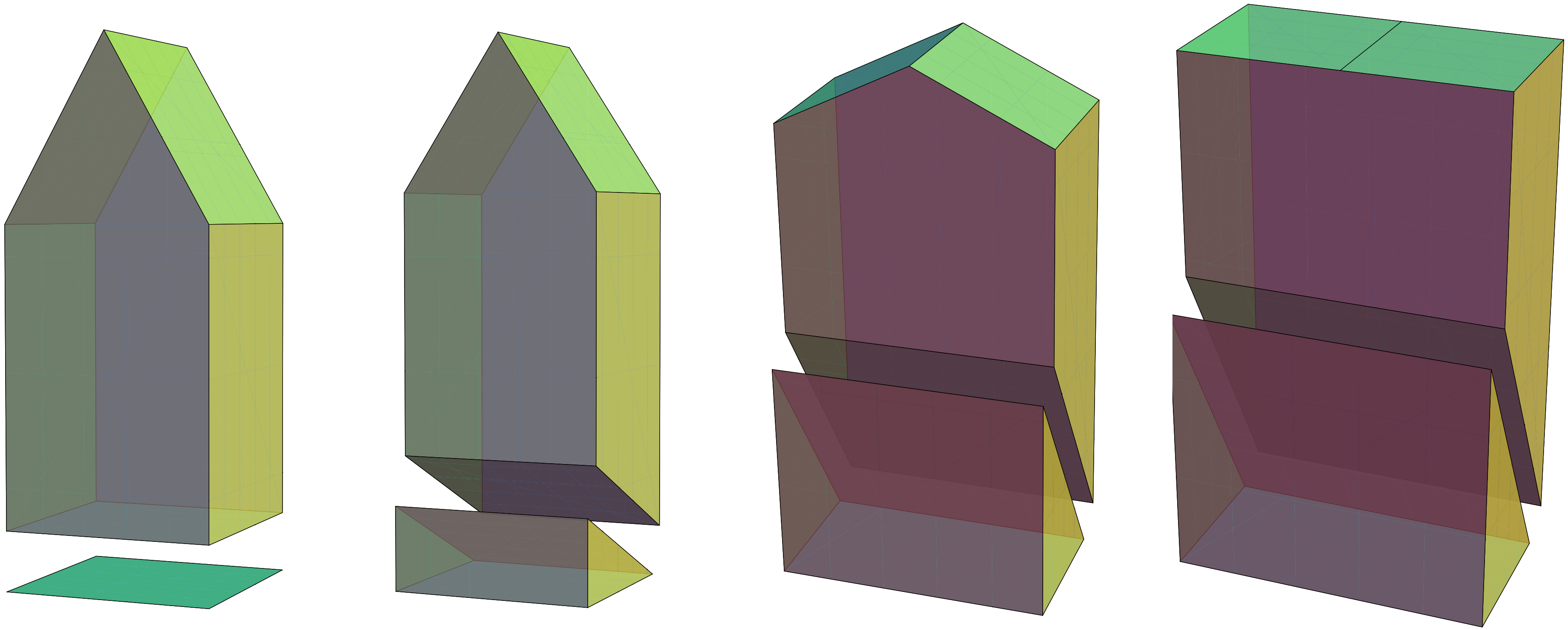
  Note: Symmetries of cut face determine growth options
- Orientation (angles) of the roof panels: could be asymmetric

