Regular 3D Polygonal Circuits of Constant Torsion

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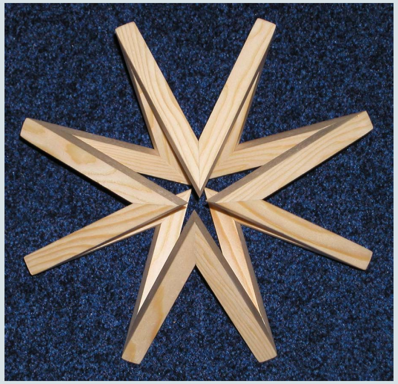

Stichting Wiskunst Koos Verhoeff
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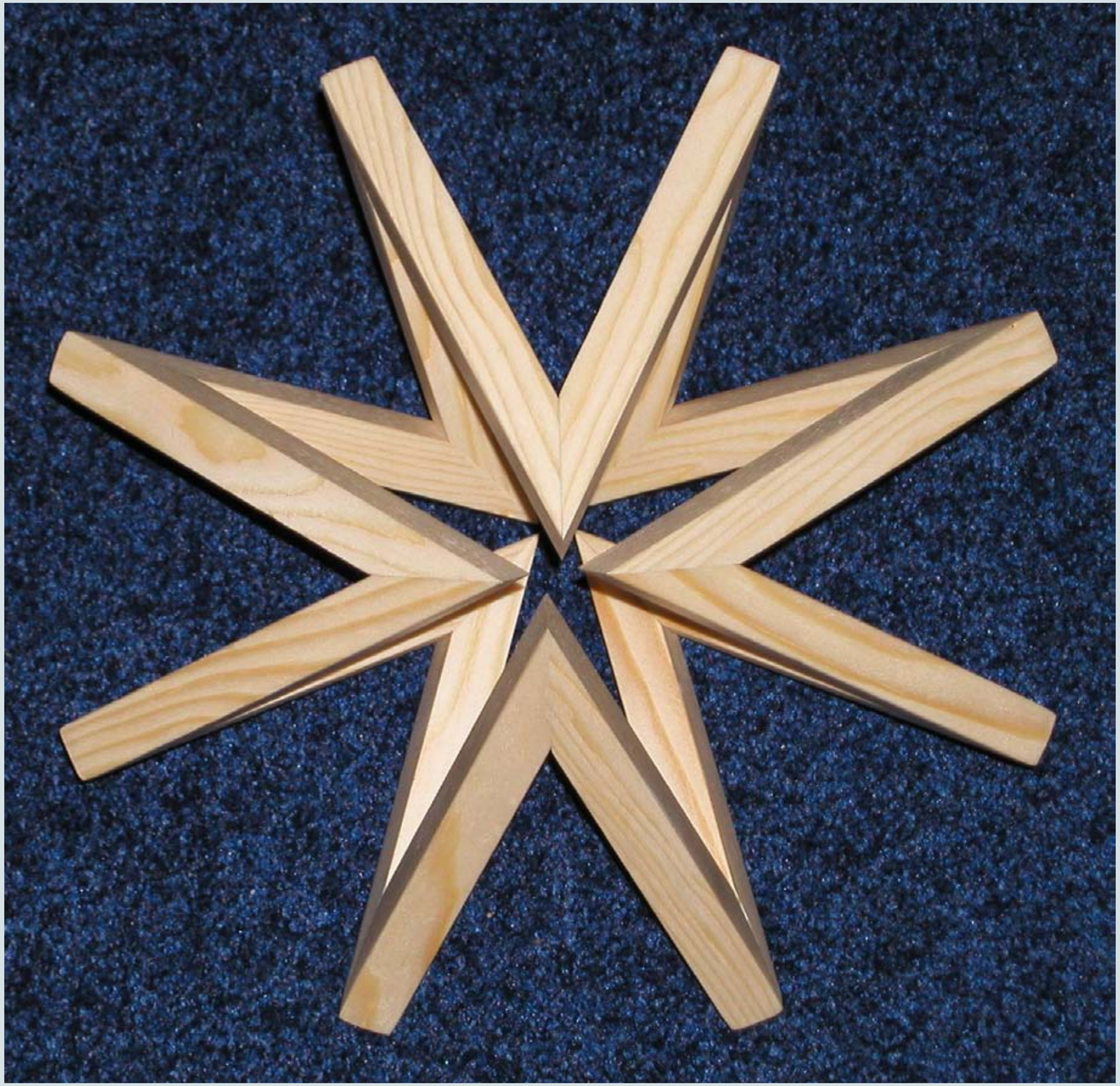
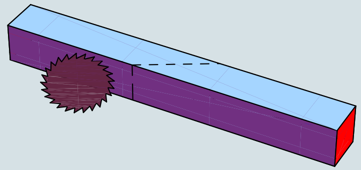
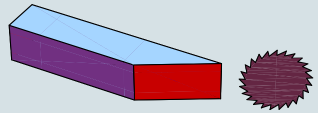
Mathematical Art by Koos Verhoeff



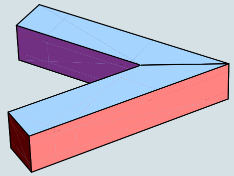
Miter Joints



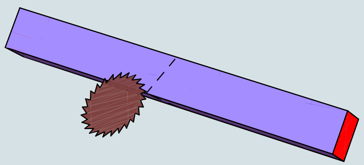
intact beam



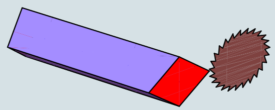
beveled at 30°



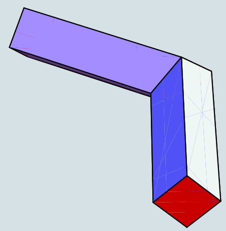
60° miter joint



intact beam
(rolled 45°)



