Folded Strips of Rhombuses

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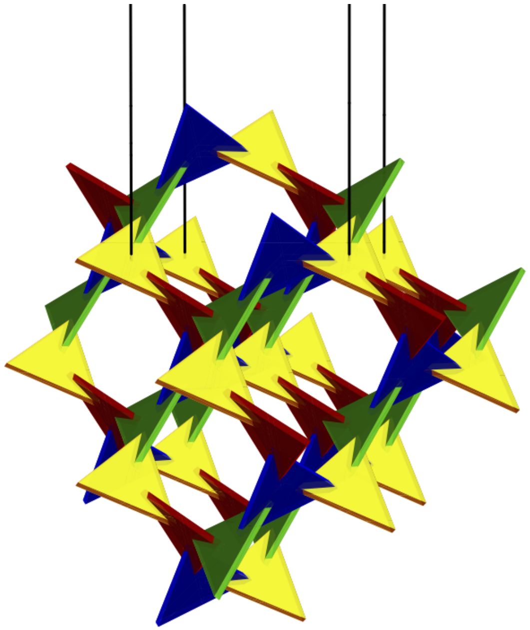
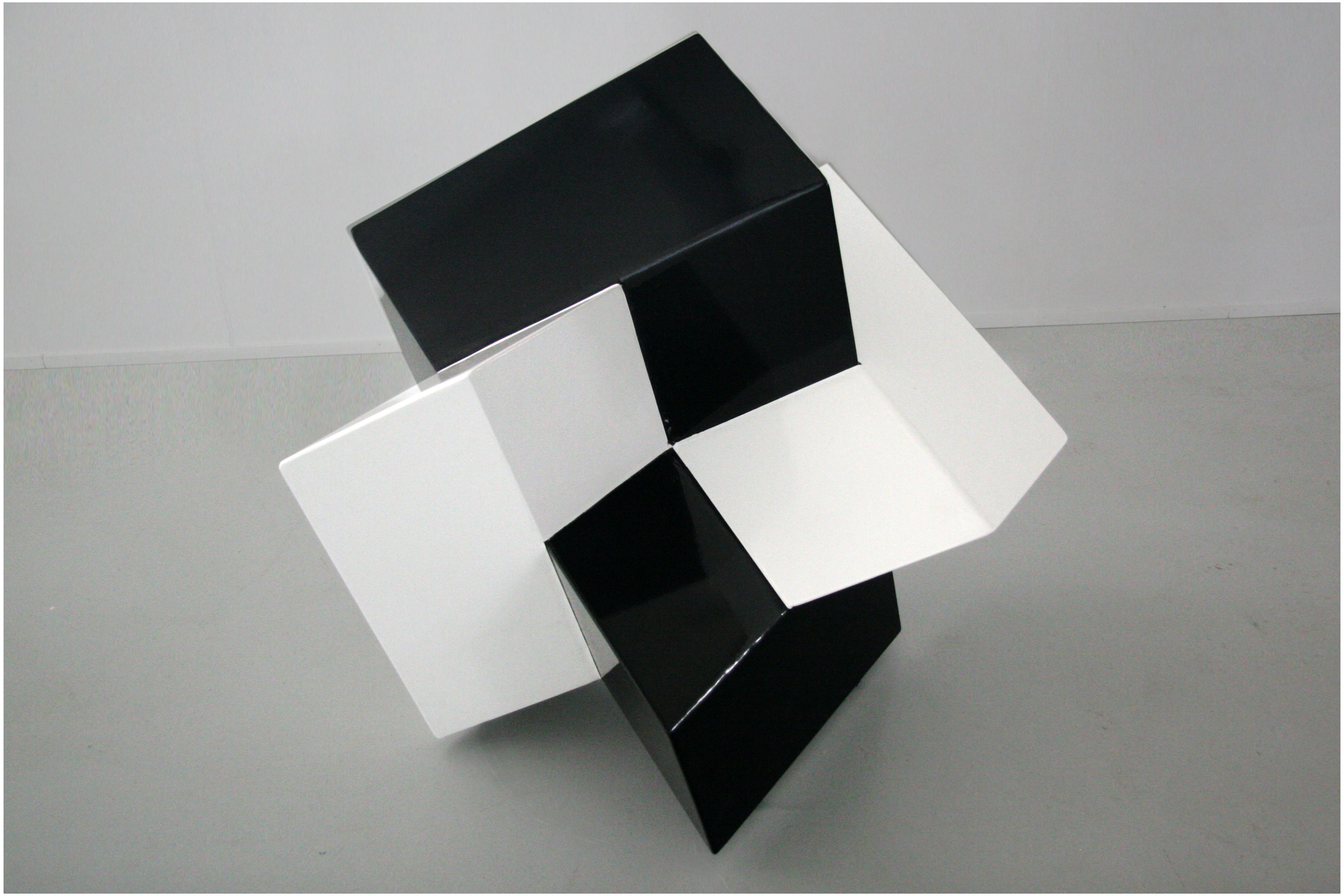


Koos Verhoeff Art Exhibition in Hengelo

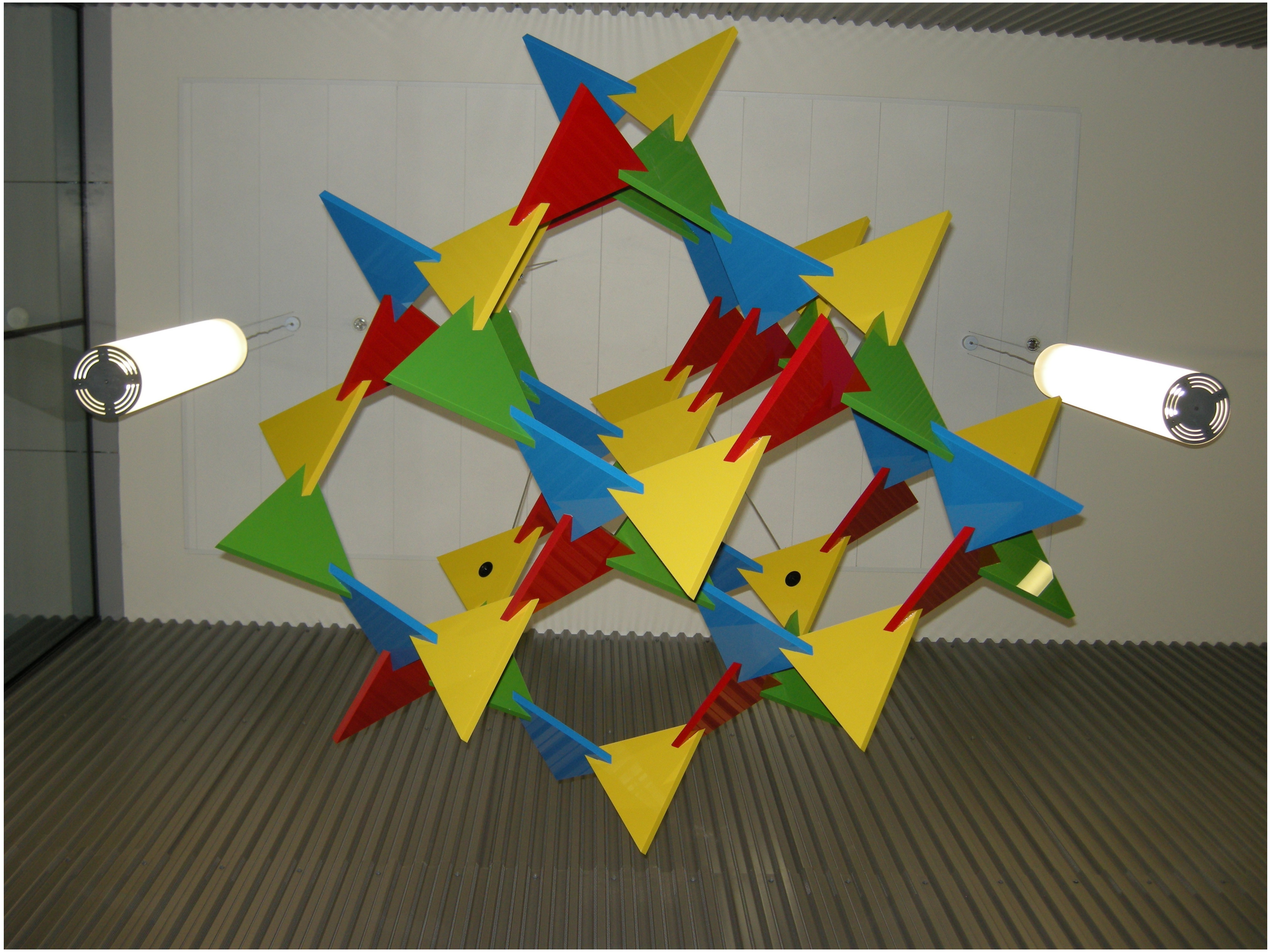
- Option 2 on Excursion Day: *A Lovely Place*
- About 150 objects on display

Mitered Trefoil Knot, Corten Steel (2013)

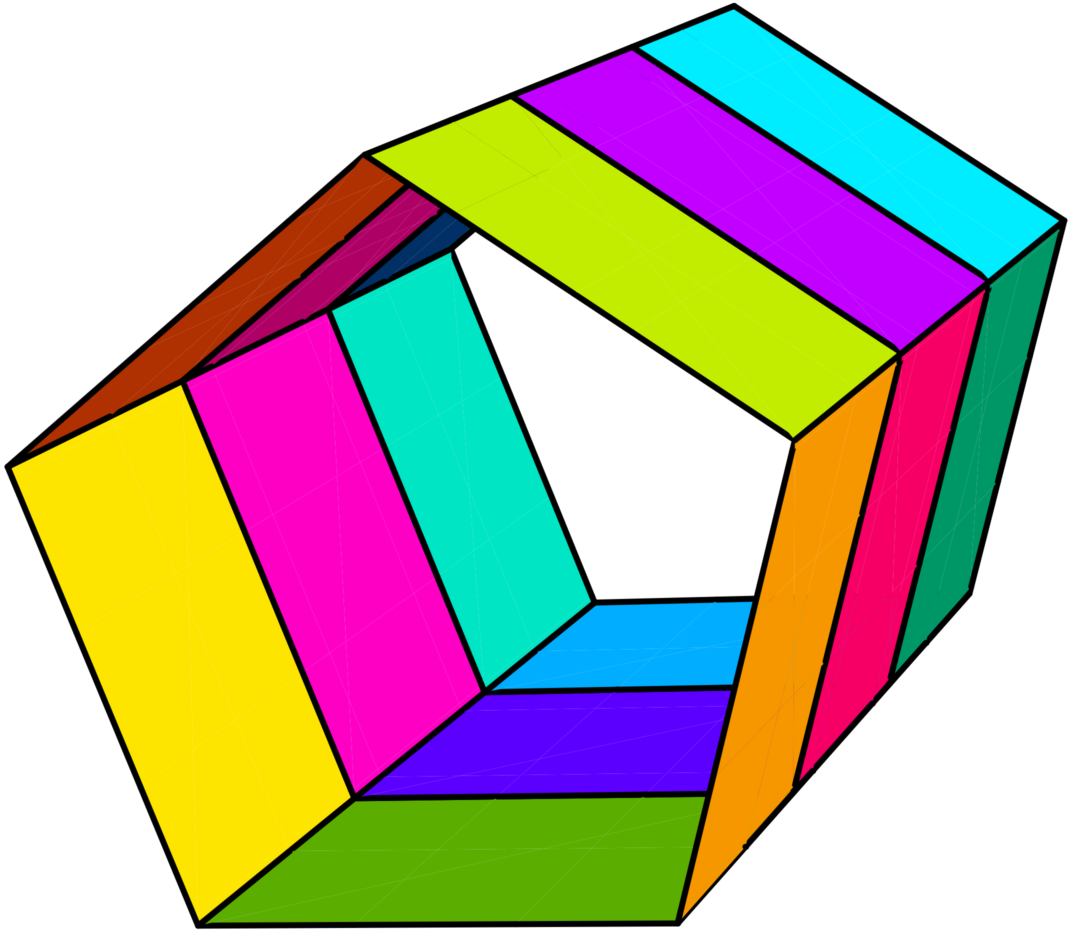
Pair of Linked Octagons, Powder-Coated Corten Steel (2013)



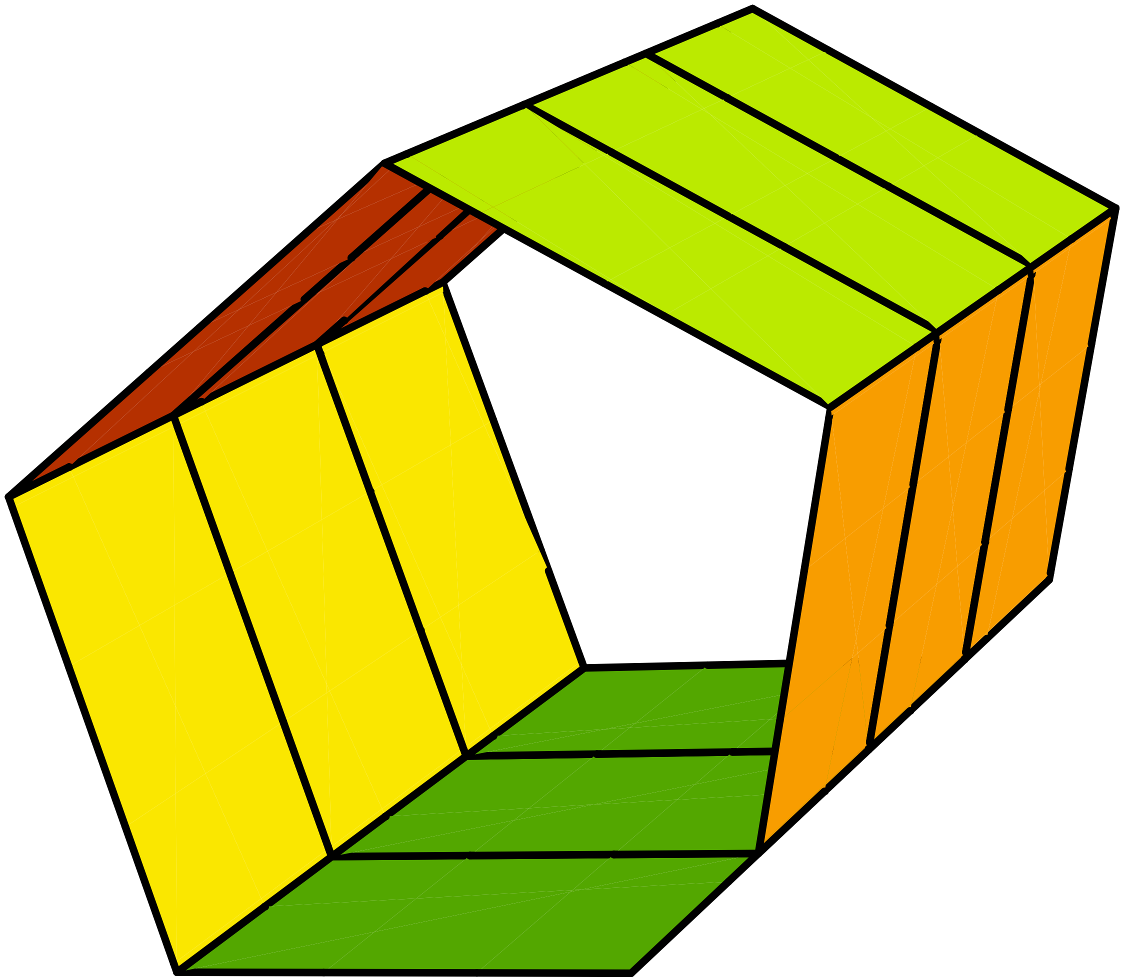
Bamboozle, Polished Acrylic (Installed January 2013)



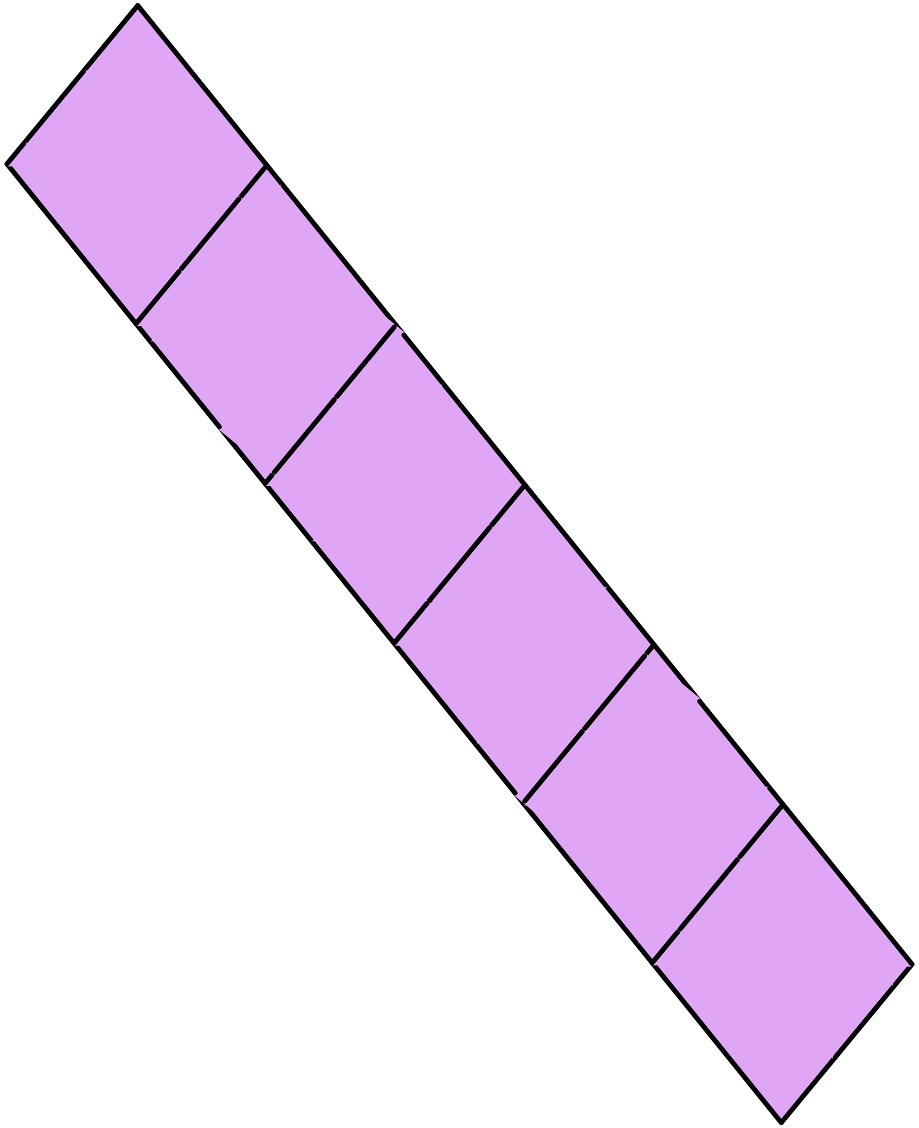
Constructing Polygonal Tubes: The “Standard” Way



