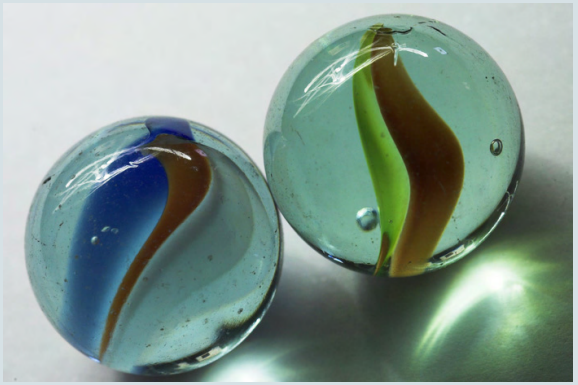
# Decisions and Games

Lunch Lecture for *W.I.S.V. Christiaan Huygens* at TU Delft
28 November 2008



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## First Game: Choosing Digits

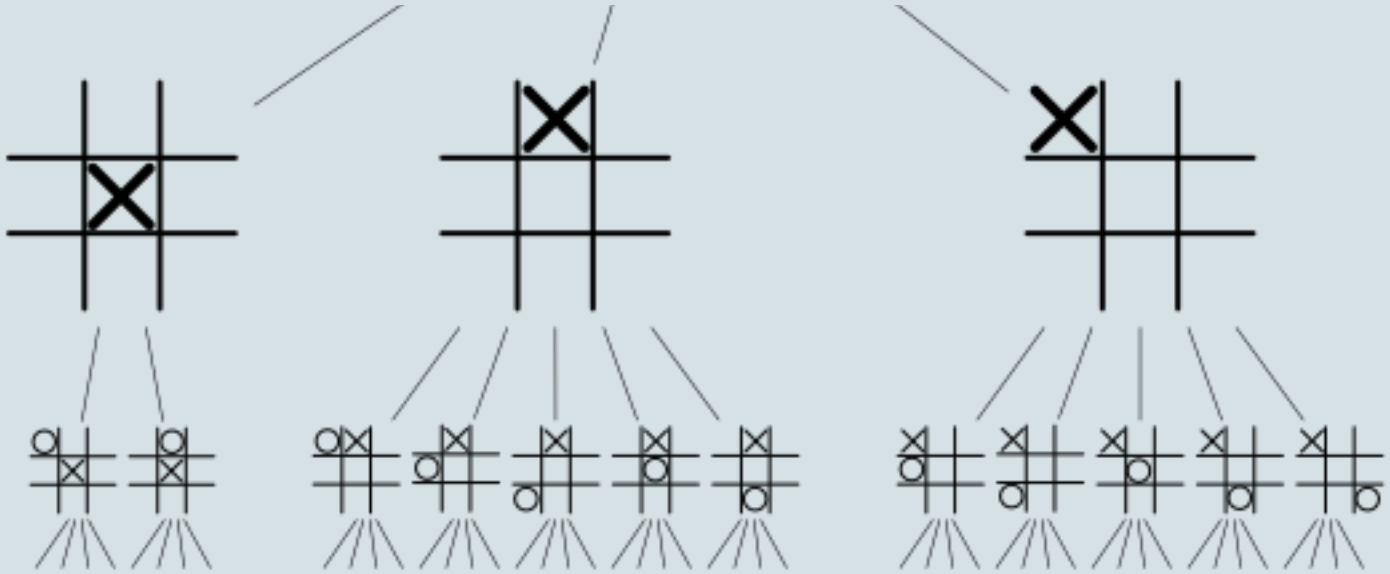

Two players: Choose a digit, taking turns.

Earlier chosen digits are no longer available.