We restrict ourselves to *rectangular* beams

Hence, cut face is *square*, or *rectangle*, or *rhombus*, or *parallelogram*

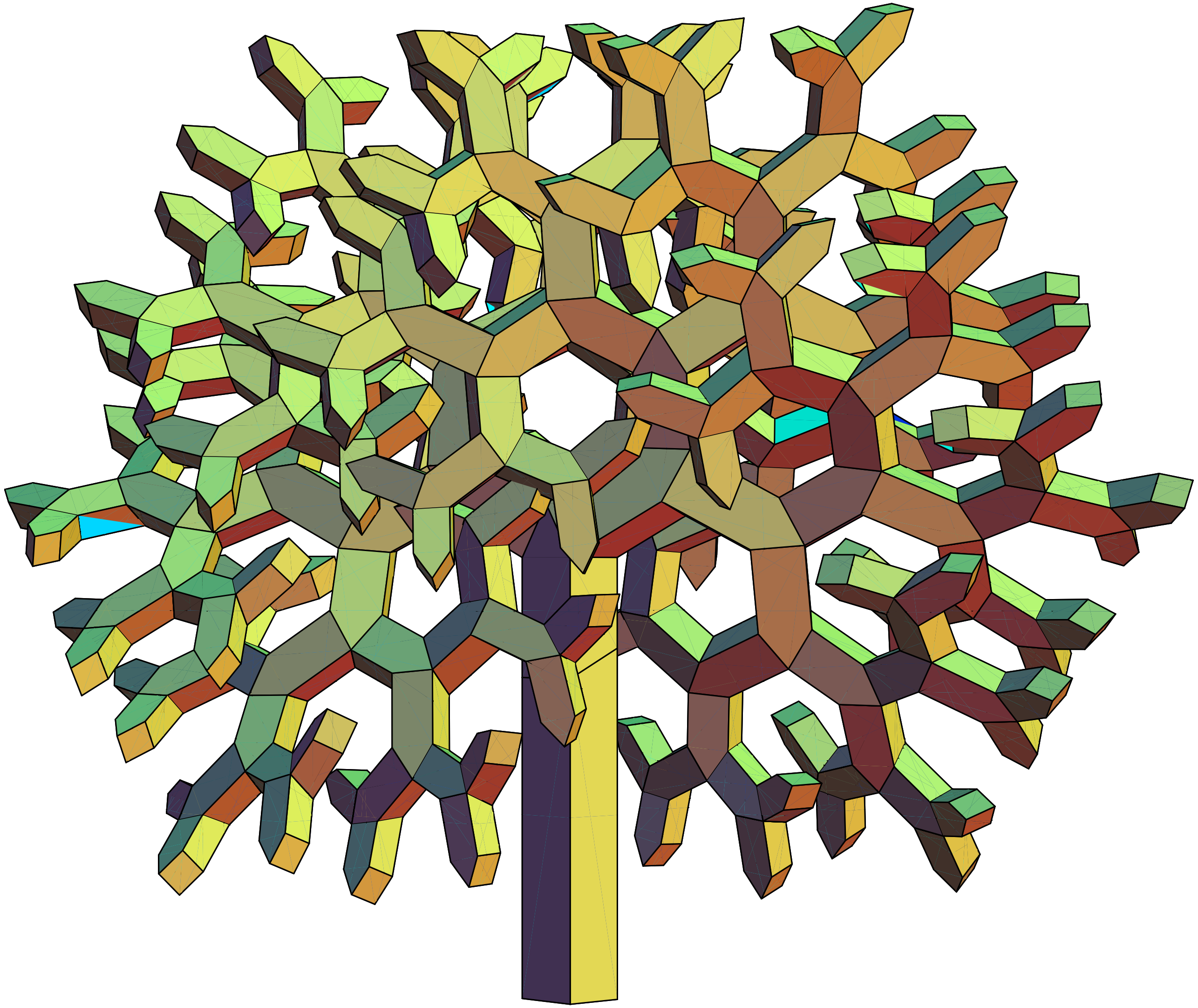
Tree Construction from $1 : a$ Rectangular Beams: The Piece