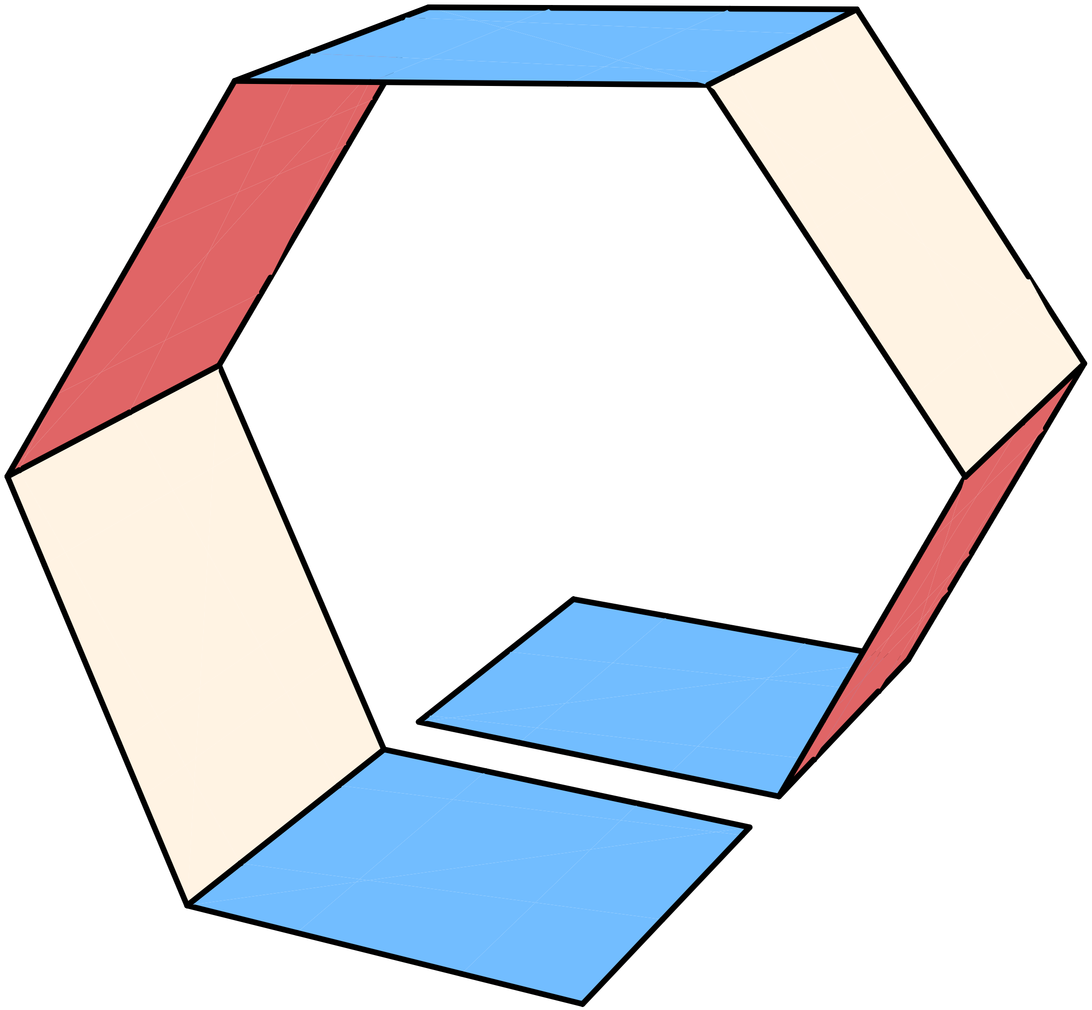
Constructing Polygonal Tubes: Folded Strip of Rhombuses



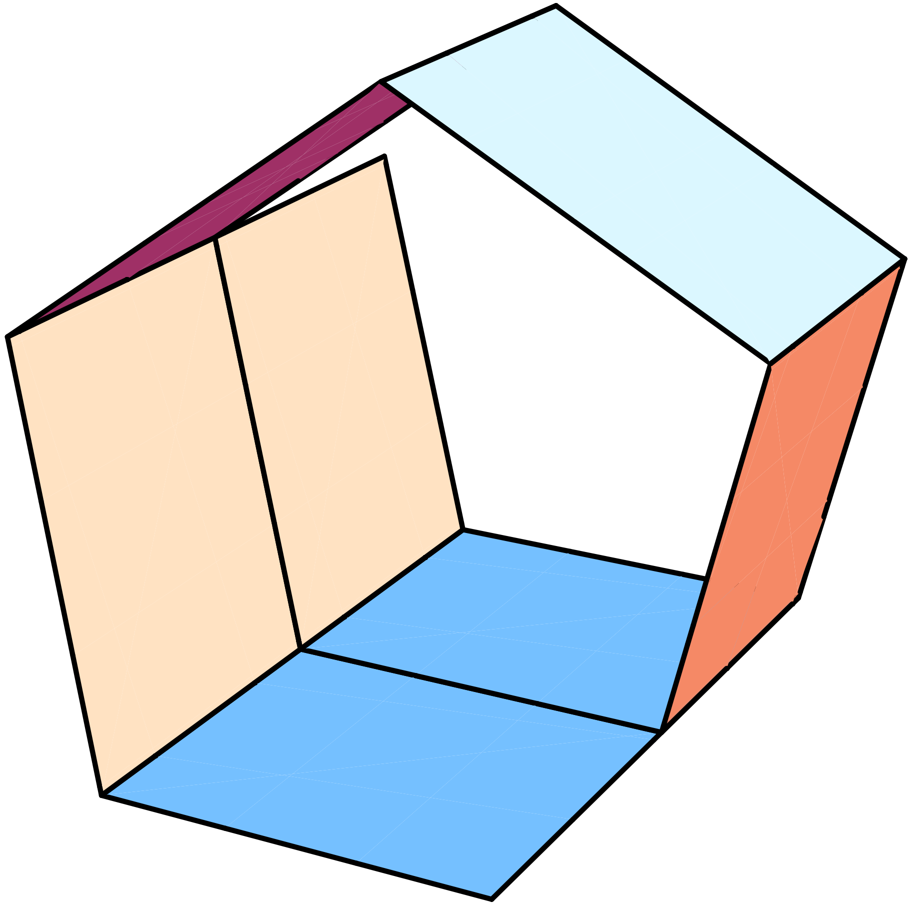
Strip of Rhombuses



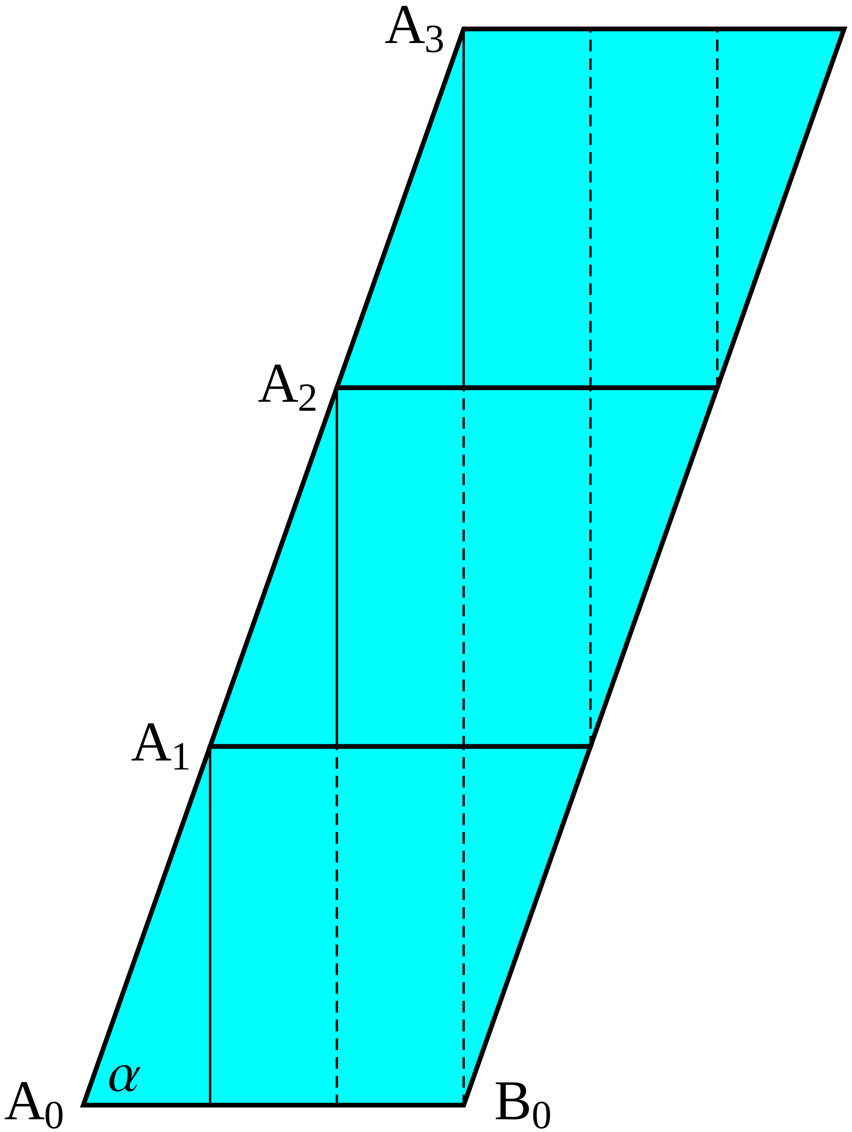
Strip of Rhombuses Loosely Folded into a Discrete Helix



Strip of Rhombuses Tightly Folded into a Tube



Strip of Rhombuses: Criteria for Tightness



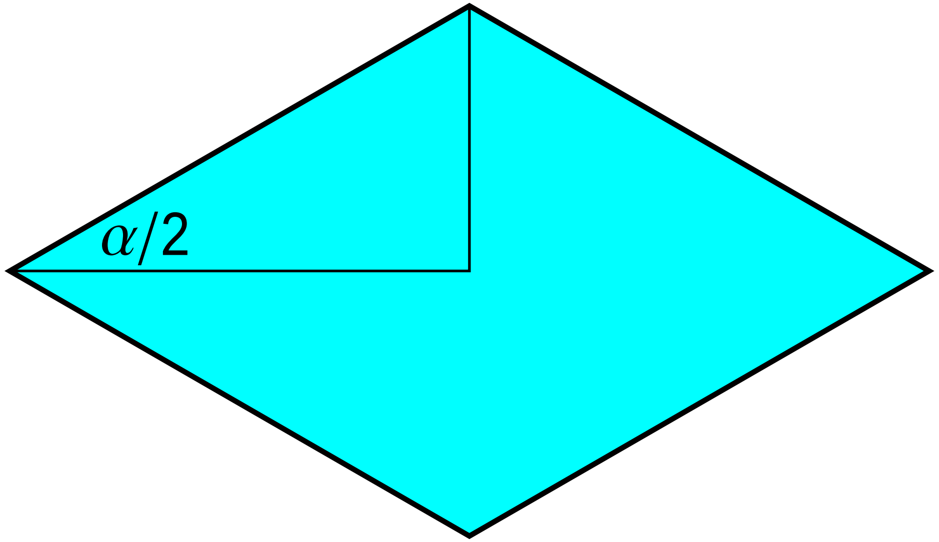
Tight: A_n folds to B_0

Cross section is n-gon

Acute angle of rhombus $= \alpha$

$$\cos \alpha = \frac{1}{n}$$

Aspect Ratio of Rhombus



$$\text{Aspect ratio} = a = 1 : \tan\frac{\alpha}{2} = \cot\frac{\alpha}{2} : 1$$

Interesting Rombuses

$$a \;=\; \cot\left(\frac{1}{2}\arccos\frac{1}{n}\right) \;=\; \sqrt{\frac{n+1}{n-1}}$$

n	3	4	5	6	7	8
α	70.53°	75.52°	78.46°	80.41°	81.79°	82.82°
a	$\sqrt{2}$	$\sqrt{\frac{5}{3}}$	$\sqrt{\frac{3}{2}}$	$\sqrt{\frac{7}{5}}$	$\frac{2}{\sqrt{3}}$	$\frac{3}{\sqrt{7}}$
	1.41421	1.29099	1.22474	1.18322	1.1547	1.13389

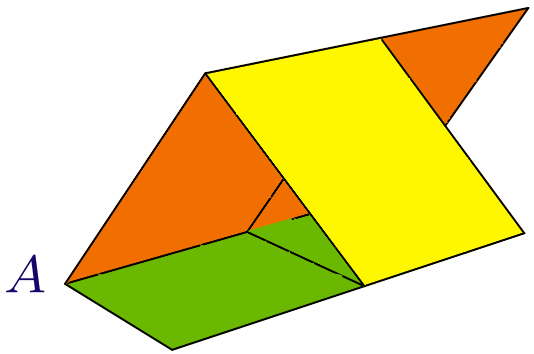
Polydron™ offers Golden Rhombus: $a = \Phi = \frac{1}{2} + \frac{1}{2}\sqrt{5} \approx 1.61803$

Polydron™ used to offer $\sqrt{2}:1$ Rhombus (discontinued)

Joining Two Beams Constructed from Rombuses

- 180° joint is not interesting

- Other joint angles require angle at A to fit a rhombus

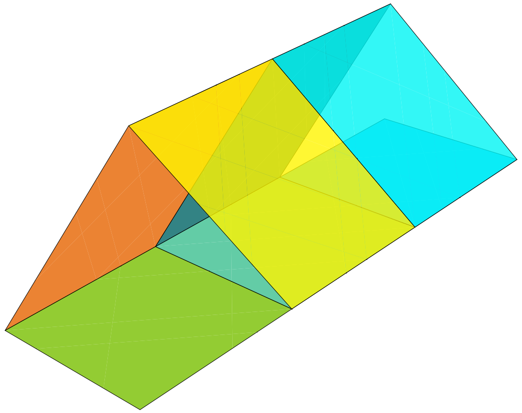


Only works for $\sqrt{2} : 1$ rhombus

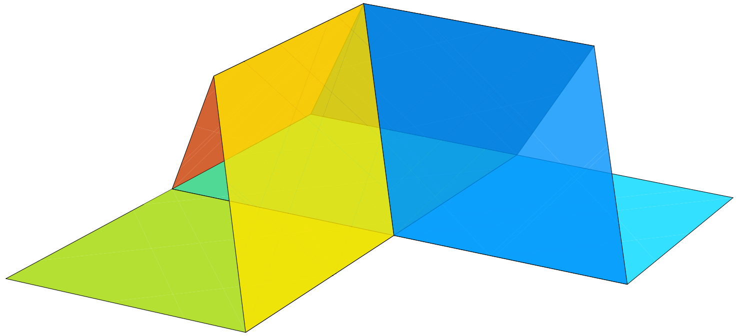
- Hence, we restrict ourselves to this rhombus

Joining Triangular Beams from $\sqrt{2} : 1$ Rombuses

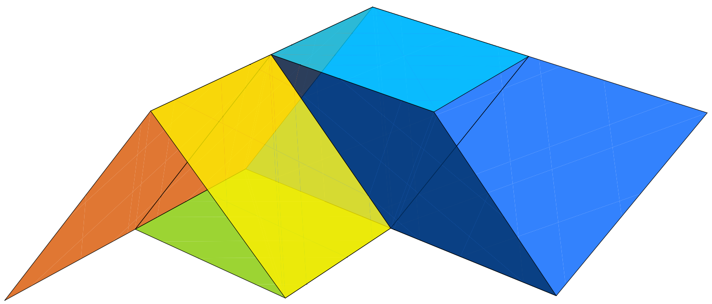
(a) 180° straight joint



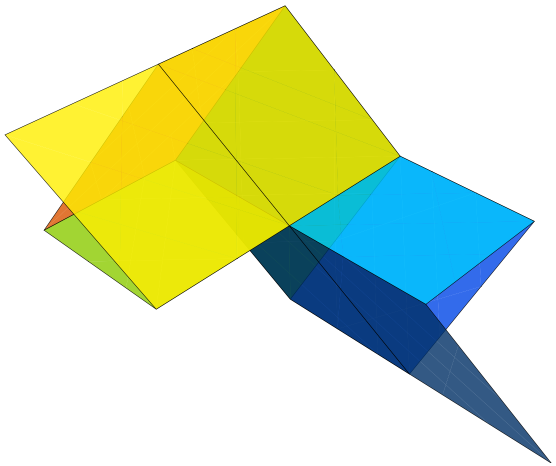
(b) 109.5° regular miter joint



(c) 70.5° joint



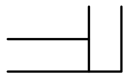
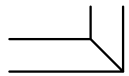
(d) 70.5° joint



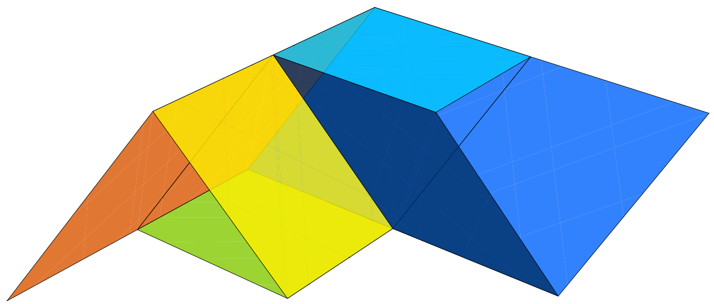
What Is the Nature of Joints (c) and (d)?