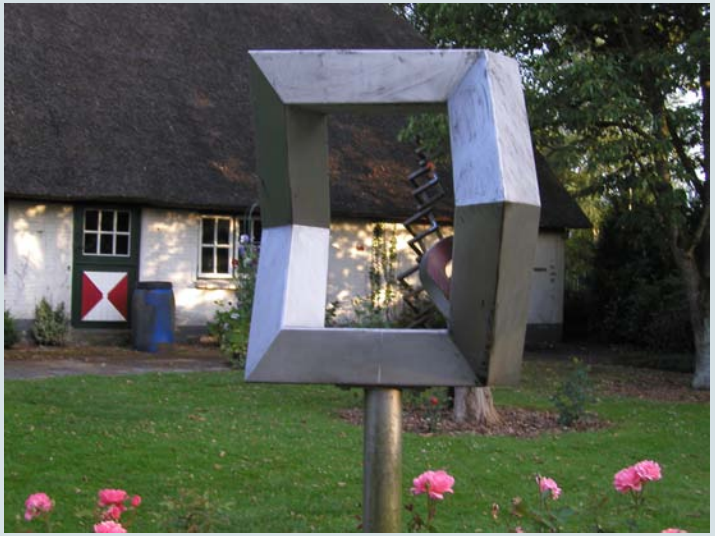
beveled at 60°

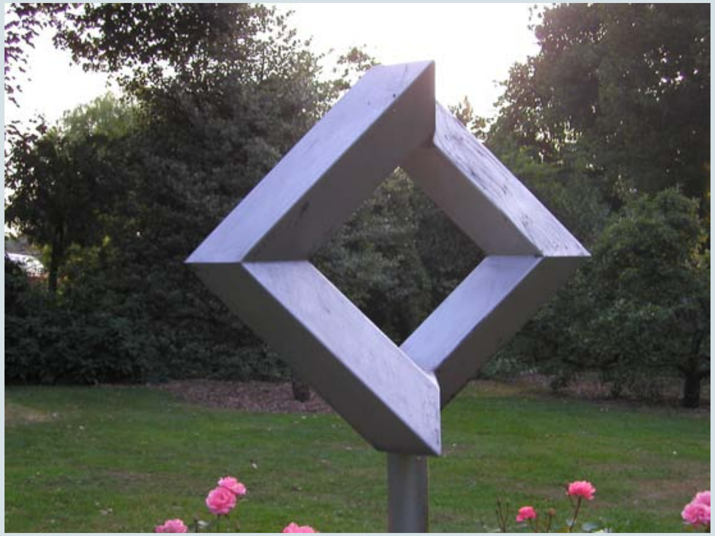


120° miter joint

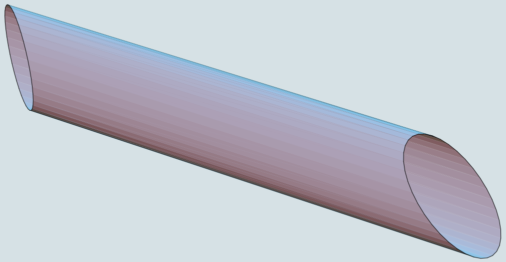
Mathematica Demonstrations Project: *Miter Joint and Fold Joint*

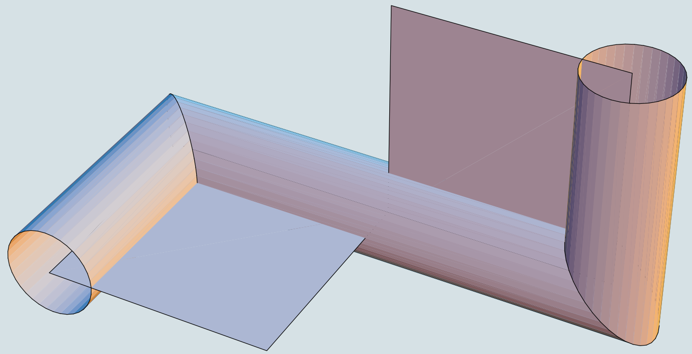
‘Kliekje’ (Eng.: ‘Left over’)