Square cut faces: symmetric roof; roof angle at ridge makes squares

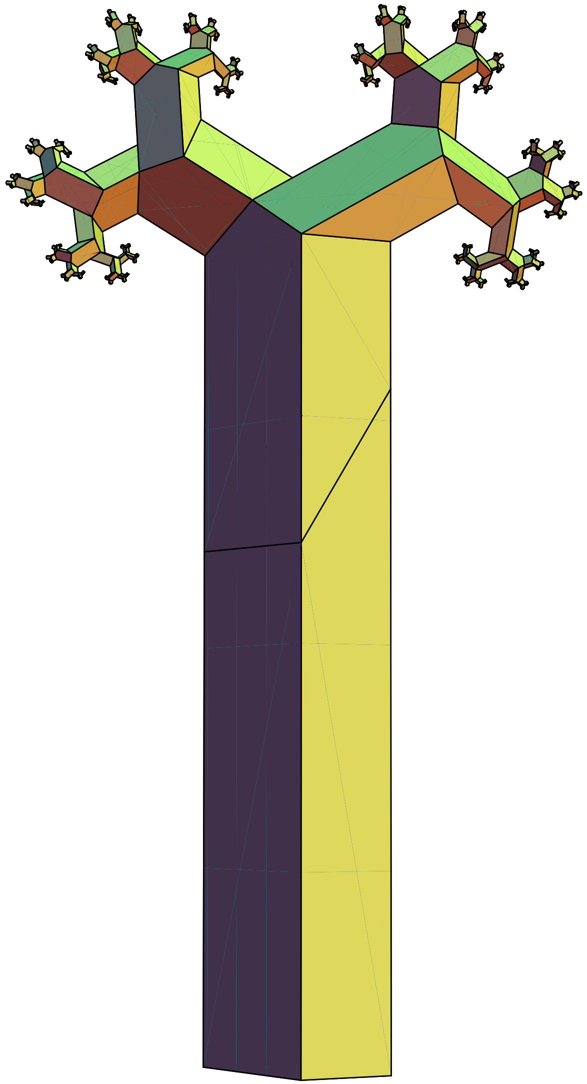


$a = 1$ $a = 1.1$ $a = 1.8$ $a = 2$

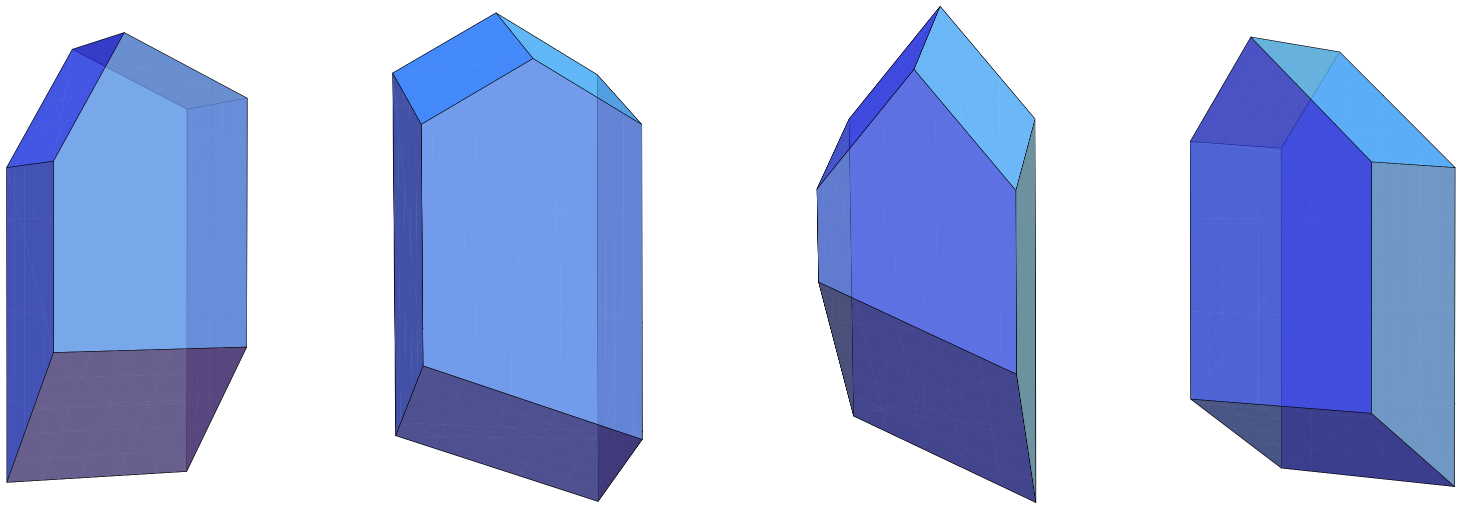
Tree Construction from $1:1.1$ Rectangular Beams