- They are not (regular or skew) miter joints: no cut plane

  The joint helixes have same handedness

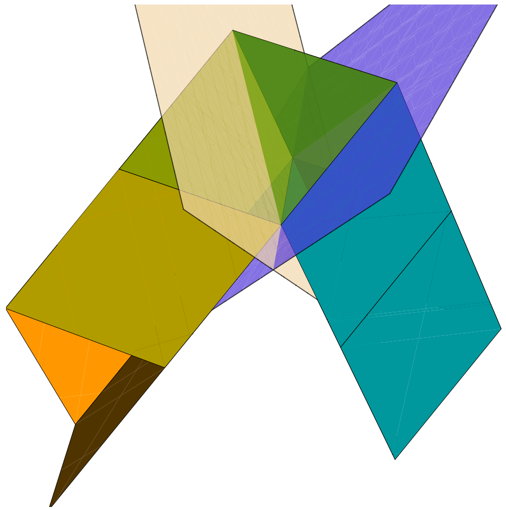
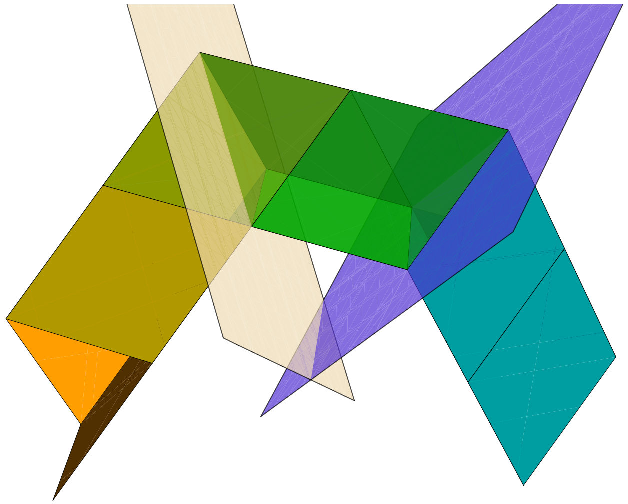
- They can be viewed as *false miter joints*:  instead of 

- Two pairs of (c) beam faces are joined by regular *fold* joints



But the bottom-left–top-right pair then does not meet at all

Acute Joint (c) Analyzed

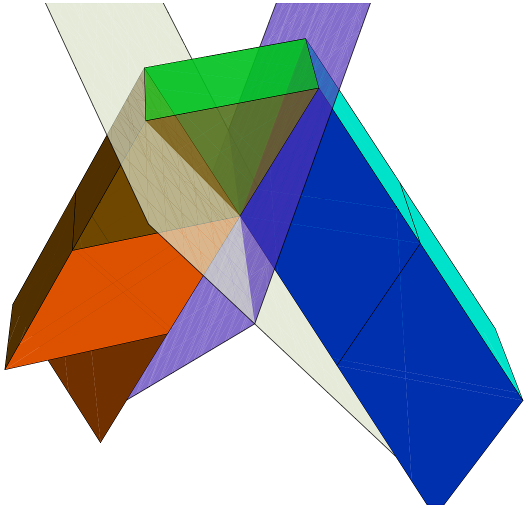
 

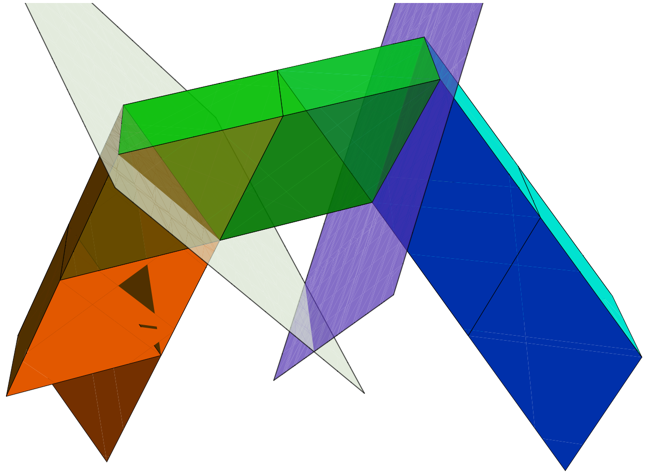
Can be viewed as a *pair* of type (b) regular 109.5° miter joints

with *degenerate* middle segment

In the middle segment, a face and two edges disappeared

Acute Joint (d) Analyzed



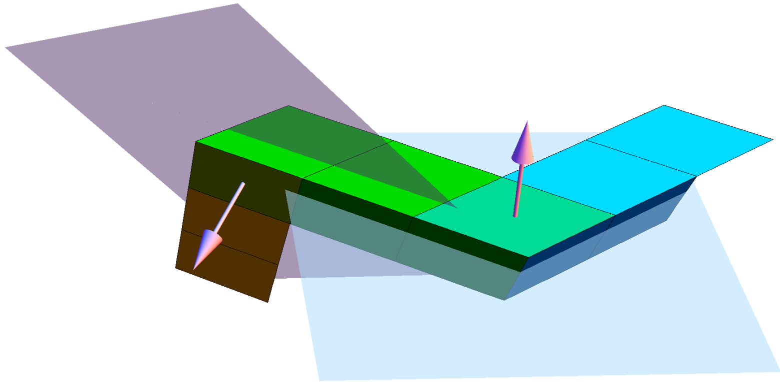


Can be viewed as a *pair* of type (b) regular 109.5° miter joints

with *degenerate* middle segment

In the middle segment, an edge disappeared

Roll Angle (Torsion) Between Consecutive Joints



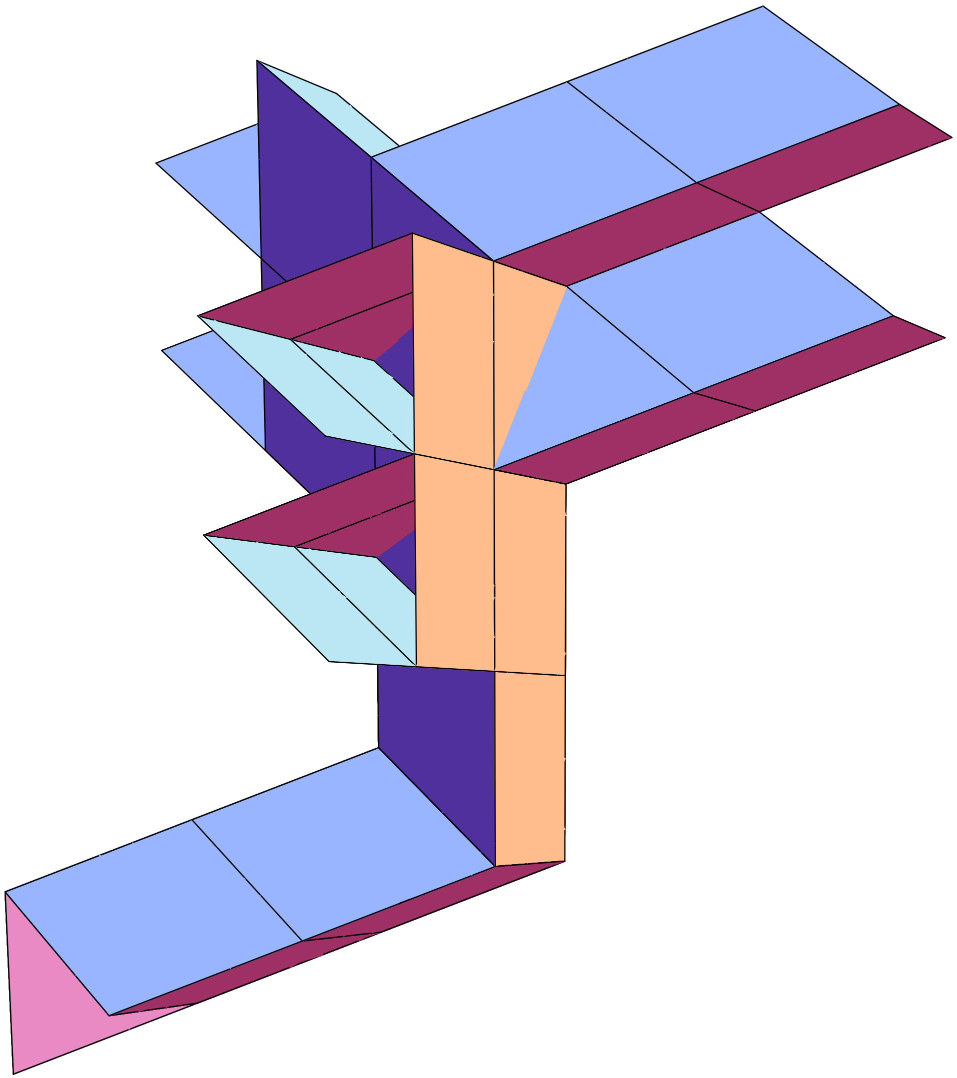
The *roll angle* between consecutive miter joints is a multiple of 120°

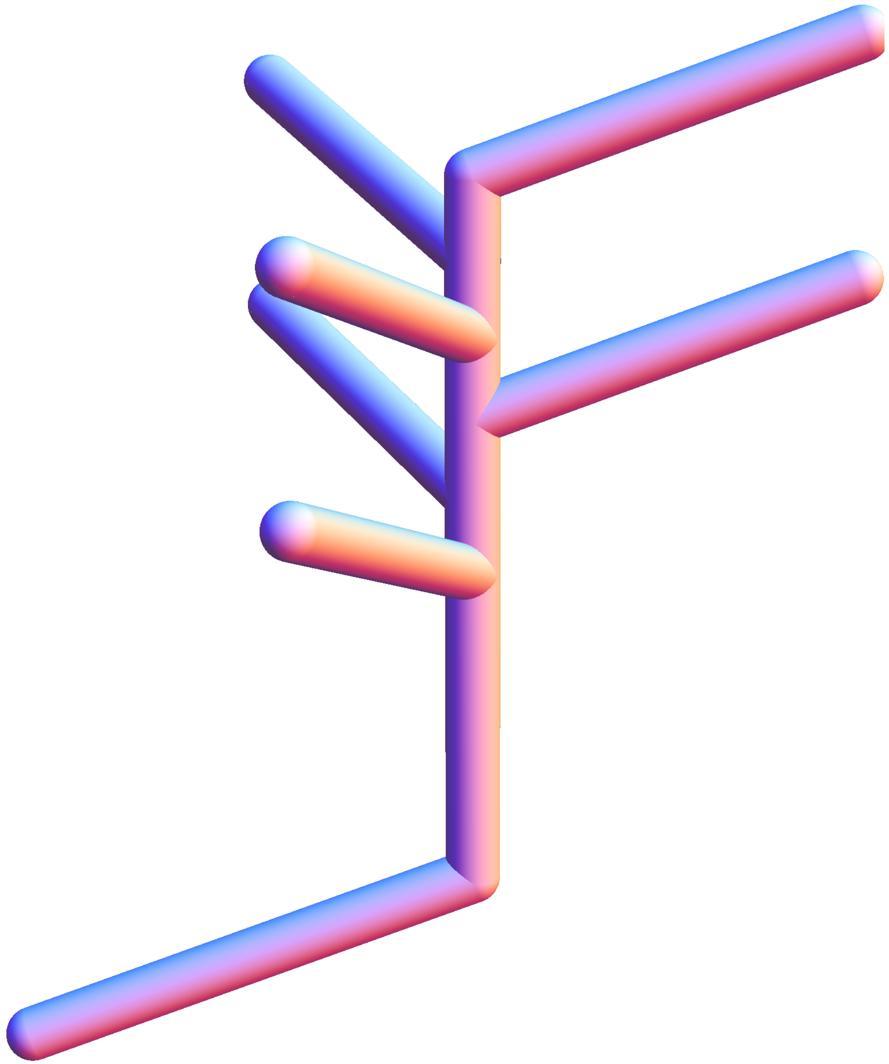
Hence, total torsion along path is multiple of 120°

120° rotation is a symmetry of the triangular cross section

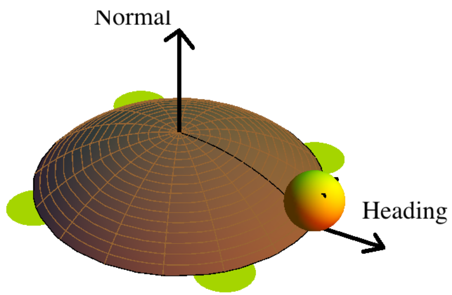
Hence, when beam path closes onto itself, all edges properly meet

6 Superimposed $\sqrt{2} : 1$ Rhombus Paths of 3 Segments





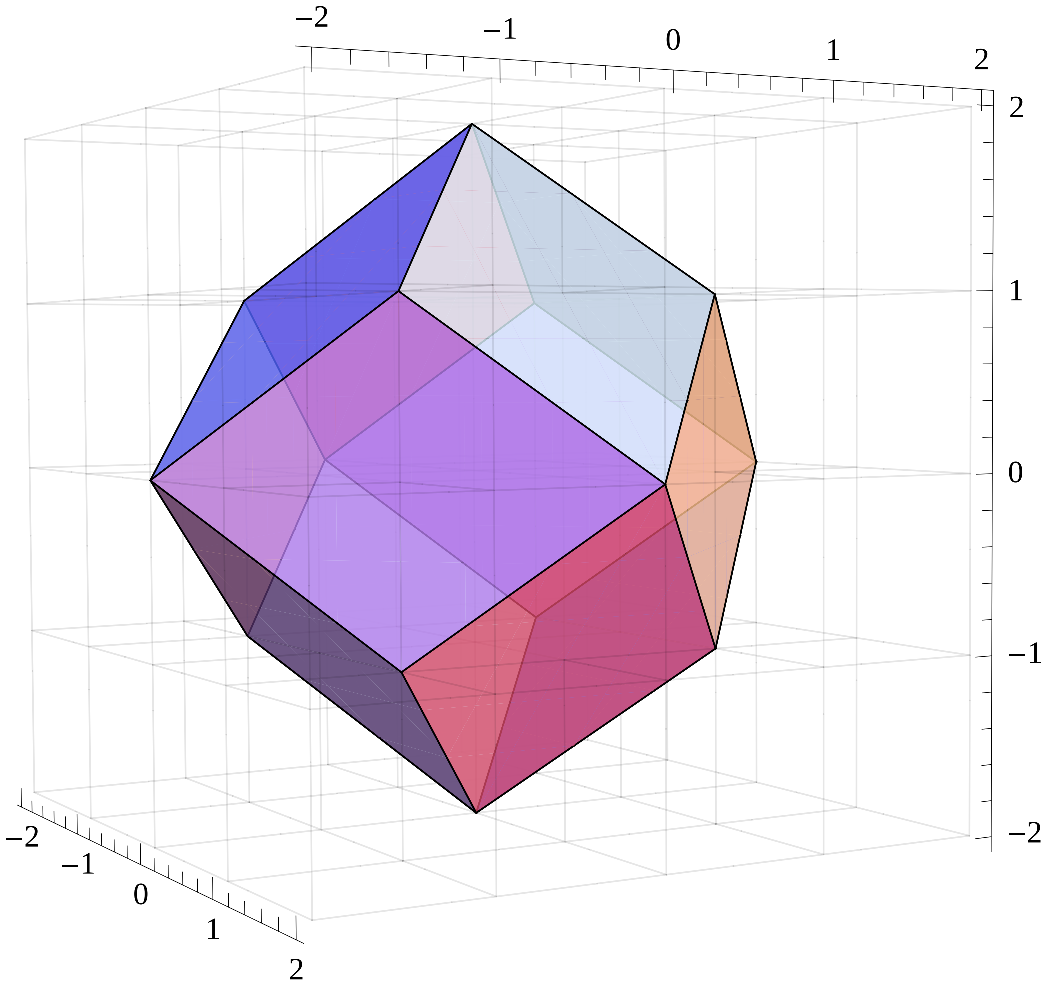
3D Turtle Description of Paths with $\sqrt{2}:1$ Rhombuses



- 3D turtle: forward motion and rolling motion are coupled
- Turtle screws (lit.) forward, rolling at a rate of 120° per rhombus

  Turtle turns 109.5°
- Consequently
  - Beams in 4 directions: main diagonals of cube
  - Can construct *constant-torsion* paths