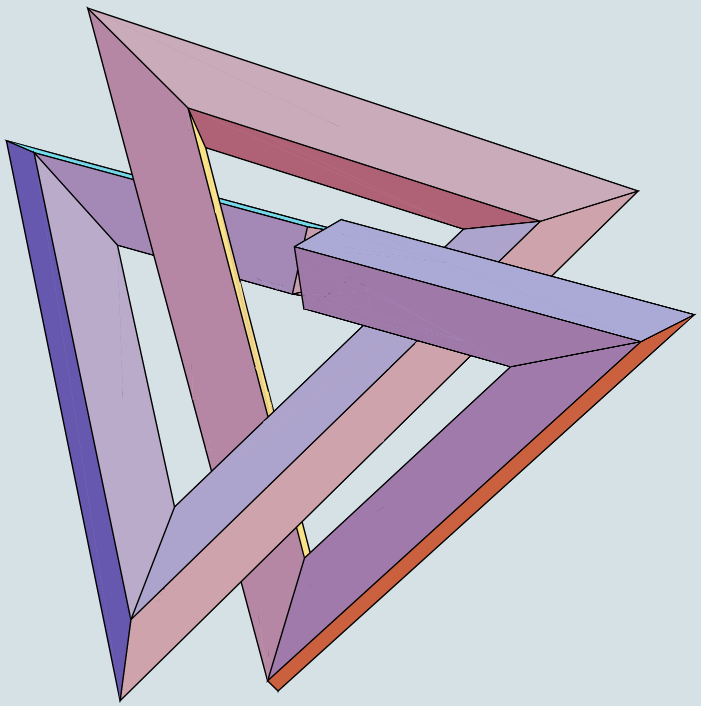
Spatial Mitering



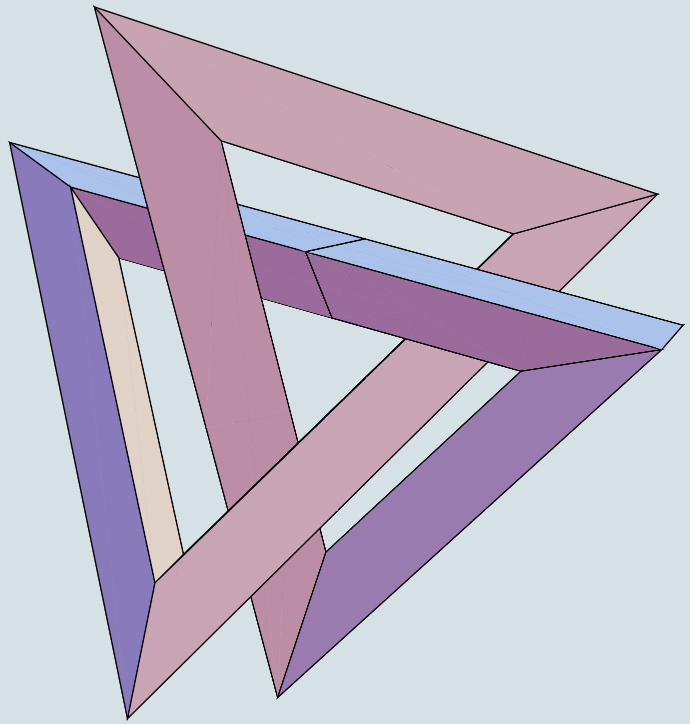


- **Corner plane** = plane spanned by adjacent segments
- **Torsion angle** = dihedral angle between adjacent corner planes

Closing the 3D Path



Square Cross Section



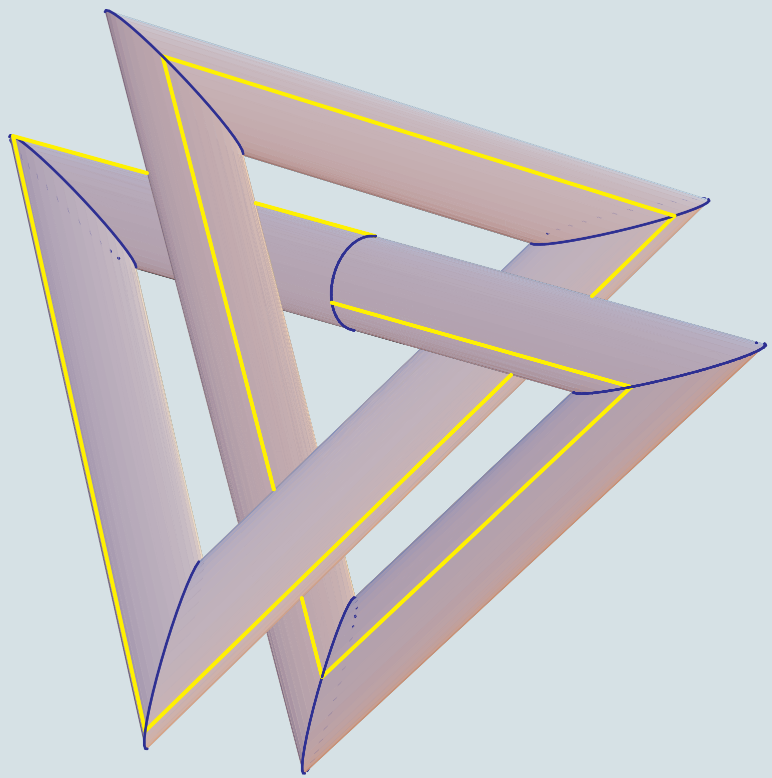
Triangular Cross Section

Mathematica Demonstrations Project: *Mitering a Closed 3D Path*

Miter Joint Rotation Invariance Theorem

Total amount of torsion is inherent property of polygonal path and does *not* depend on

- choice of initial segment
- initial rotation of cross section about center line
- shape of cross section

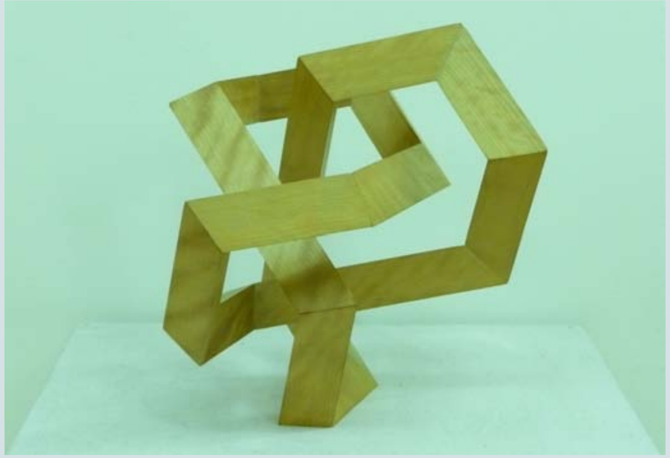


Mitering matches $\Longleftrightarrow$ total torsion is symmetry of cross section