Tree Construction from $1 : 1.8$ Rectangular Beams

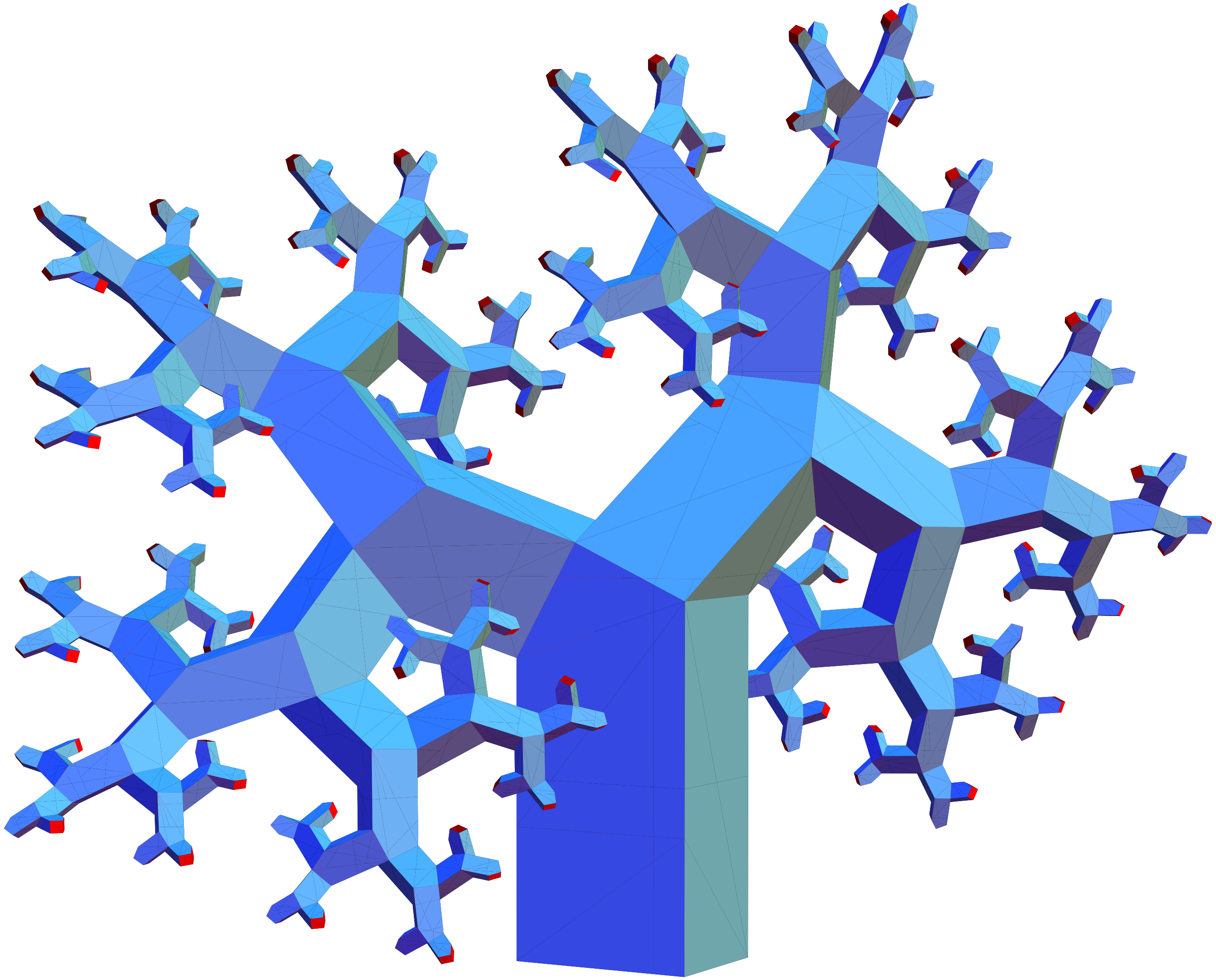


Tree Construction from $1 : \sqrt{2}$ Rectangular Beams: Variants

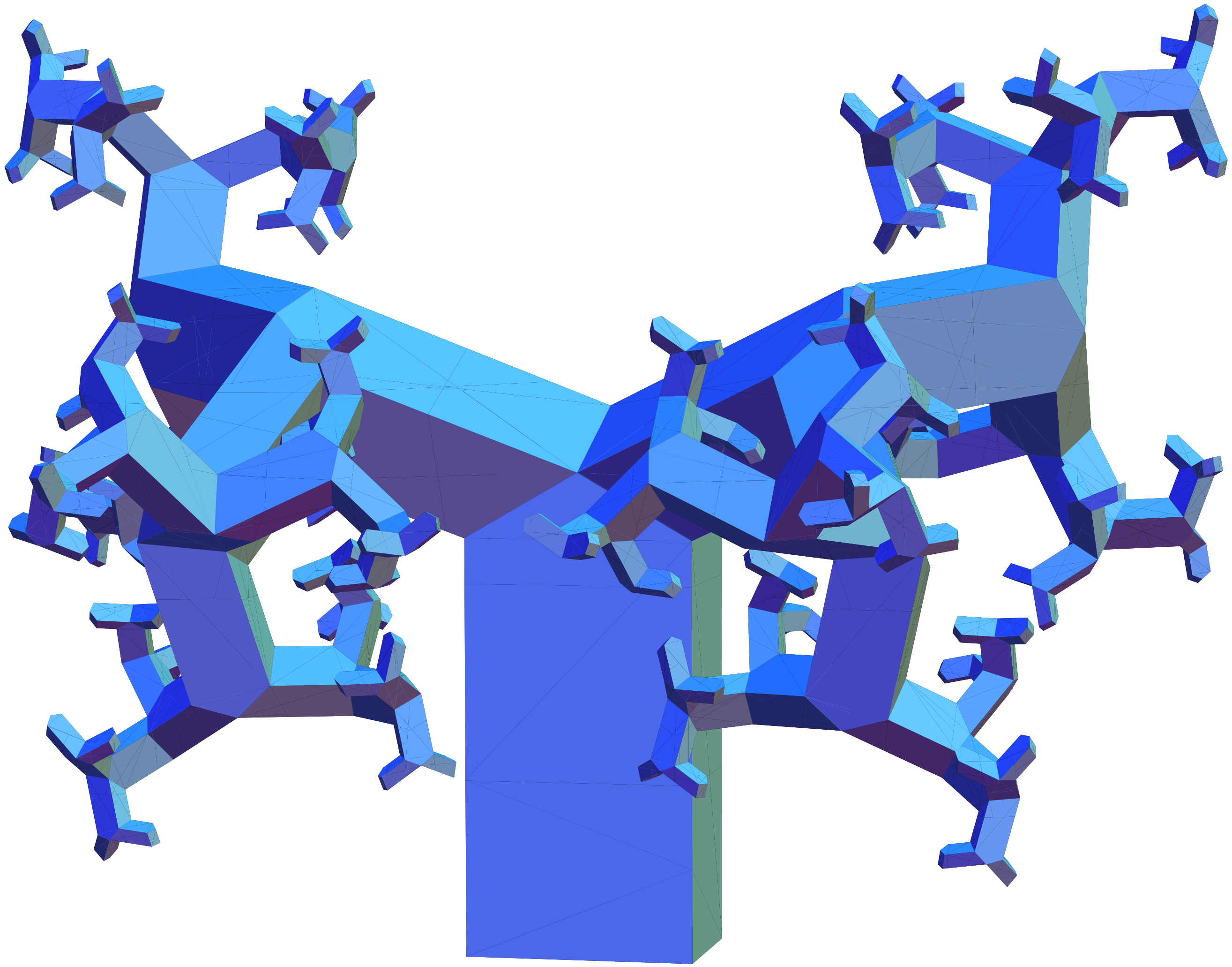


cut	square	parallelogram	rhombus	rectangle
ridge	horizontal	slanted	slanted	horizontal
roof	asymmetric	symmetric	symmetric	incongruent