Rhombus Orientations



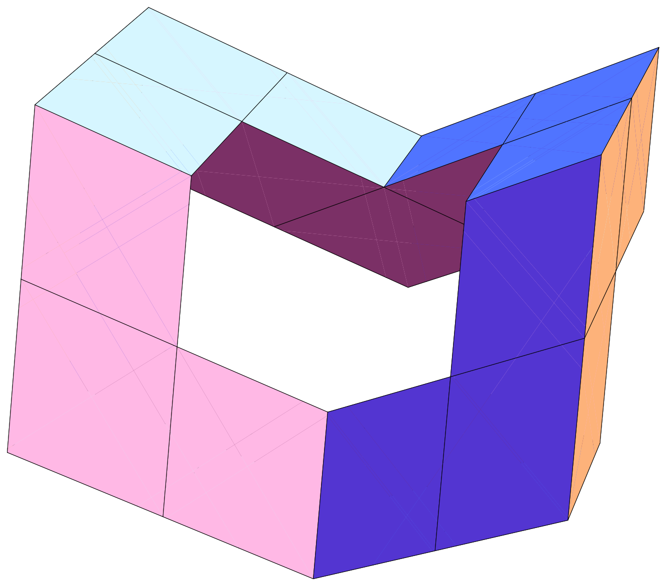
12 rhombus orientations

Rhombic Dodecahedron

All vertex coordinates can be integers

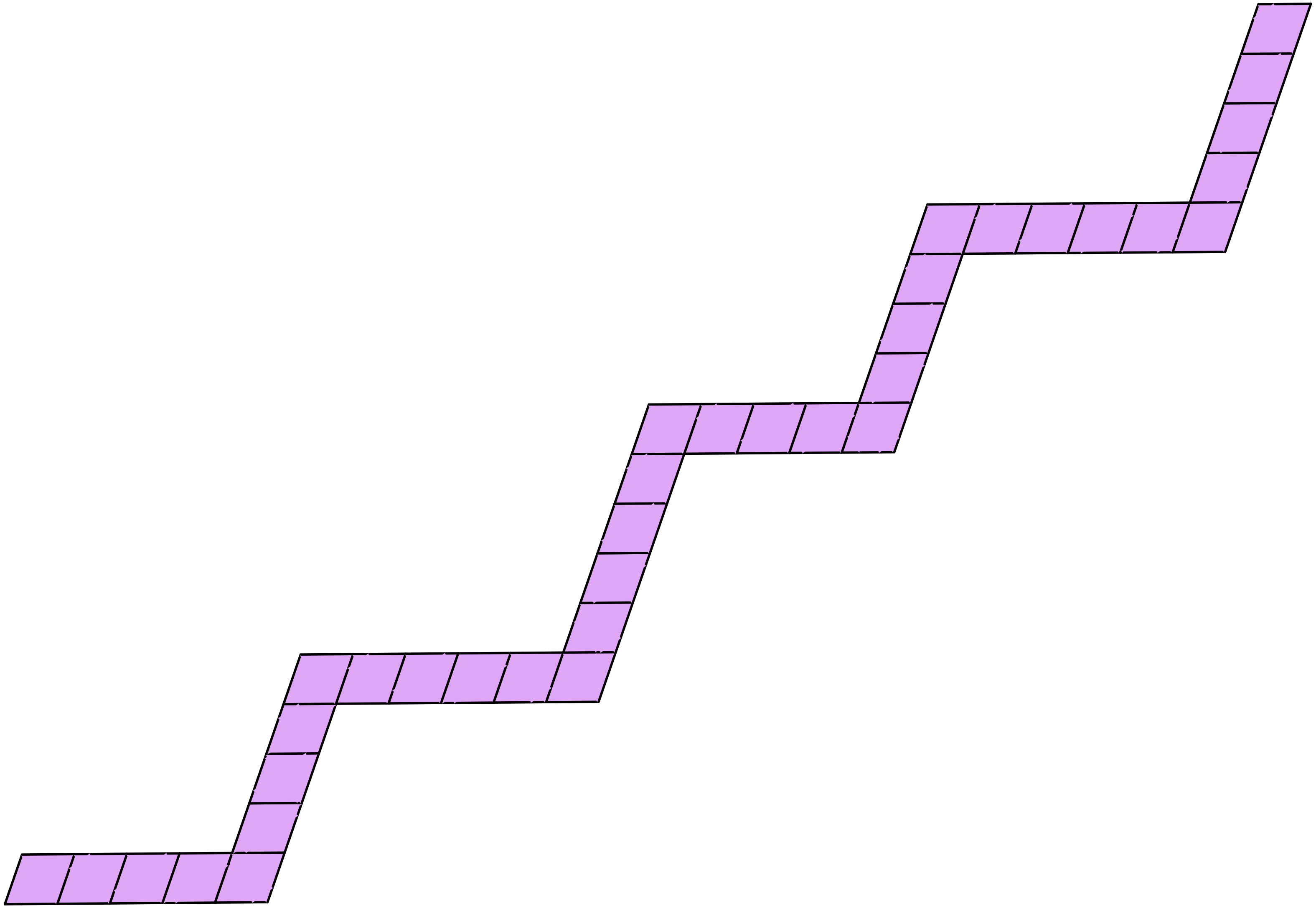
Closed Shapes of Triangular $\sqrt{2}:1$ Rhombus Beams

- Measure beam length in terms of *number of rhombuses*
- Sequence of beam lengths uniquely defines the shape

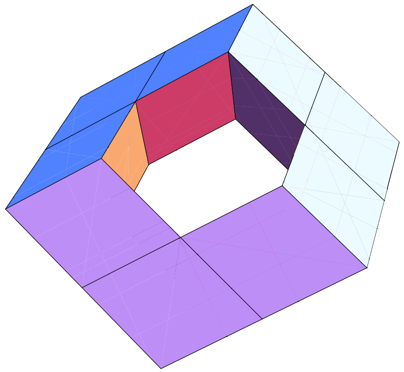


octagon: $(4, 4, 5, 5, 4, 4, 5, 5)$

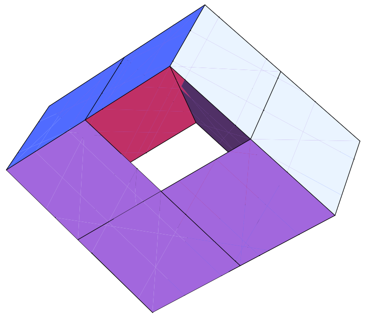
Octagon $(4, 4, 5, 5, 4, 4, 5, 5)$ **Unfolded**



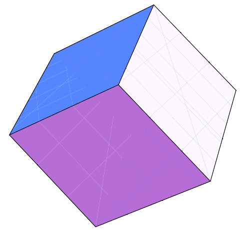
Hexagons



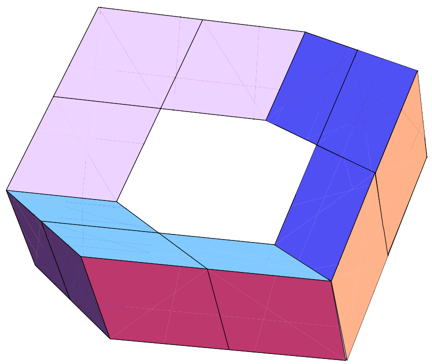
(4, 4, 4, 4, 4, 4)



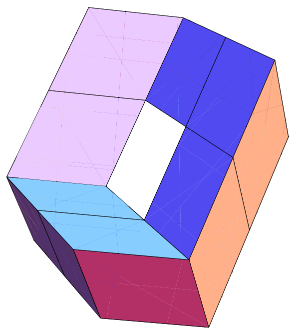
(4, 4, 1, 4, 4, 1)



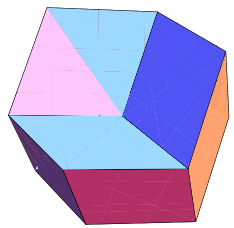
(1, 1, 1, 1, 1, 1)



(5, 5, 5, 5, 5, 5)

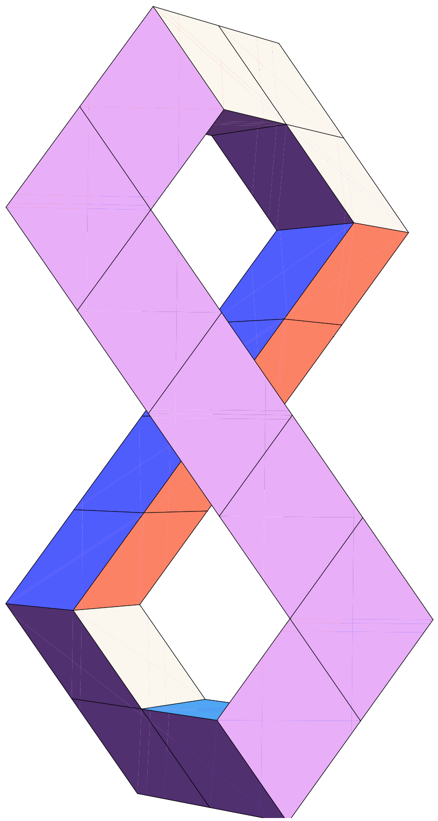


(5, 5, 2, 5, 5, 2)

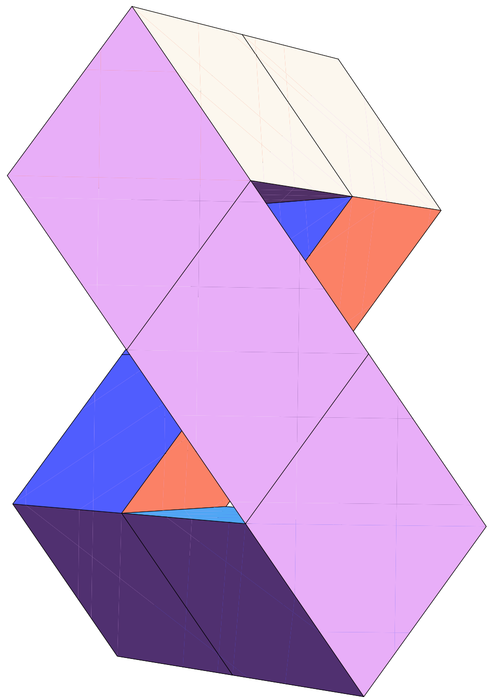


(2, 2, 2, 2, 2, 2)