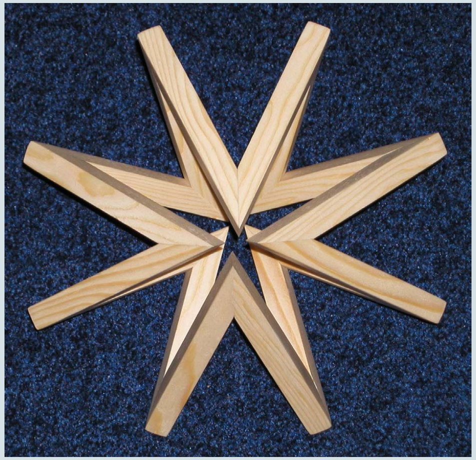
Three Techniques to Tackle Torsion

‘Tinkering’

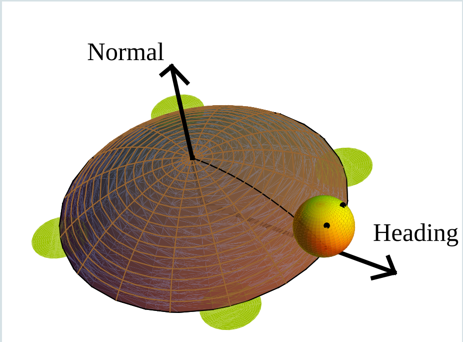
Lattice walking



Constant torsion



3D Turtle Geometry



State :

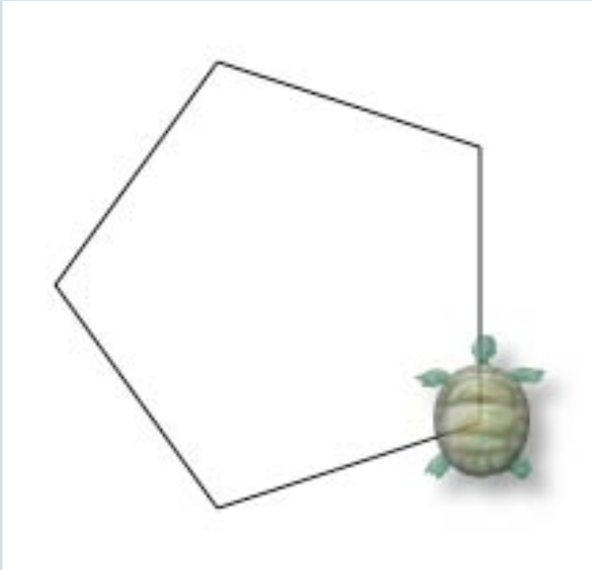
- **Position** in space
- **Attitude** = (heading vector, normal vector)

Commands :

- $Move(d)$: move distance d in direction of heading
- $Turn(\varphi)$: turn clockwise by angle φ about normal
- $Roll(\psi)$: roll clockwise by angle ψ about heading

Mathematica Demonstrations Project: *3D Flying Pipe-laying Turtle*

Regular 2D Polygons



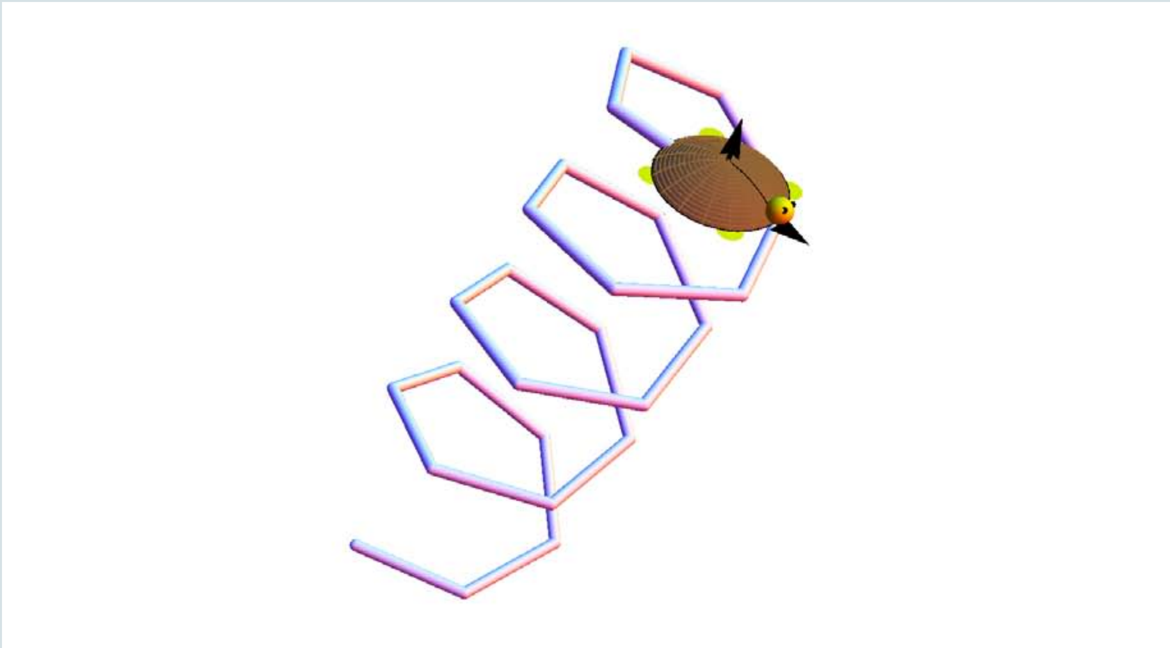
- All edge lengths are equal: $\mathit{Move}(d)$
- All corner angles are equal: $\mathit{Turn}(360°/N)$