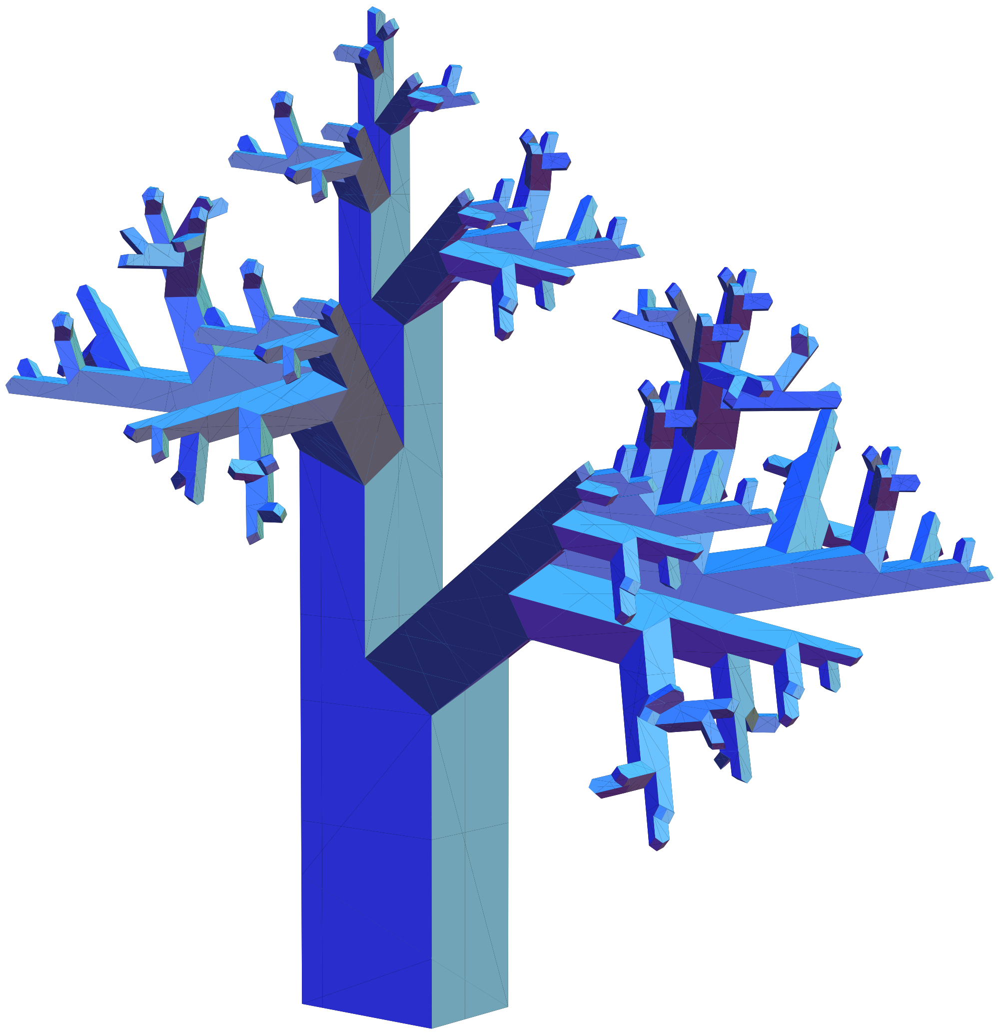
Squares as Cuts, Horizontal Ridge, Asymmetric Roof