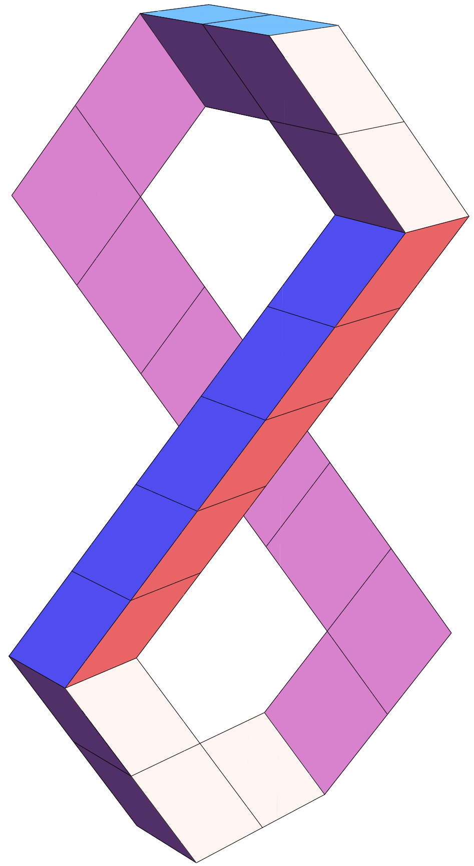
More Octagons



$(12, 4, 4, 4)^2$

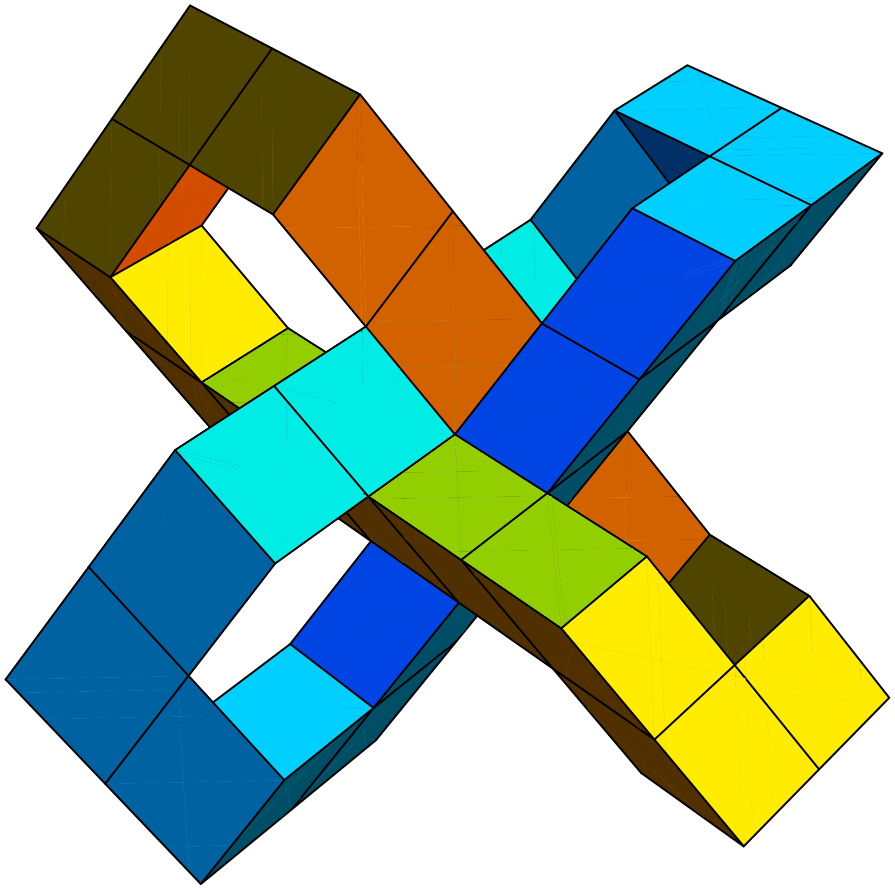


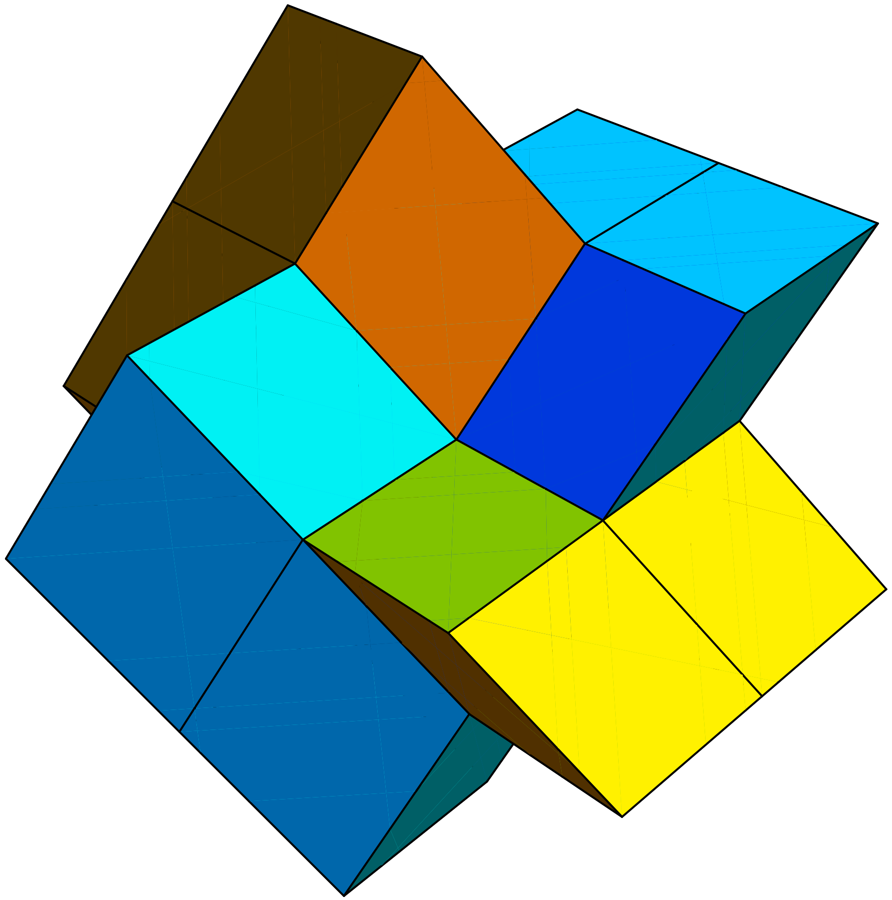
$(6, 1, 4, 1)^2$



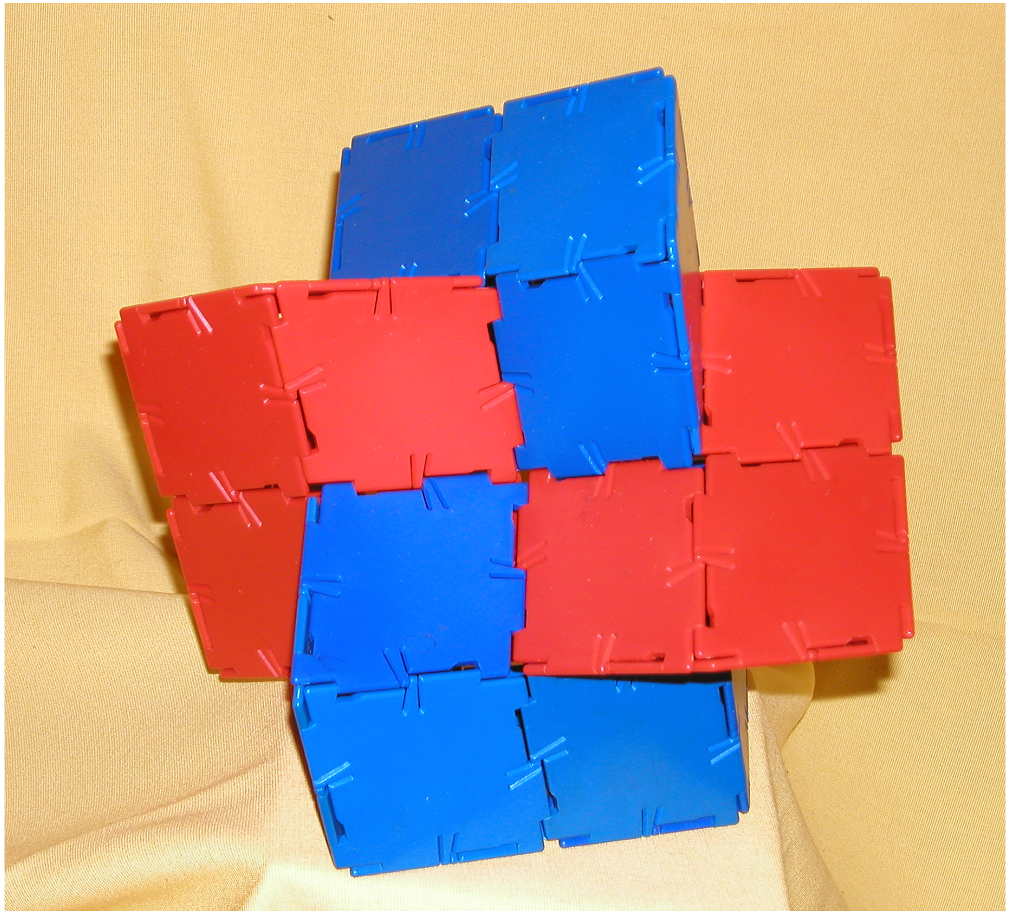
$(15, 5, 5, 5)^2$

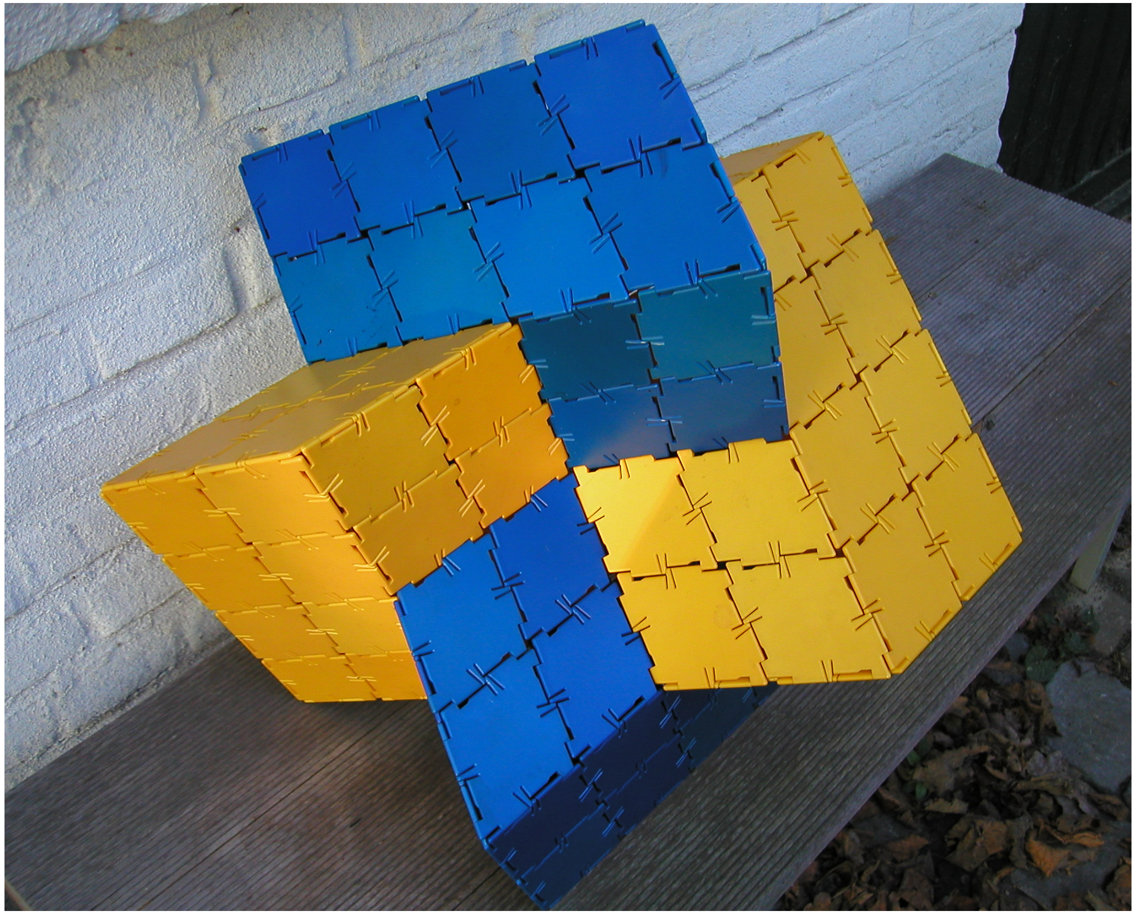
Pairs of Linked Octagons



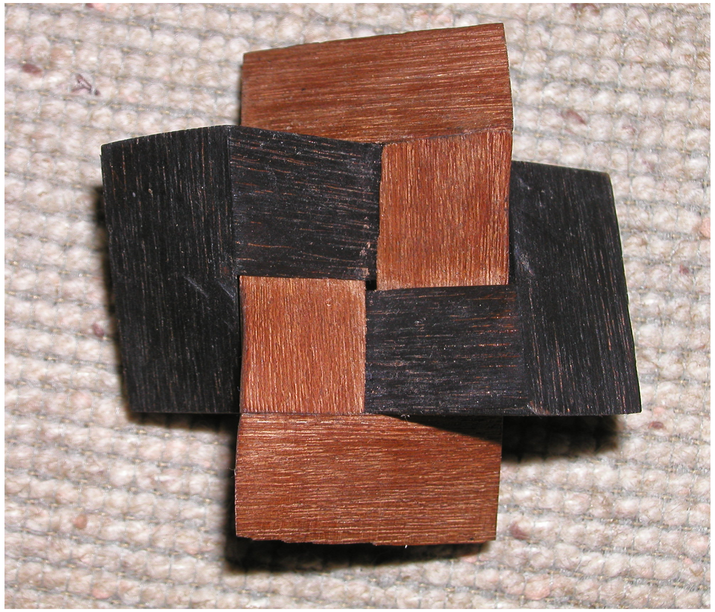


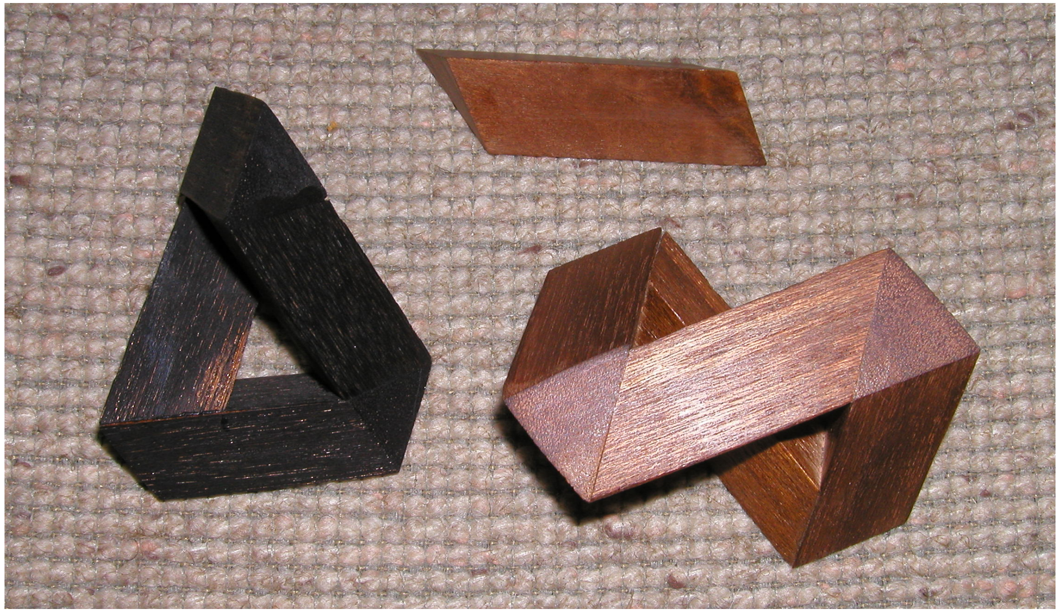
Linked Octagons: Polydron^{TM}



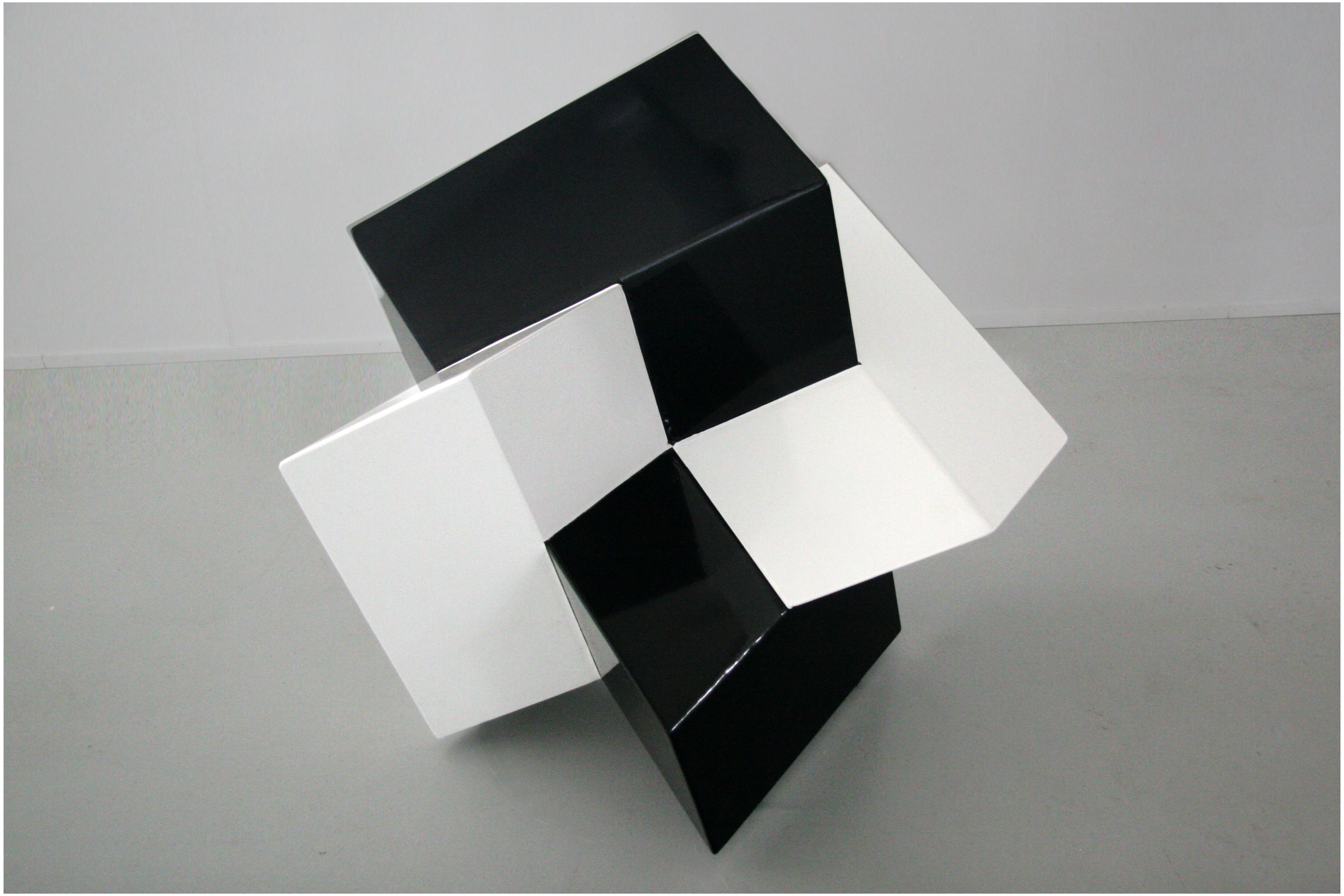


Linked Octagons: Wood

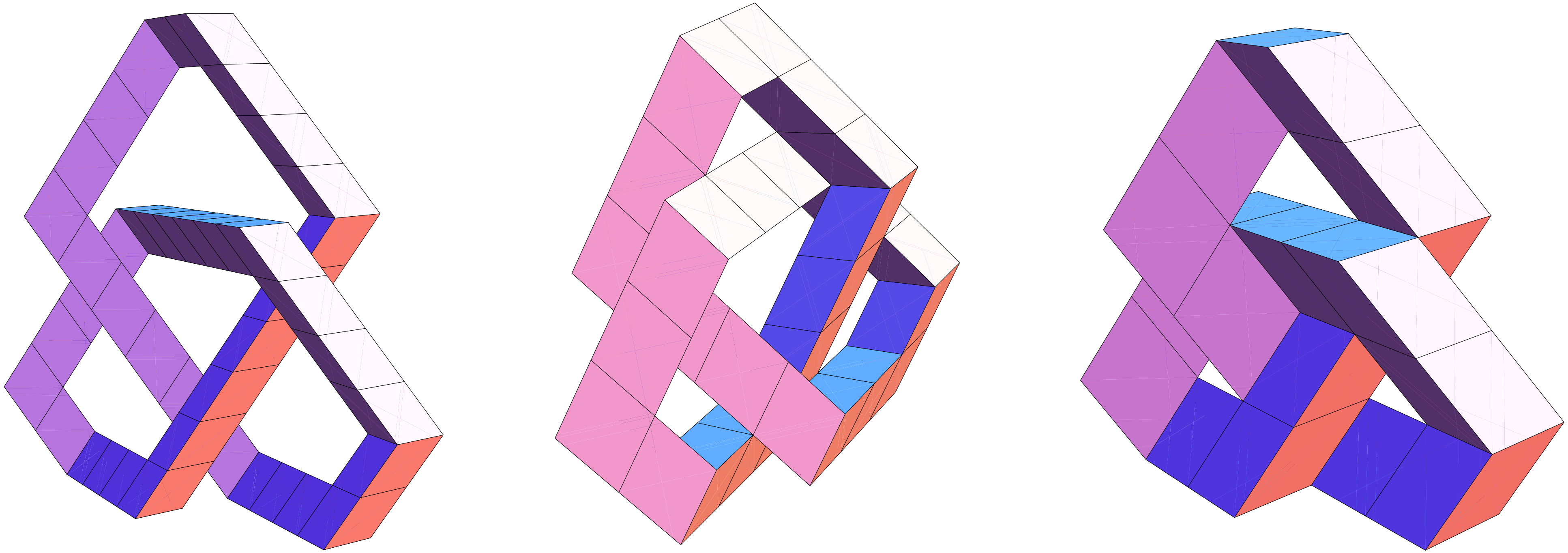




Pair of Linked Octagons, Powder-Coated Corten Steel (2013)