Logo program: `Repeat N [ Forward 100 Left 360 / N ]`

Generalization to 3D

In 3D: roll angles provide additional freedom

All roll angles equal — $Roll(\psi)$ — yields a helix, which never closes



Added requirement: all torsion angles are equal in *absolute value*

Constant Torsion Polygons

Define

$$\mathit{Segment}(d, \psi, \varphi) \;=\; \mathit{Move}(d)\ ;\ \mathit{Roll}(\psi)\ ;\ \mathit{Turn}(\varphi)$$

Regular path: path produced by sequence of $\mathit{Segment}(d_i, \psi_i, \varphi_i)$ with all $d_i = d > 0$ and all $\varphi_i = \varphi$ for $0 < \varphi < 180°$

Constant-torsion (CT) path: all $d_i > 0$, $0 < \varphi_i < 180°$, and $|\psi_i| = \psi$

3D Polygon: path produced by *properly closed* turtle program

Turtle program is **properly closed** when turtle returns to initial state (both initial position *and* initial attitude)

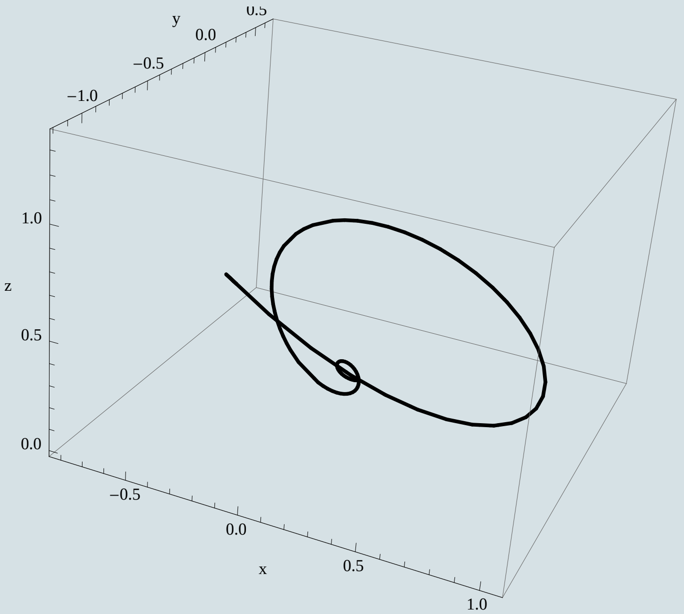
Regular CT polygon is determined by d, ψ, φ and sequence of roll *signs* (N.B. d is only a scale factor; w.l.o.g. assume $d = 1$)