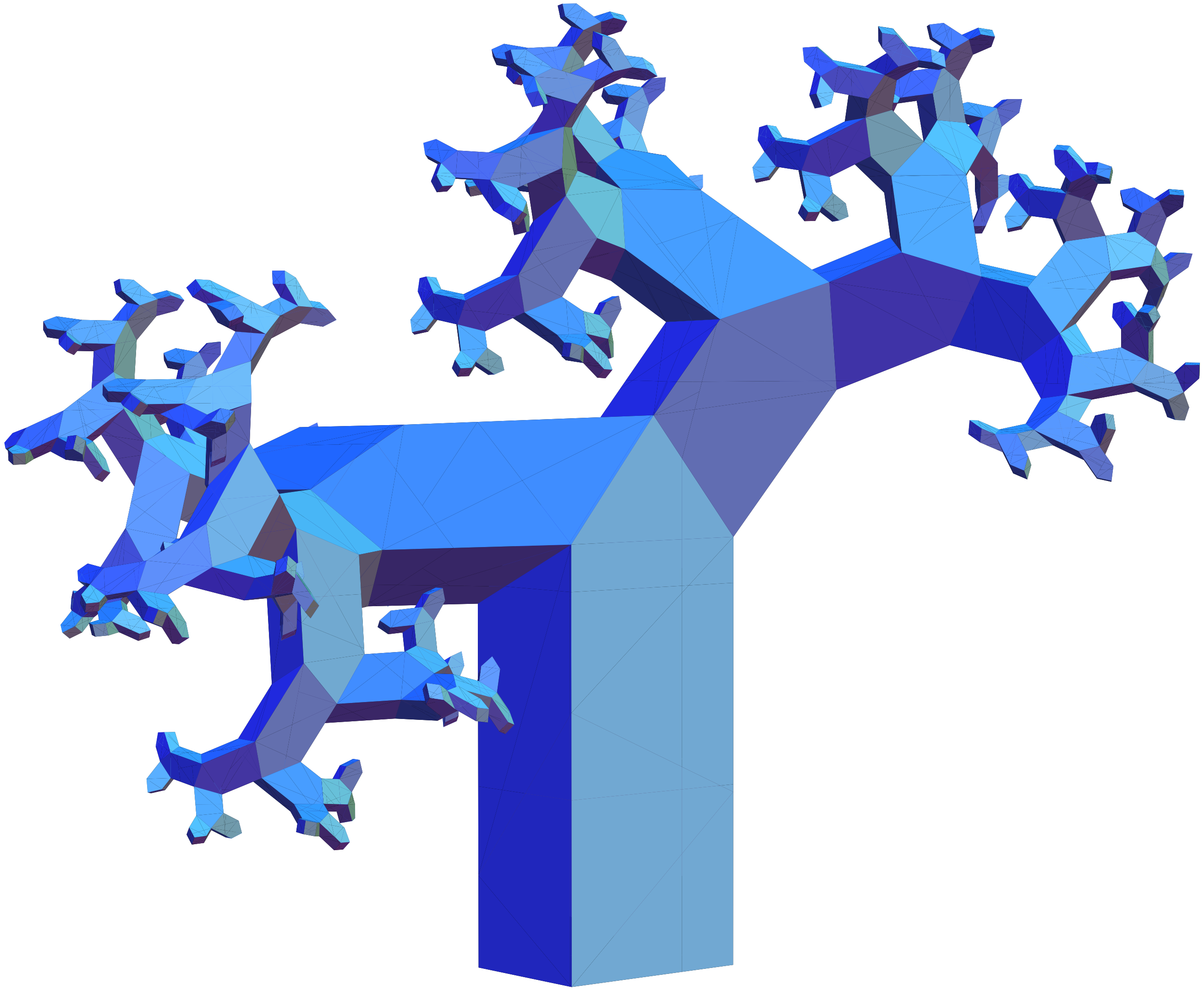
Parallelograms as Cuts, Slanted Ridge, Symmetric Roof (1)



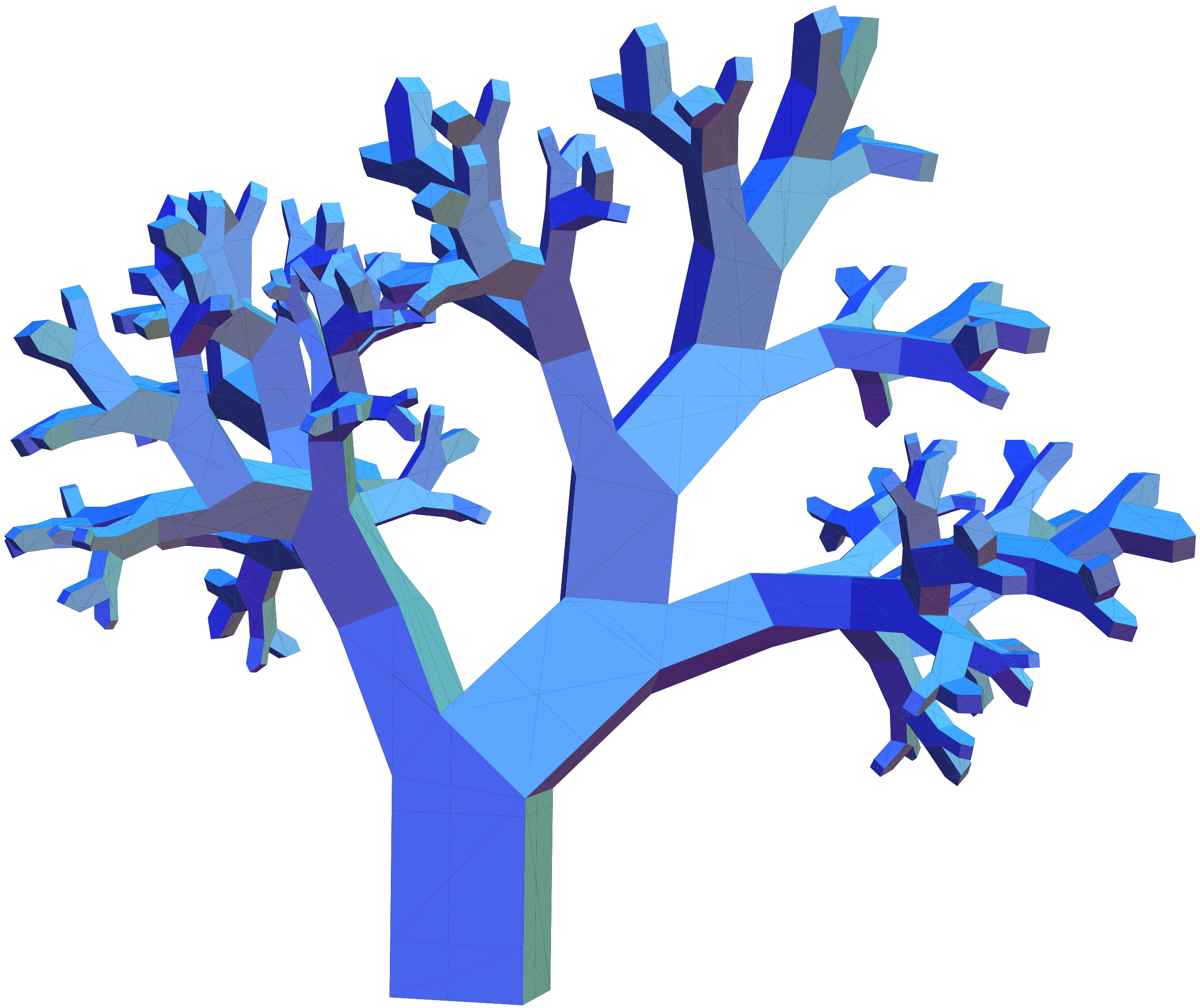
Parallelograms as Cuts, Slanted Ridge, Symmetric Roof (2)