**Goal**: Having chosen *three* digits that sum to **15**.

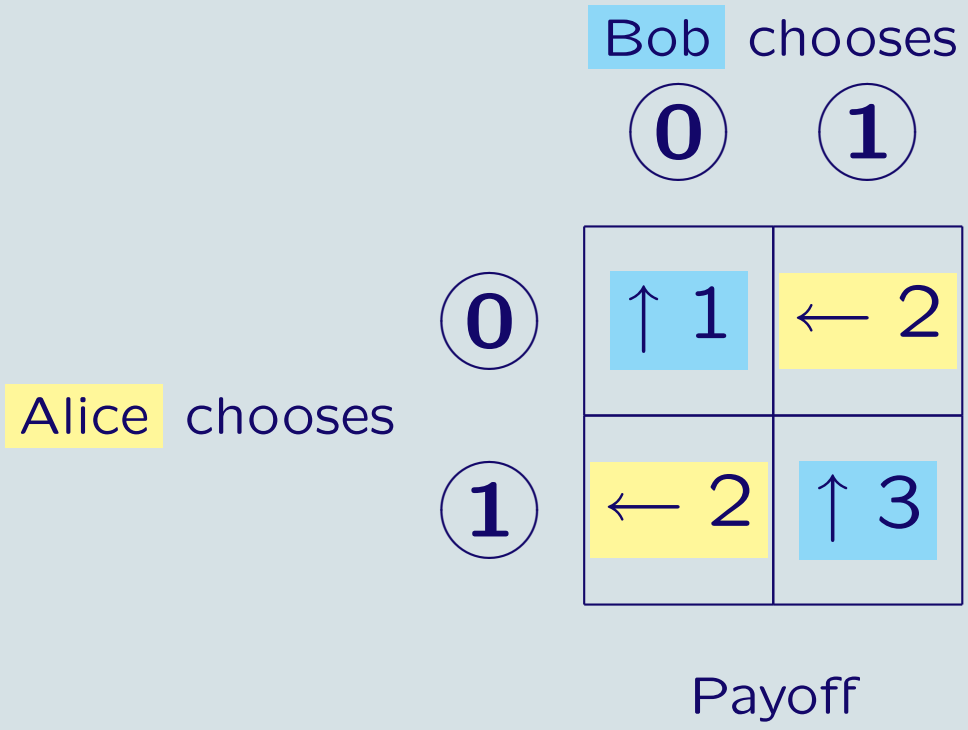
# Logical Combinations

| | | |
|---|---|---|
| 6 | 1 | 8 |
| 7 | 5 | 3 |
| 2 | 9 | 4 |



Analysis can be extremely hard!

## Second Game: Choosing Heads or Tails



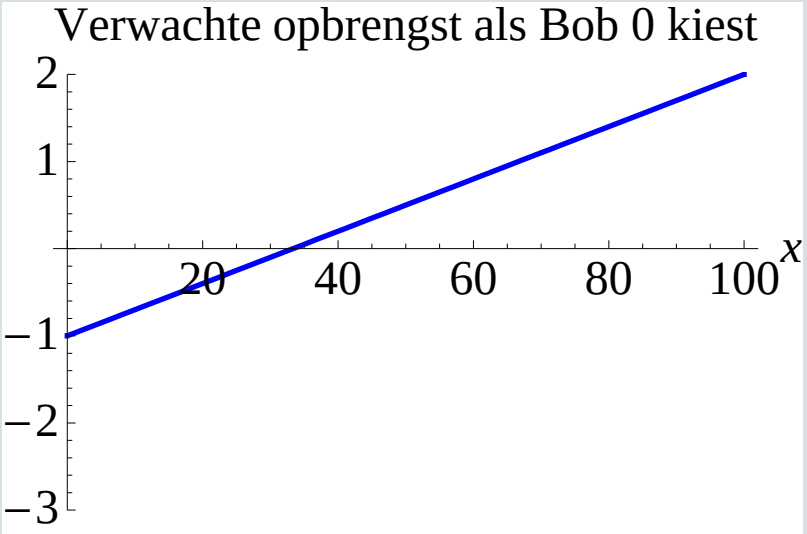

Two players: Each chooses **0** (head) or **1** (tail), *secret* the other.

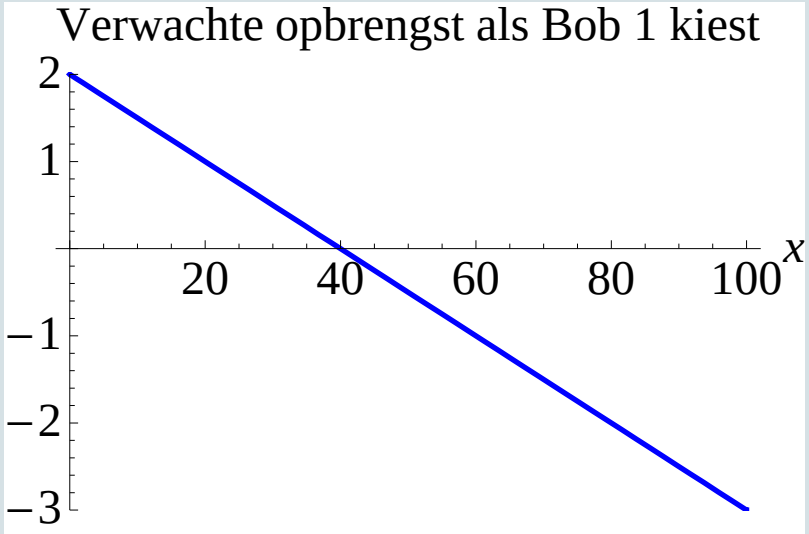
**Goal**: Maximize total payoff under repeated play.

## Strategic Bluffing: The Mixed Strategy

Analysis for Alice: she chooses 1 with probability $x$%.

Alice wants to determine optimal $x$, not knowing Bob's choice.