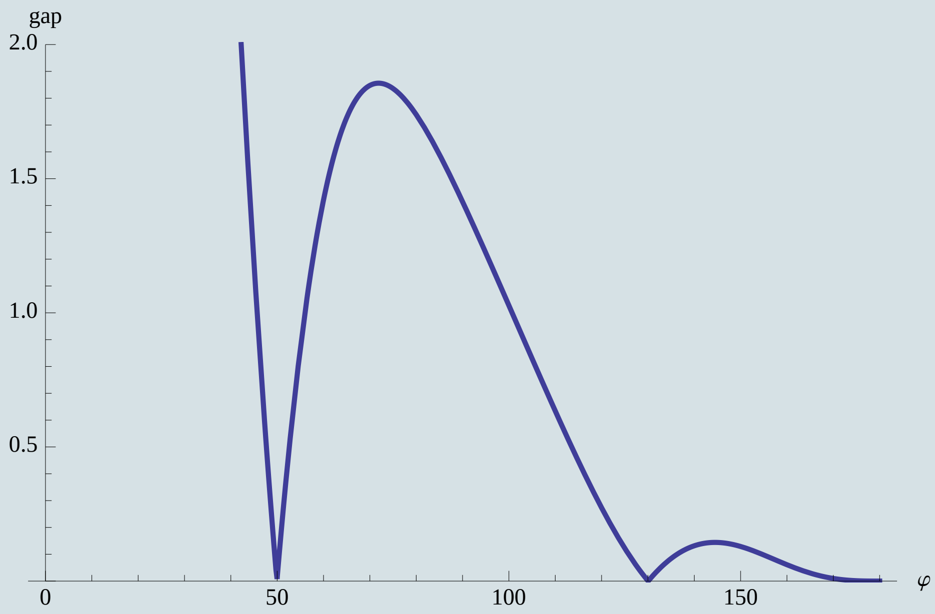
Existence and Construction

Existence of sign sequence and values for angles ψ, φ not evident

Method: Choose signs and one of ψ, φ, then determine other angle

Movie: Given sign sequence $(++--)^4$, $\psi = 90°$, determine φ for closure

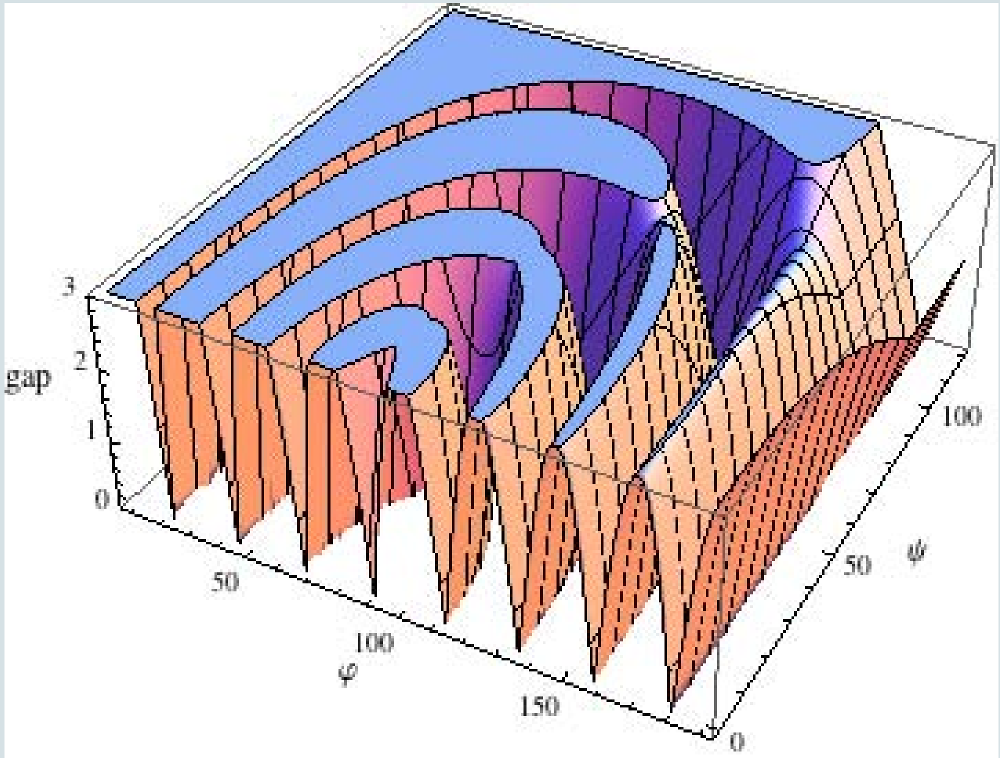




ϕ-ψ Landscapes

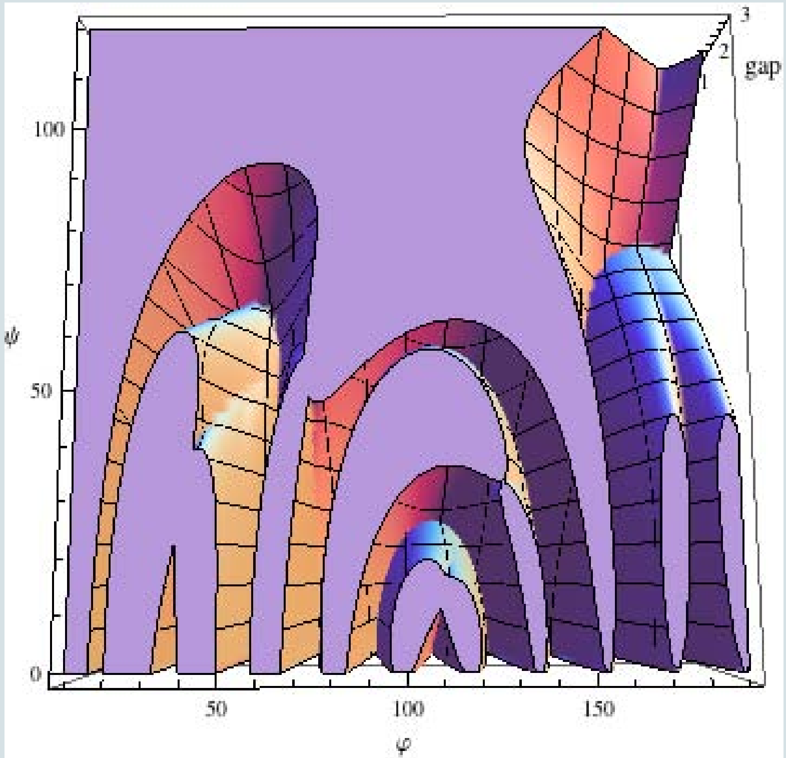
++--++--++--++--

16 segments



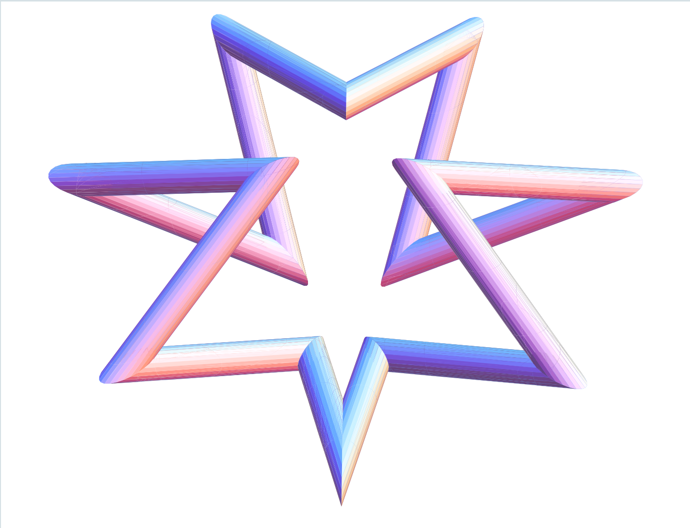
++-++--+--++-++--+--

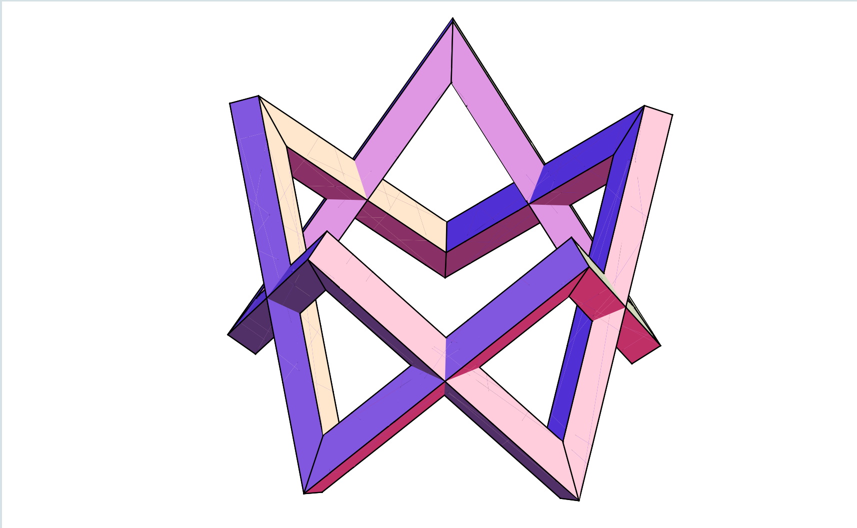
20 segments



Some Observations about Regular CT Polygons

Closed regular CT path is not necessarily a regular CT polygon
$\psi = 90°$, $\varphi = 120°$, sign sequence +--++--++--++-
(In 2D: closed regular $\Rightarrow$ properly closed)