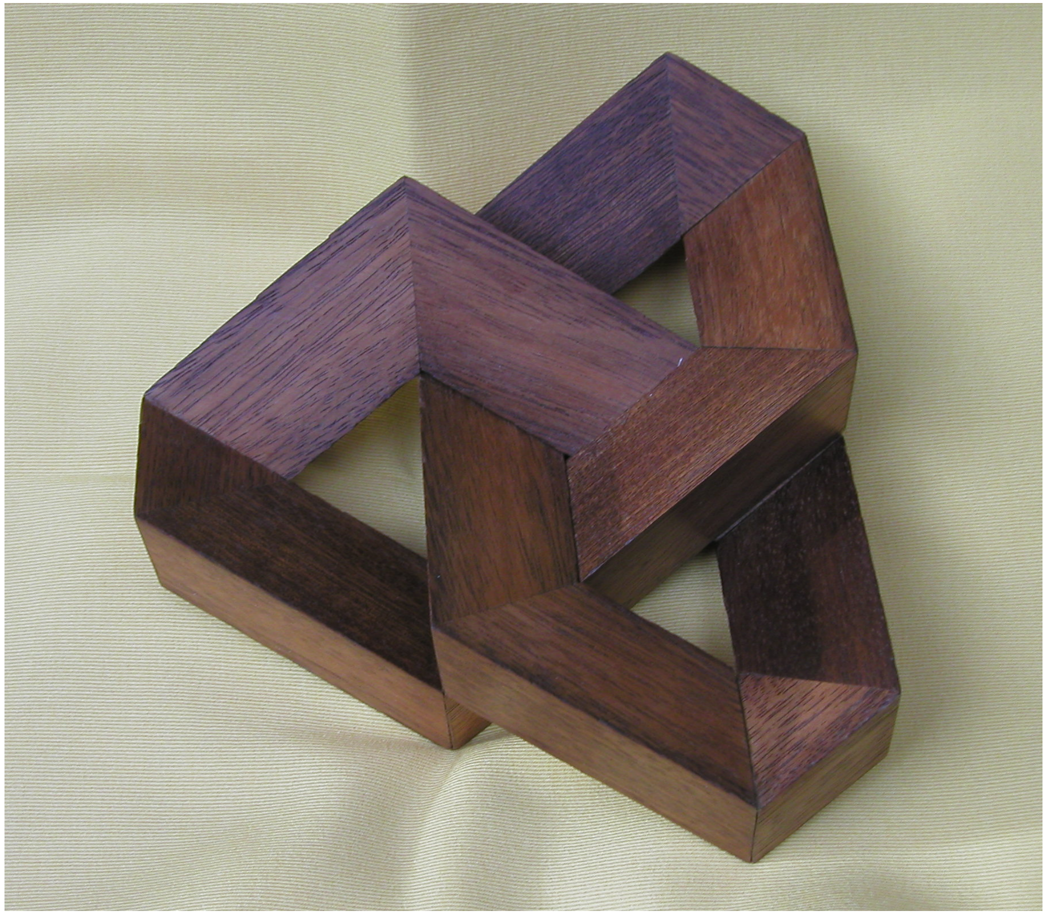


Trefoil Knots



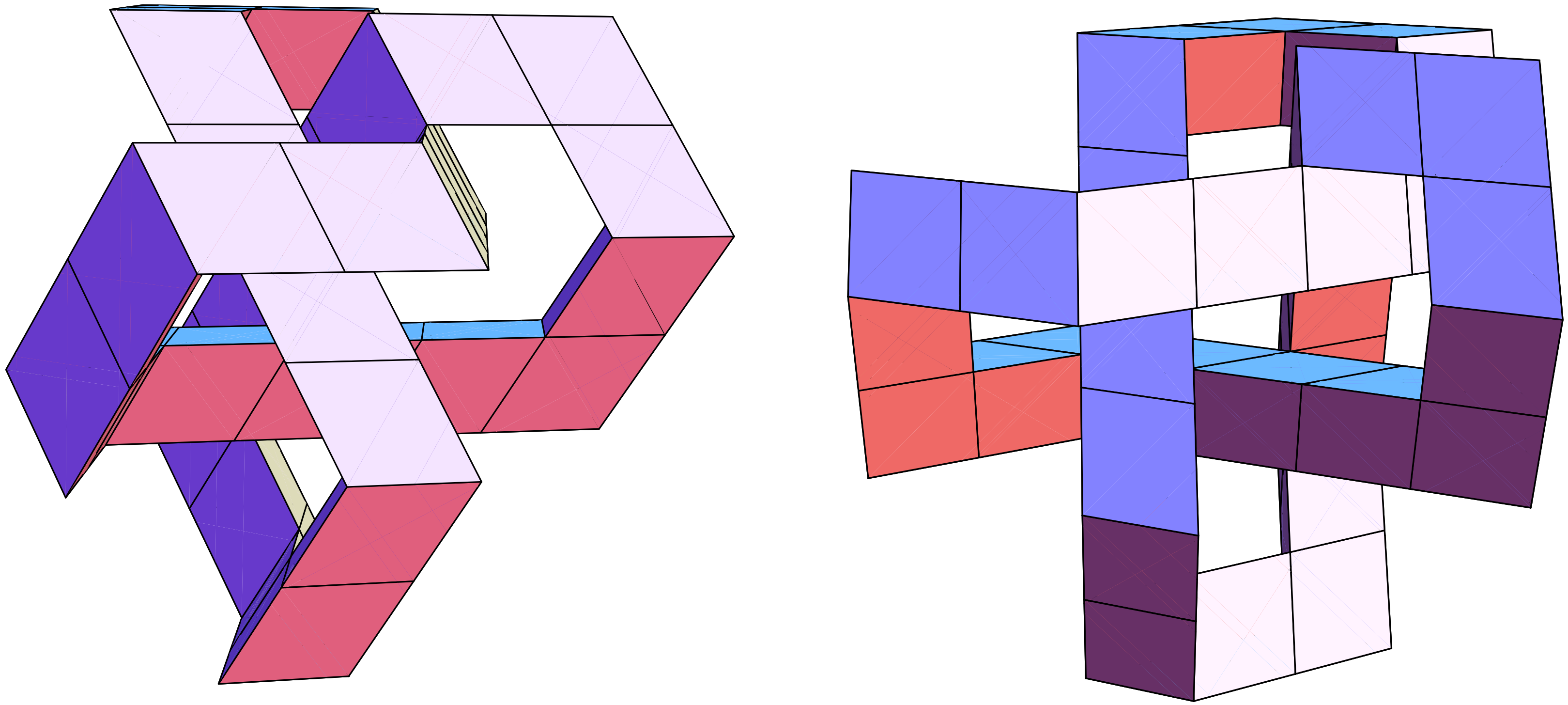
$(5, 11, 17, 11)^3$ $(4, 7, 10, 7)^3$ $(2, 5, 8, 5)^3$

Trefoil Knots: Wood



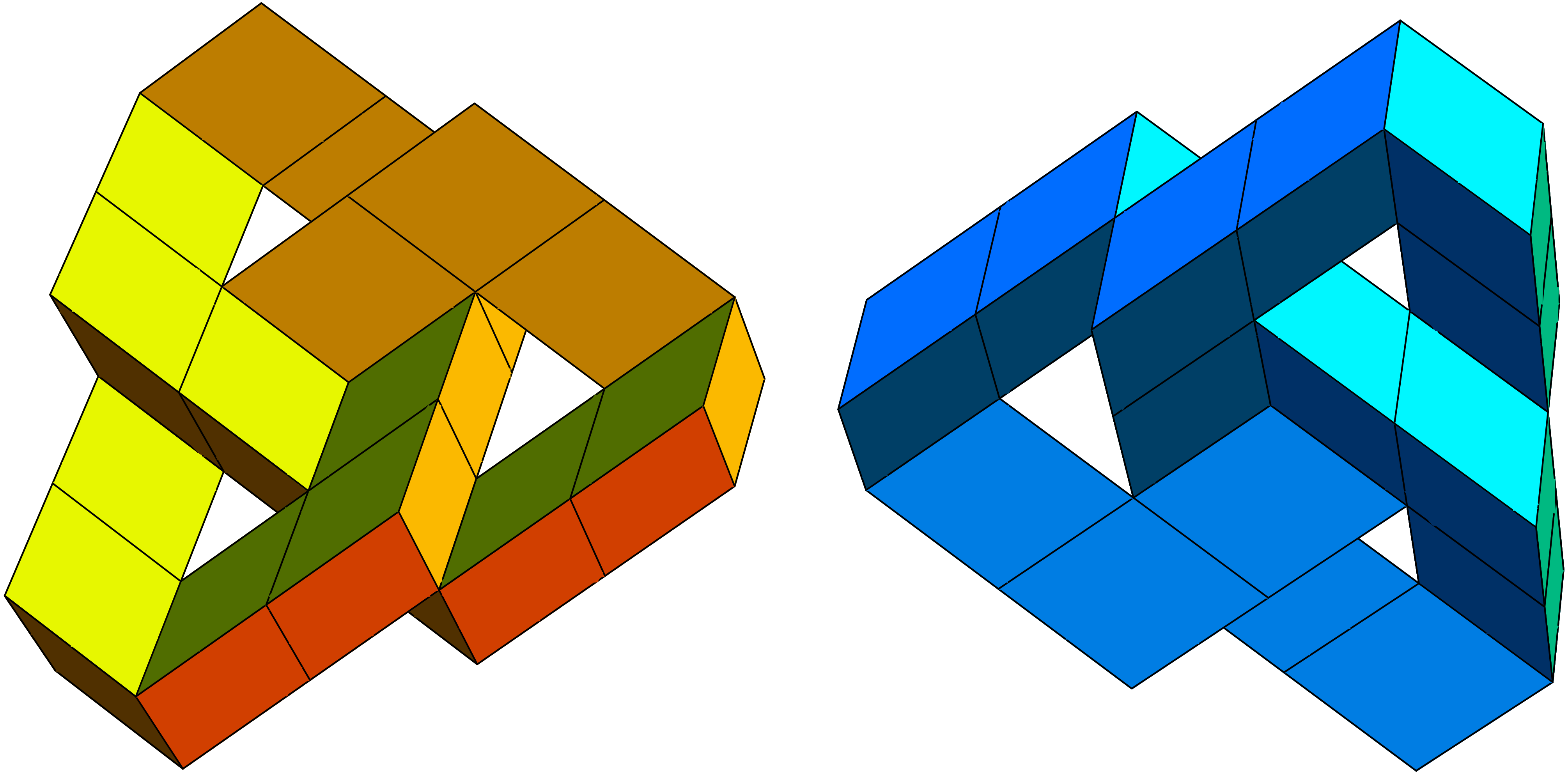
Mitered Trefoil Knot, Corten Steel (2013)