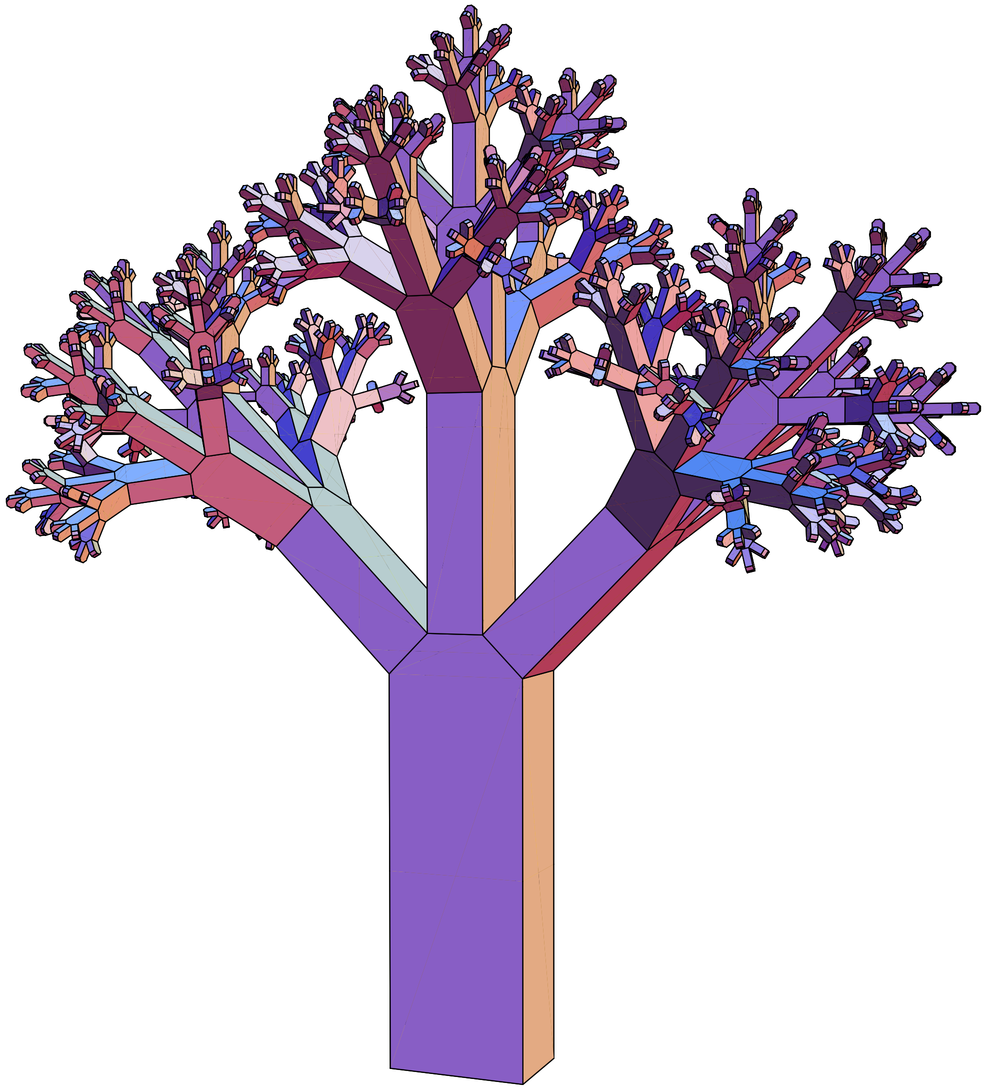
Rhombi as Cuts, Slanted Ridge, Symmetric Roof