Regular CT polygon can be self-intersecting
$\psi = 90°$, $\varphi = 112.456°$, signs $(+-)^5$

Some Theorems about Regular CT Polygons

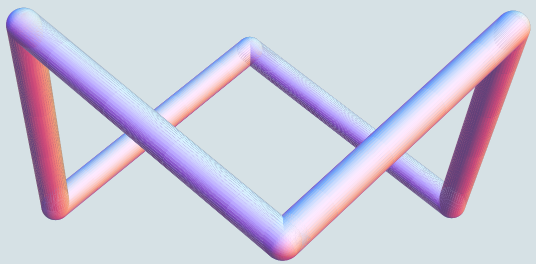
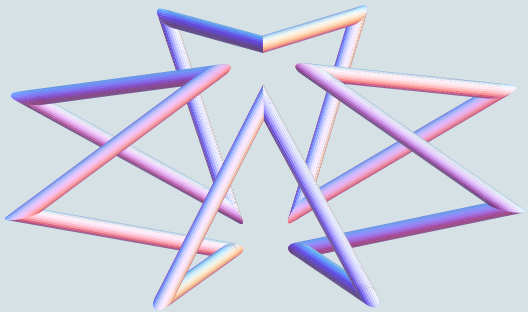
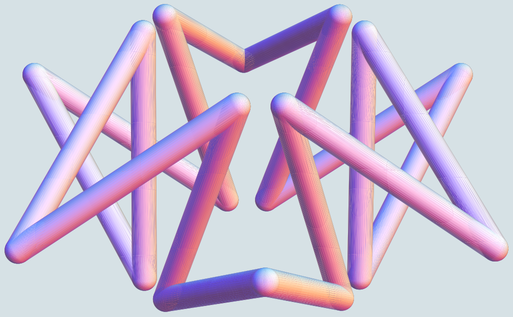
Distance between vertices k edges apart is constant, for $k = 1, 2, 3$

Total torsion $= \sum_{i=1}^{n} \psi_i \equiv 0 \pmod{\psi}$

Corollary: Mitering matches $\Longleftrightarrow$ ψ is symmetry of cross section

For square cross section, $\psi = 90^\circ$ is practical choice

Some Infinite Families of Regular CT Polygons

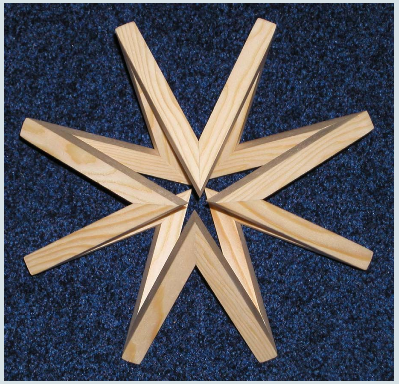
- Alternating signs $(+-)^n$: crowns, vertices in two layers
- $(++--)^n$: vertices in three layers
- $(+++---)^n$: vertices in four layers