Figure-Eight Knot

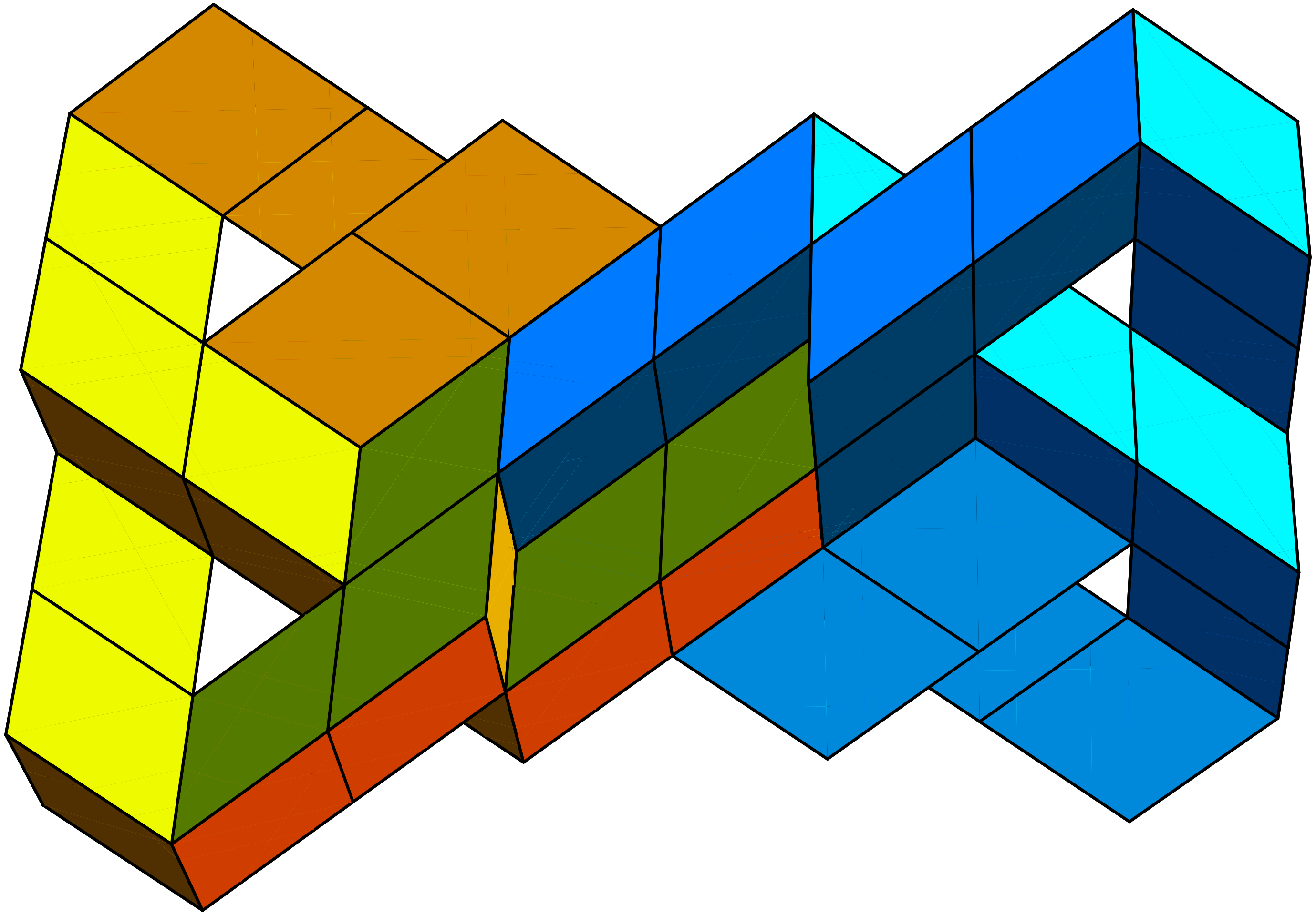


$$(4, 4, 4, 11, 5, 5, 5, 10)^2$$

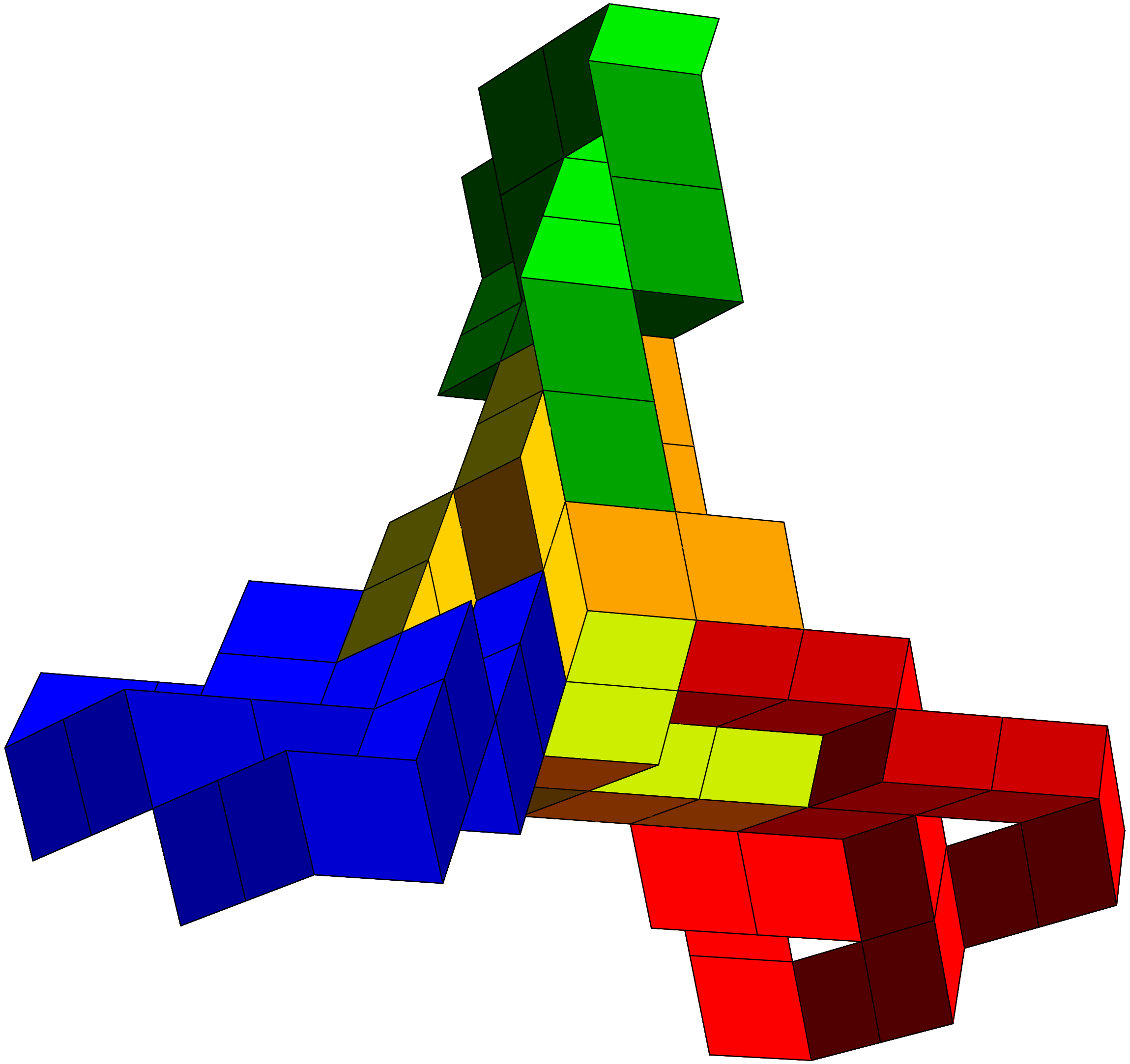
Pair of Unlinked Trefoil Knots



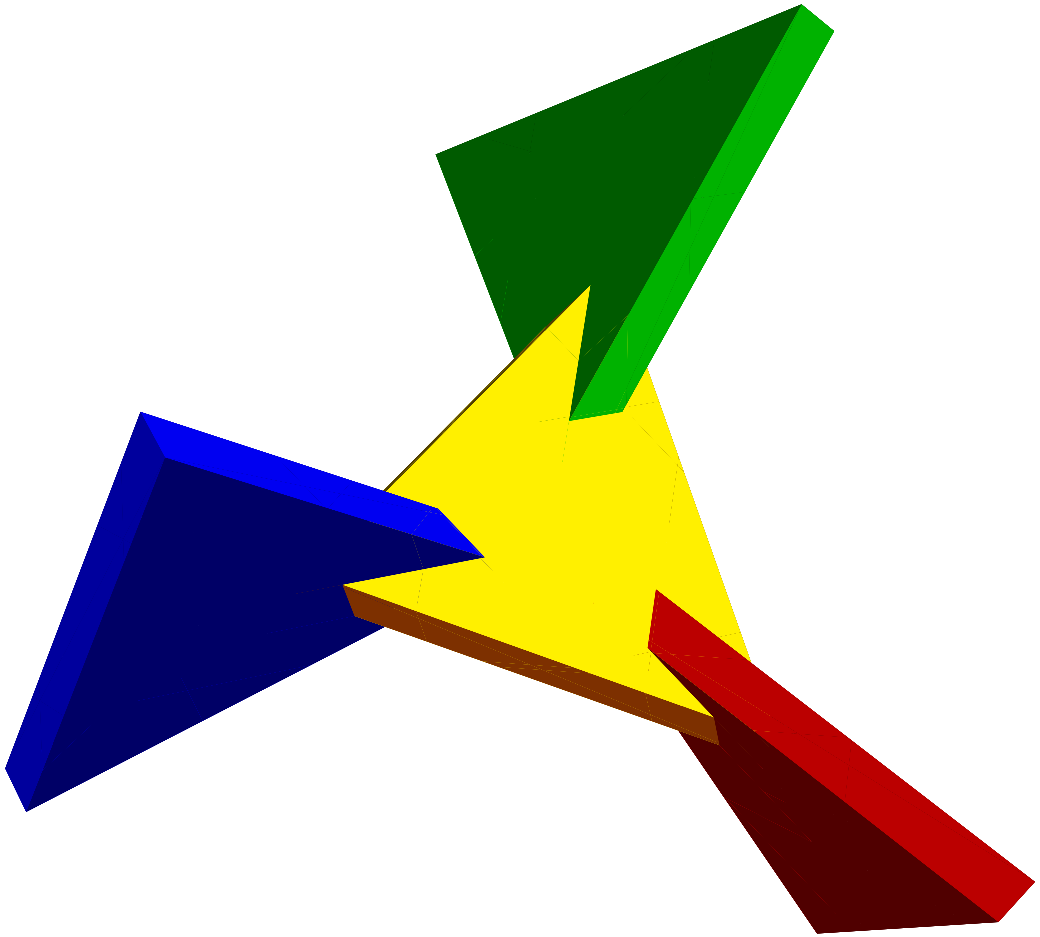
Pair of Linked Trefoil Knots



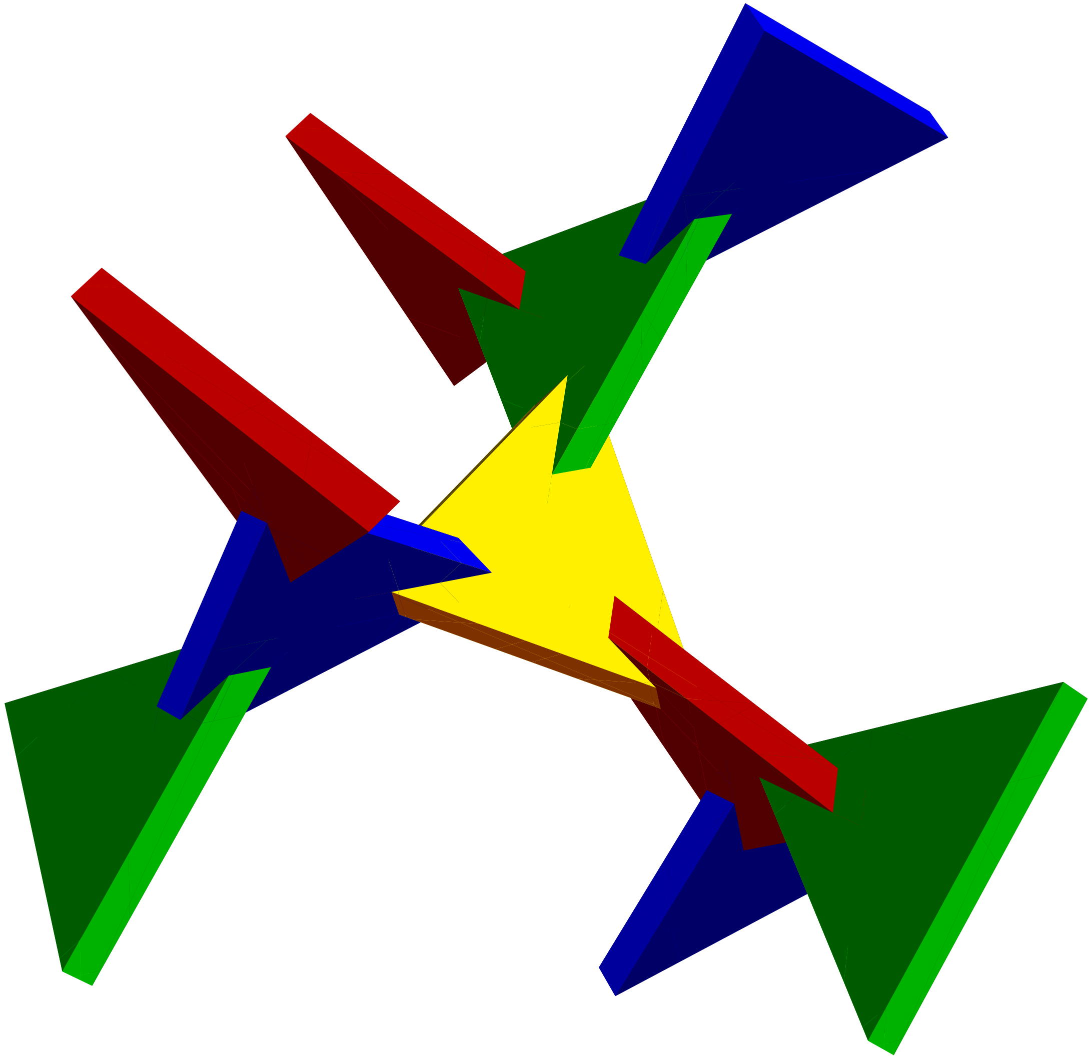
Four Linked Trefoil Knots



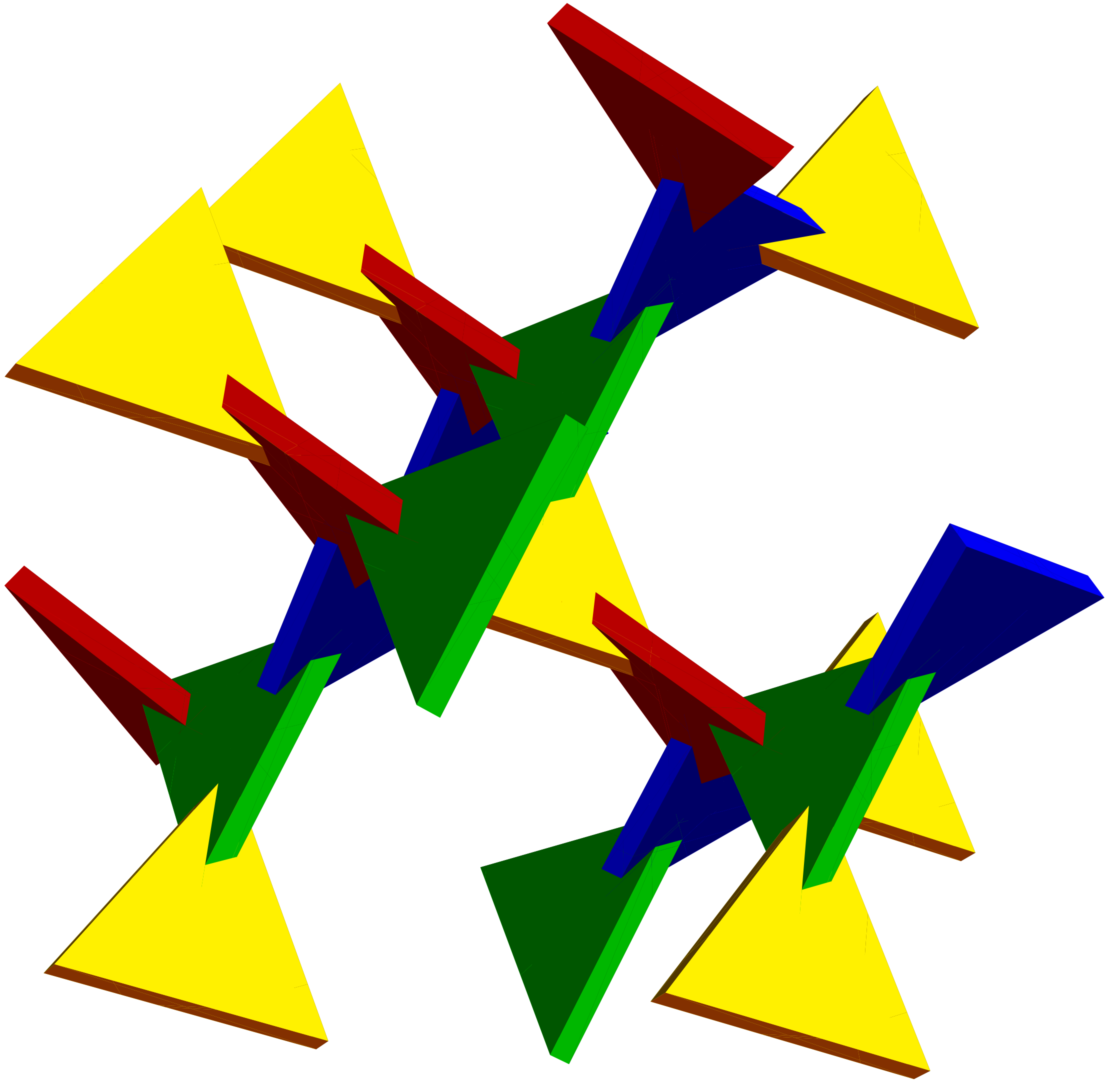
Replace Trefoil Knots by Equilateral Triangles



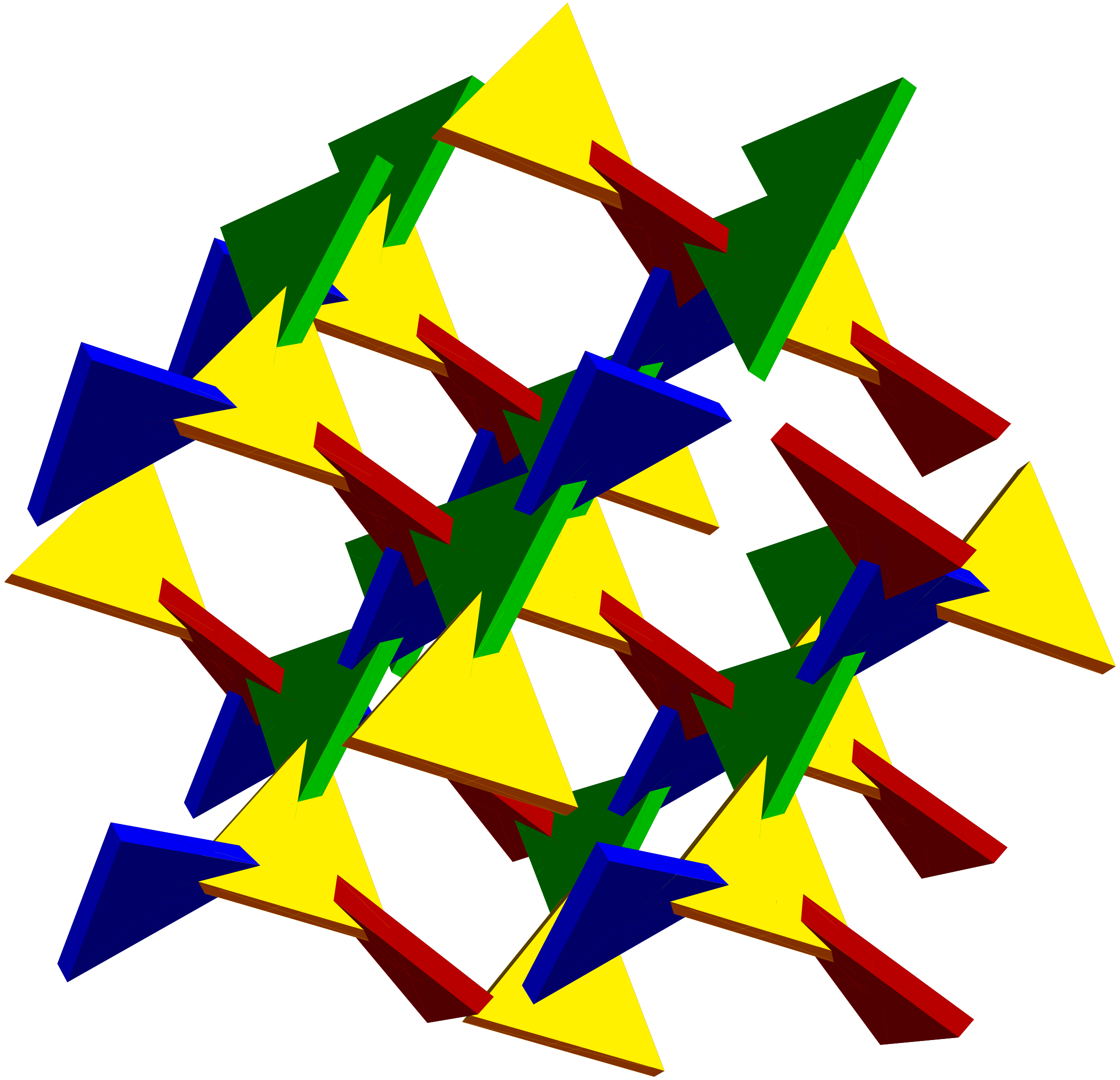
What Happens When Adding Triangles: 2nd Generation