Rectangles as Cuts, Horizontal Ridge, Asymmetric Scaling



Ternary Mitered Fractal Trees



Properties of Mitered Fractal Trees

- Fractal dimension: $D = \dfrac{\log N}{\log f}$

  where N = arity, and f = scale-down factor

  $D > 3$ implies self-intersection (for large number of generations)

- Self-intersection is hard to determine

- Symmetries: rotational, reflectional

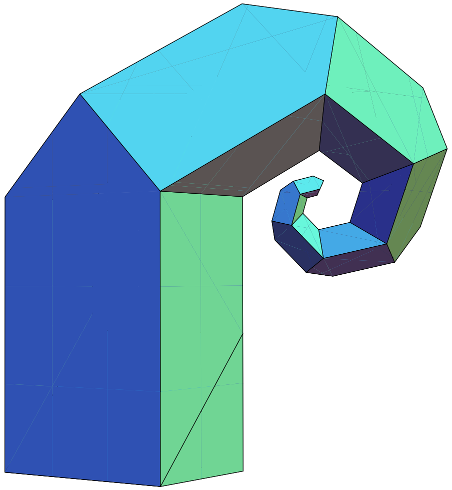
- Branch directions: repetitive or not

Branch Directions

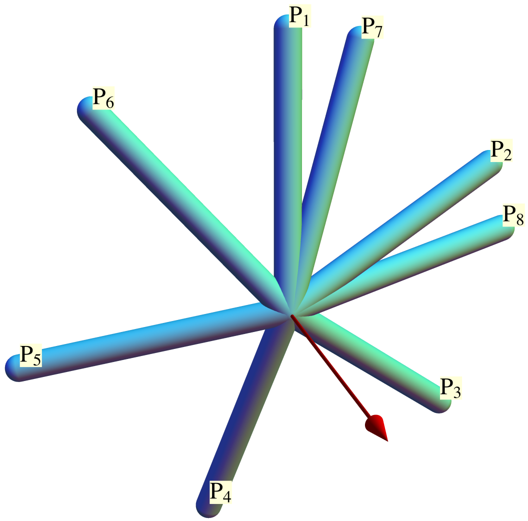
Analyze by *Turtle Geometry*: consider turn angle ϕ and roll angle ψ.

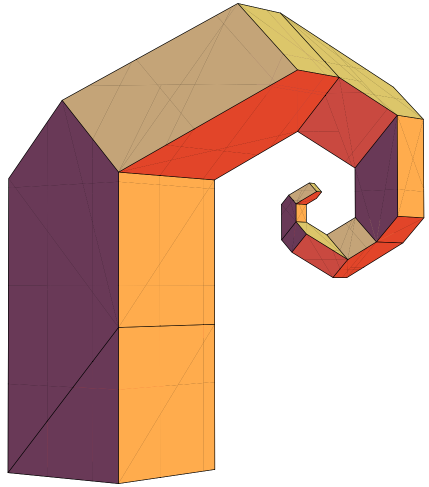
Branches on cone. Projected turn angle θ satisfies (Bridges 2011):

$$\cos(\theta/2) = \cos(\phi/2)\cos(\psi/2)$$



$\phi, \psi = 60°, 19.4712\cdots°$
$\theta = 62.7994\cdots°$





$\phi, \psi = 60°, 0$
$\theta = 60°$

Leonardo's Rule

Leonardo da Vinci writes in item 394 of his Notebook, Vol. 1:

> "All the branches of a [natural] tree at every stage of its height when put together are equal in thickness to the trunk"

Eloy (2011) rephrased this as:

> "the *total cross section of branches* is conserved across branching nodes".

Eloy proposes the theory that this property evolved to help trees withstand *wind-induced stresses*.

Conclusion

- Explained the two earliest binary mitered fractal tree designs

- Explored various generalizations

To do:

1. General quadrangle as cross section

2. Ternary and higher

3. Grow trees with ‘deeper’ patterns, or randomly

4. Sibling branches that share a cut surface (cf. ternary miter joint)

Related Work

- Tom Verhoeff & Koos Verhoeff
  "The Mathematics of Mitering and Its Artful Application"
  Bridges 2008, Leeuwarden, Netherlands, pp.225–234

- Tom Verhoeff & Koos Verhoeff
  "Branching Miter Joints: Principles and Artwork"
  Bridges 2010, Pécs, Hungary, pp.27–34

- Tom Verhoeff & Koos Verhoeff
  "From Chain-link Fence to Space-spanning Helical Structures"
  Bridges 2011, Coimbra, Portugal, pp.73–80

Also see: `http://www.win.tue.nl/~wstomv/publications/`