Existence and values of angles ψ, φ not obvious

Artwork

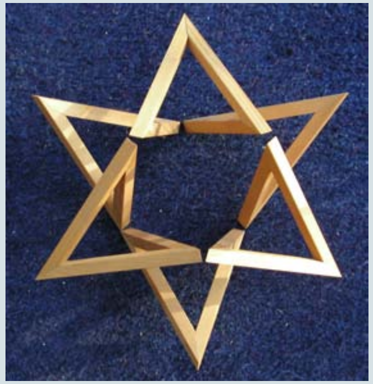
$(++--)^3$

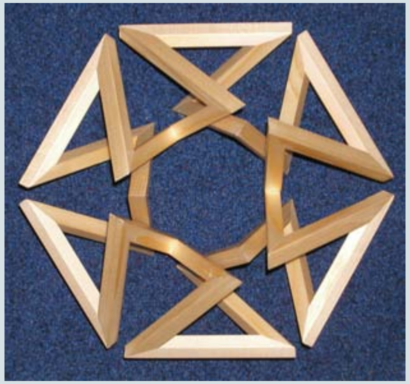
$(++--)^4$



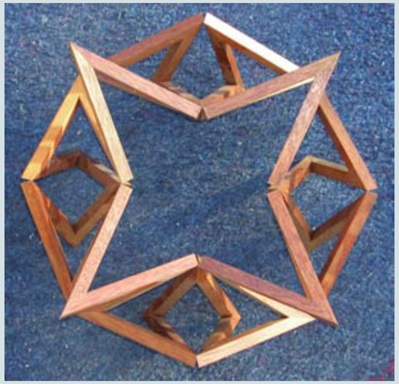
$(++--)^5$

$(+++---)^3$





$(++-++--+--)^3$



$(+++--++---)^4$

$(+++---)^4$

Conclusion

- Definition of (regular) 3D polygons of constant torsion
- Some characteristics
- Some constructions
- Some artwork based on regular CT polygons

Open problems:

- Complete characterization (easy in 2D)
- Are there knotted regular CT polygons? With $\psi = 90°$?
- Is a Möbius twist possible: total torsion $\neq 0 \pmod{360°}$

Questions?

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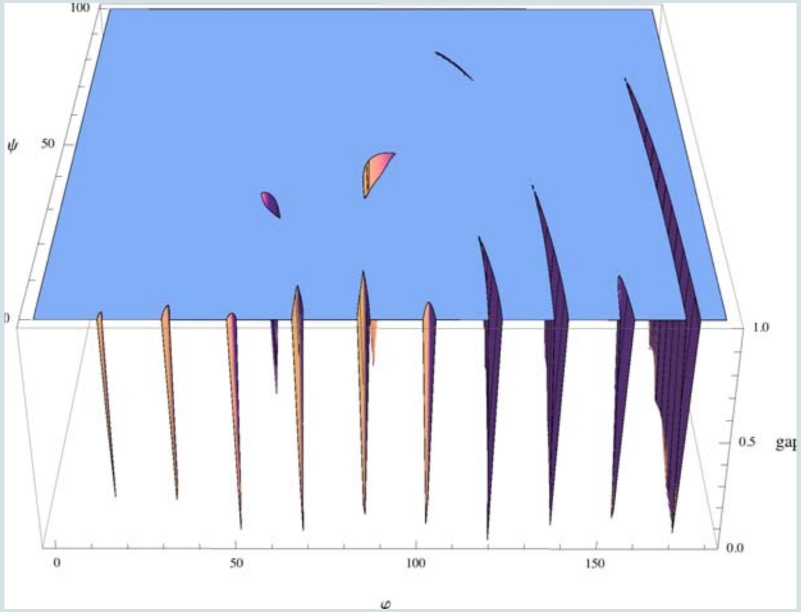
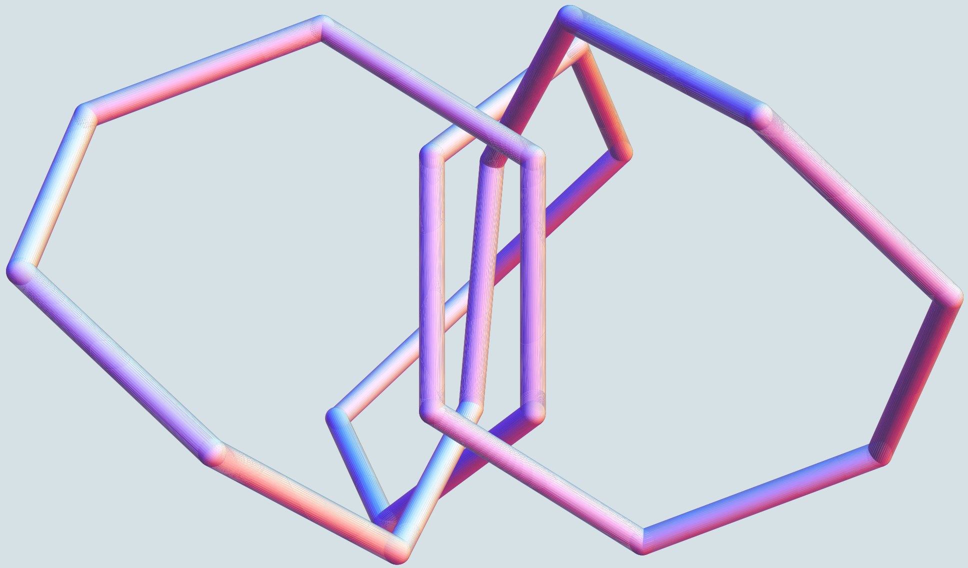


$\varphi = 58.8^\circ,\ \psi = 32.05^\circ$



RCT Trefoil Knot (21 segments)