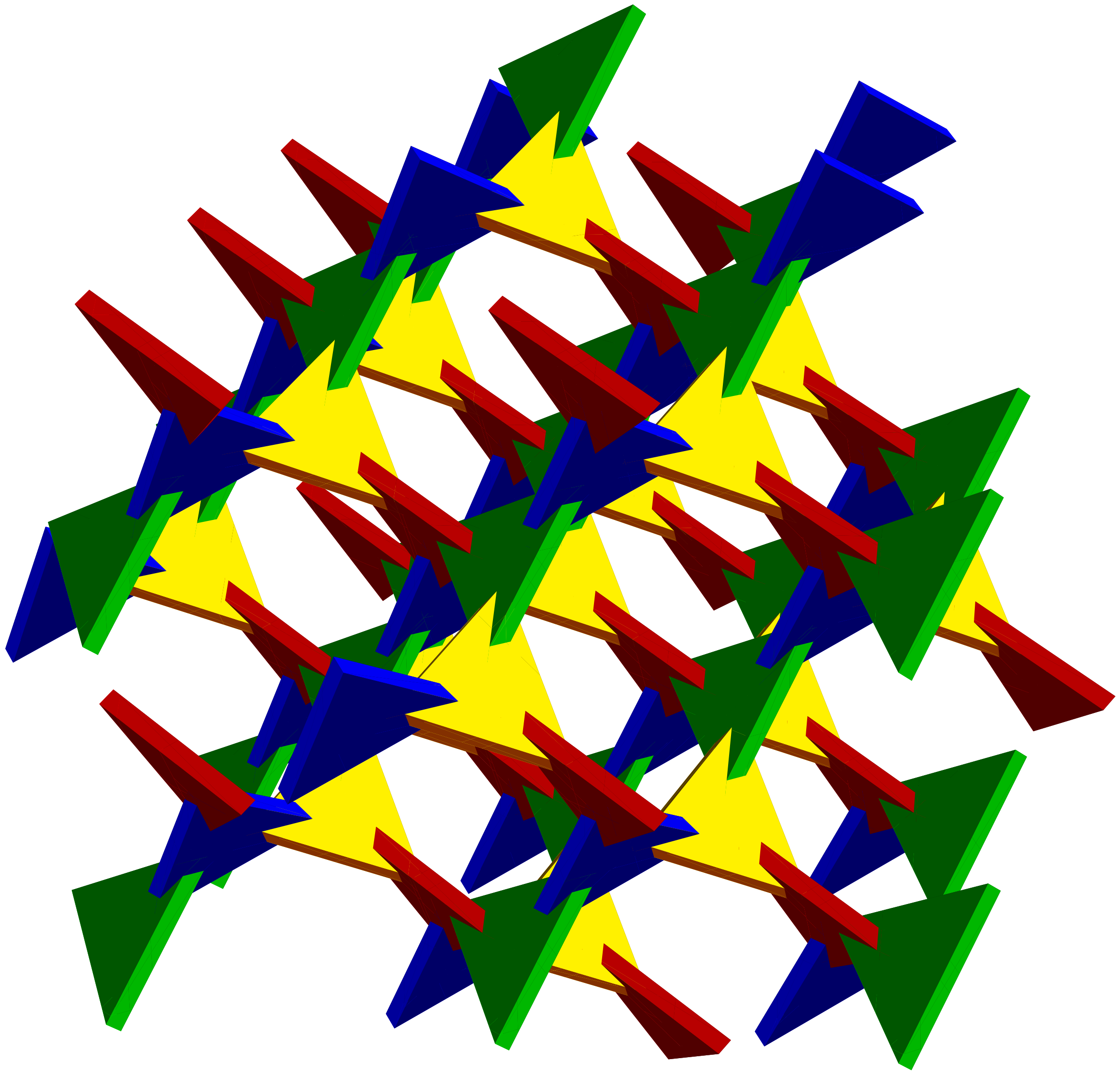
3rd Generation



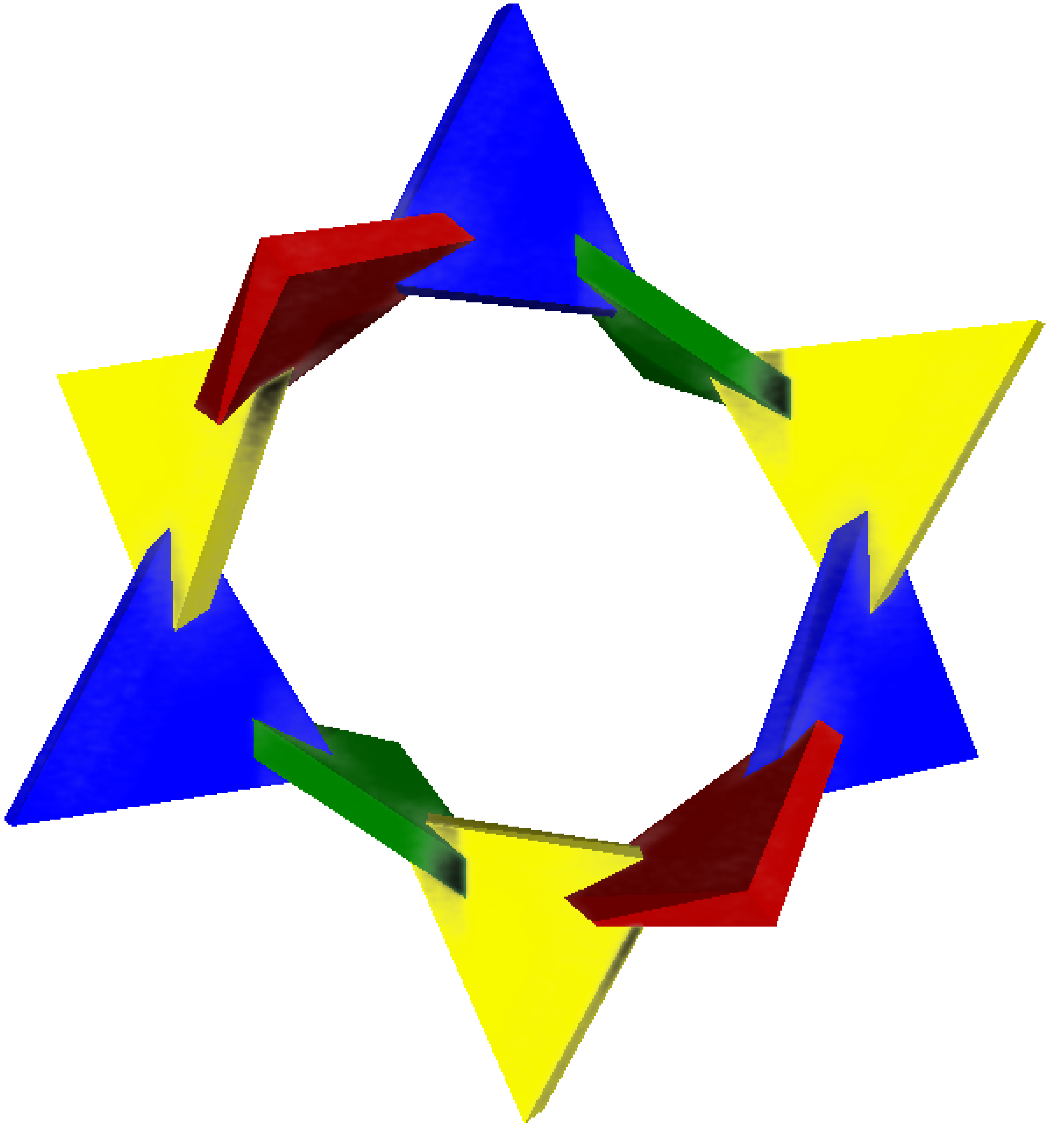
4th Generation: Still No Collision



5th Generation: First Cycles Close Nicely



Shortest Cycle Consists of 10 Triangles

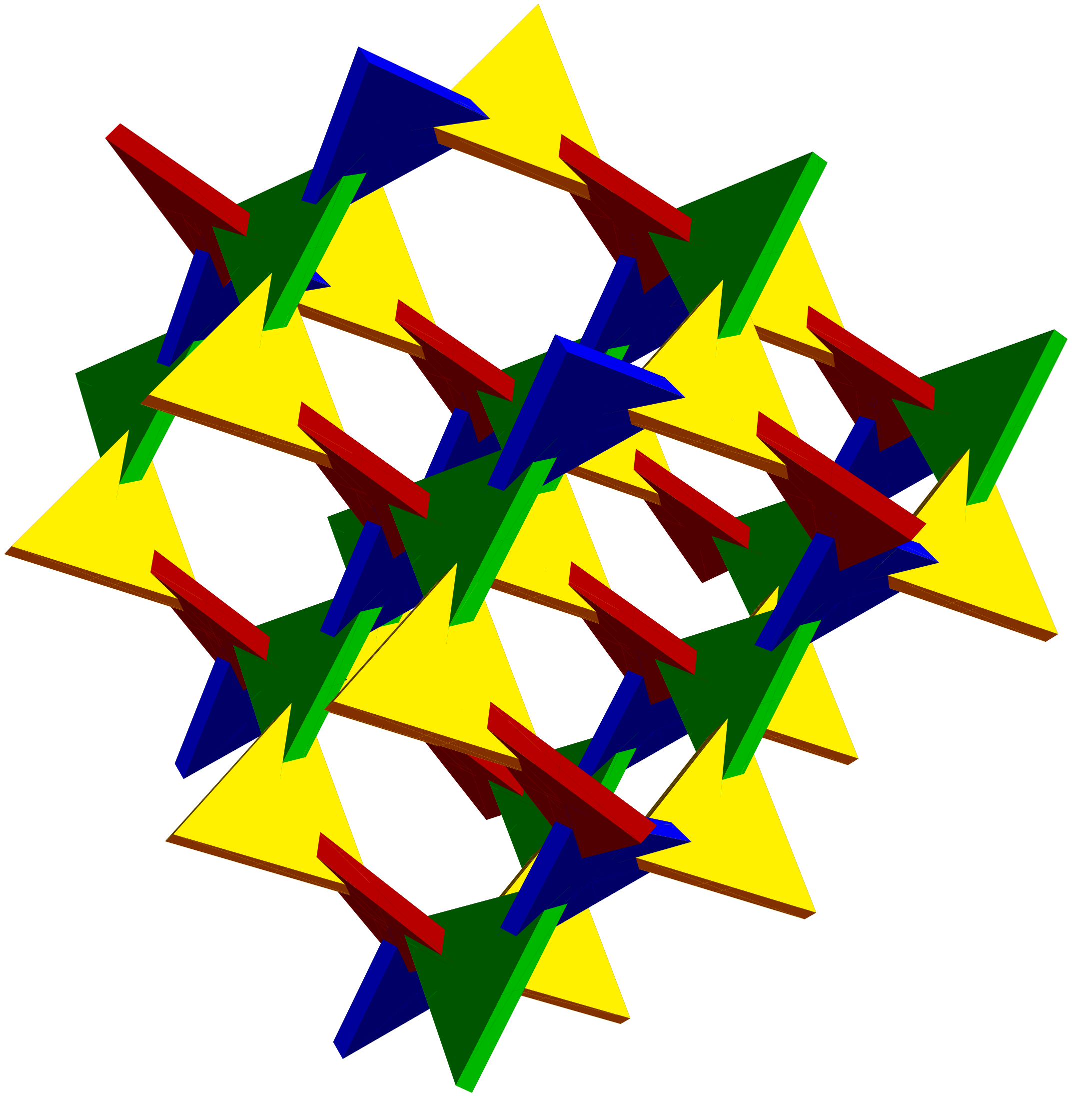


Infinite Space-spanning Structure: Triamond, . . .

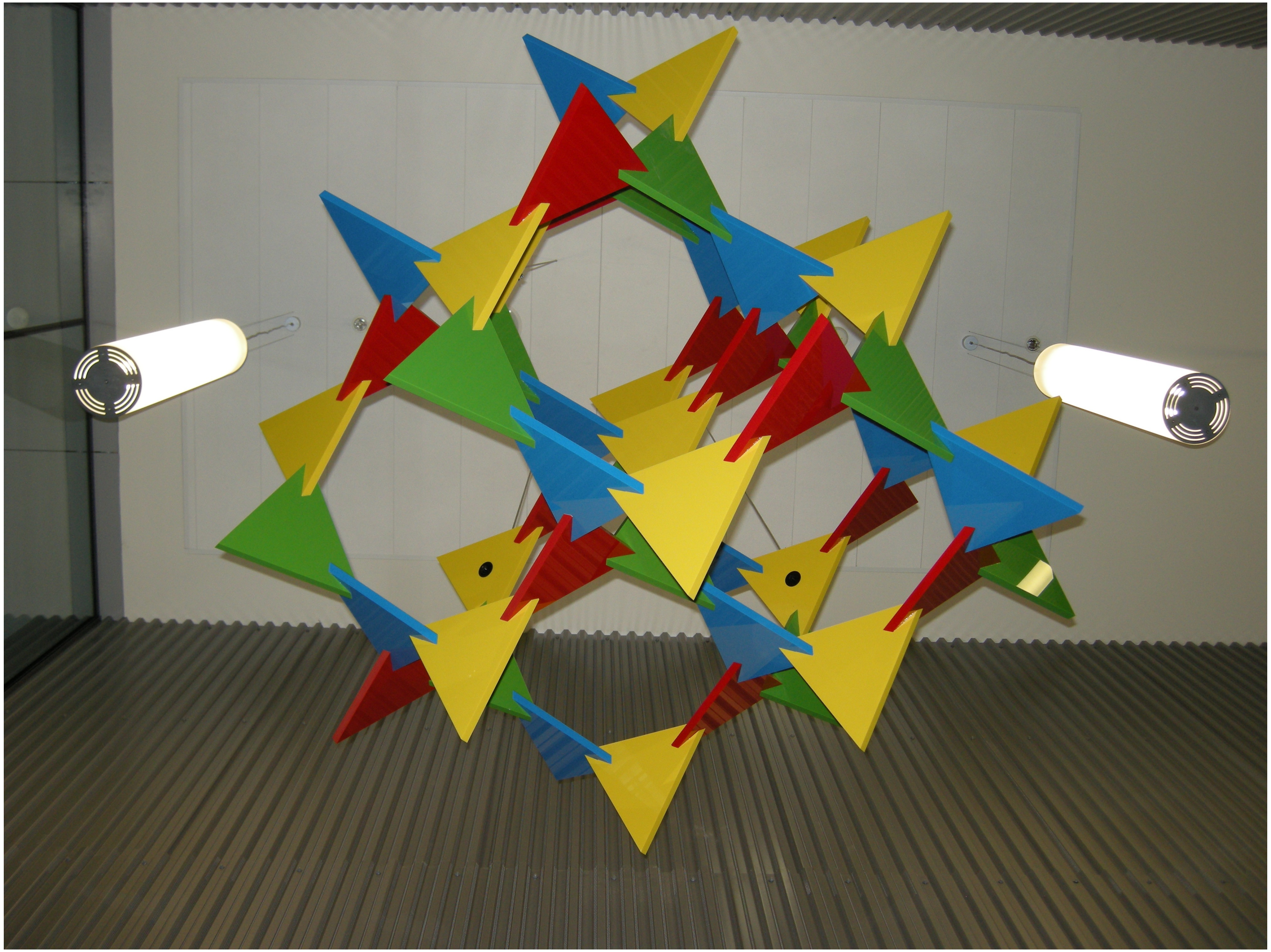
- Translational symmetries
- Rotational symmetries, of order 2 and order 3
- *No* mirror symmetries (chiral)
- *Screw axes* (glide rotations), of order 3 and order 4
- Space group 214 (of 230): $I4_332$
- T. Sunada, "Crystals That Nature Might Miss Creating" (2008)

  Very strong isotropic property; energy minimizing

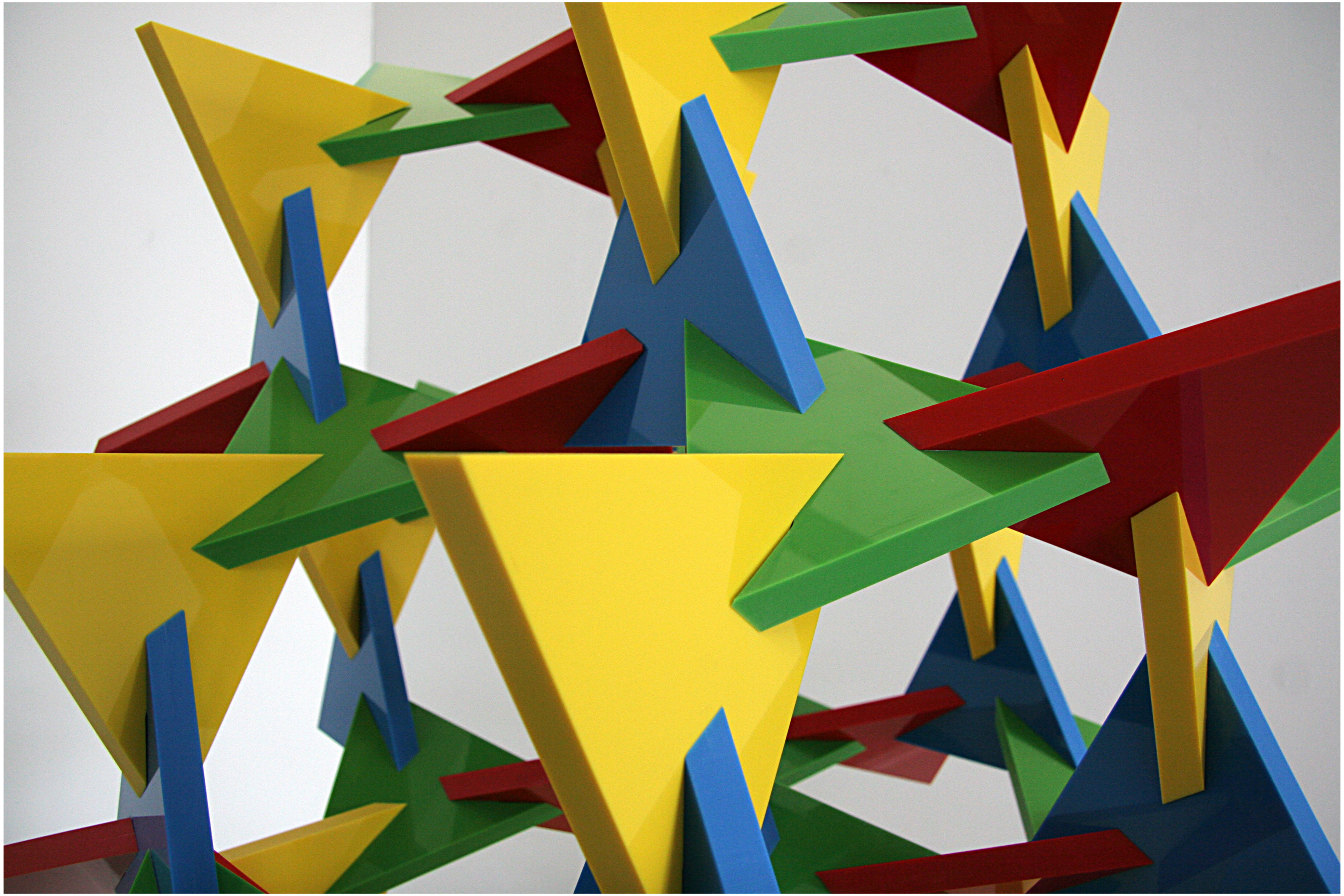
Drop Dangling Triangles, with Degree < 2: Bamboozle



Bamboozle, Polished Acrylic (Installed January 2013)



Smaller Bamboozle (July 2013)

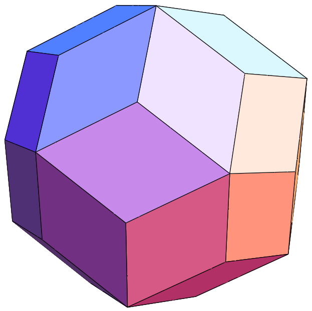


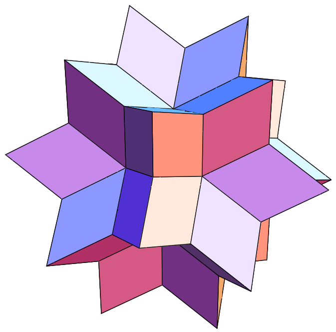
Conclusion

- Beams constructed by folding a strip of rhombuses into a helix

- $\sqrt{2}:1$ rhombus yields triangular beams, allowing versatile joints

  N.B. Golden Rhombus only useful for tria- and hexecontahedron





  Plea: Polydron™, please re-introduce the $\sqrt{2}:1$ rhombus!

- Artwork designs based on/inspired by $\sqrt{2}:1$ rhombus

Future Work

1. Half rhombuses
2. Ternary joints
3. Intertwined discrete helixes
4. Generalizations of Bamboozle

Stichting Wiskunst Koos Verhoeff

`http://wiskunst.dse.nl`

Related Work

- Tom Verhoeff & Koos Verhoeff
  "The Mathematics of Mitering and Its Artful Application"
  Bridges 2008, Leeuwarden, Netherlands, pp.225–234

- Tom Verhoeff & Koos Verhoeff
  "Regular 3D Polygonal Circuits of Constant Torsion"
  Bridges 2009, Banff, Canada, pp.223–230

- Tom Verhoeff & Koos Verhoeff
  "Branching Miter Joints: Principles and Artwork"
  Bridges 2010, Pécs, Hungary, pp.27–34

- Tom Verhoeff & Koos Verhoeff
  "From Chain-link Fence to Space-spanning Helical Structures"
  Bridges 2011, Coimbra, Portugal, pp.73–80

Also see: `http://www.win.tue.nl/~wstomv/publications/`