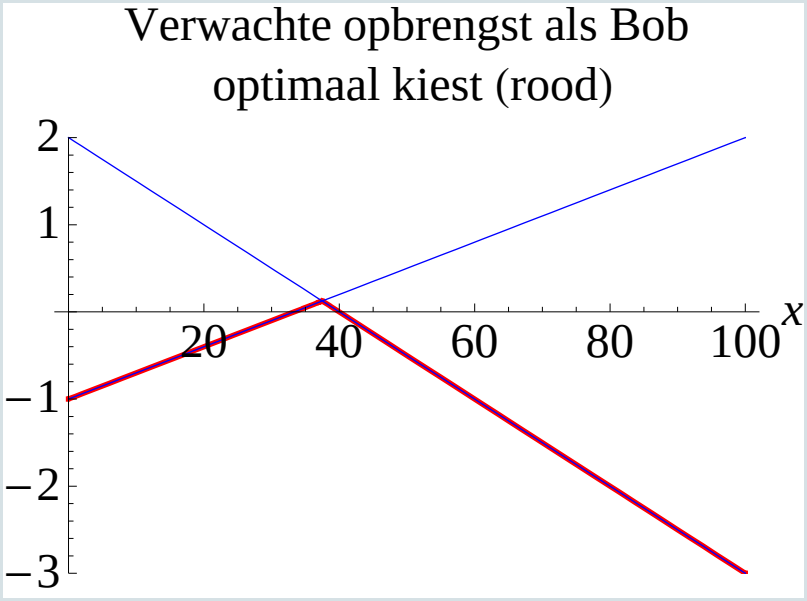


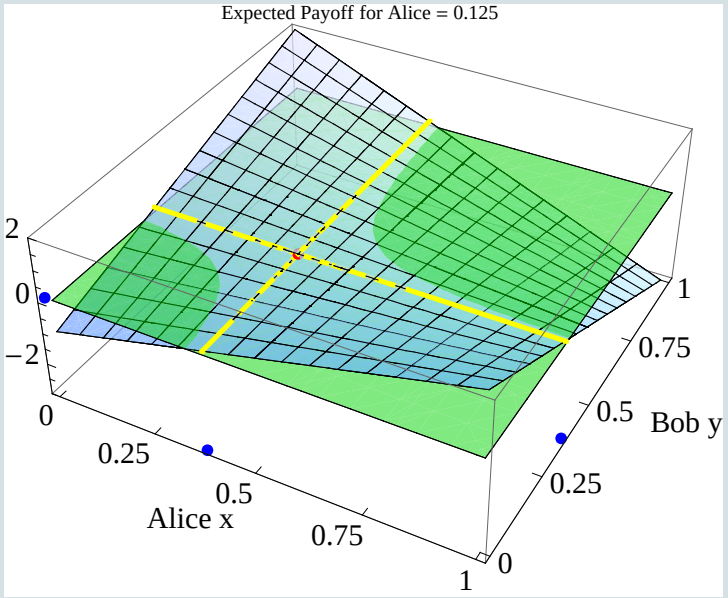




Expected payoff $= +0,5$
voor $x = 50\%$

Expected payoff $= -0,5$
voor $x = 50\%$

# Strategic Bluffing: The Nash Equilibrium




Optimal for Alice:

choose 0 with probability 5/8 & choose 1 with probability 3/8

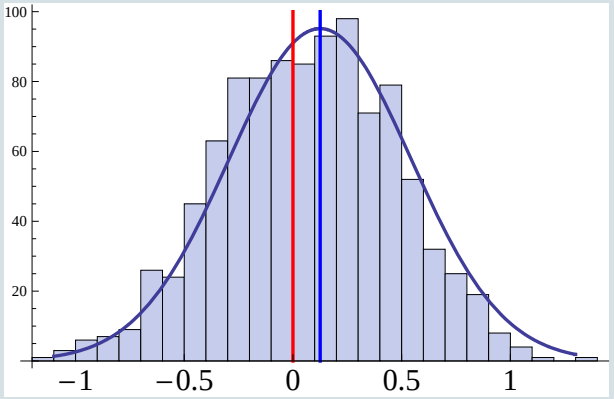
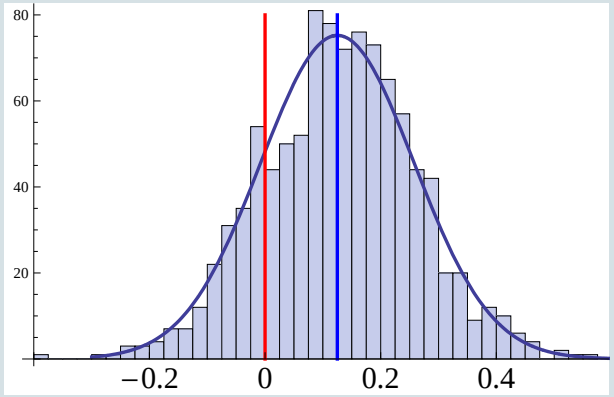
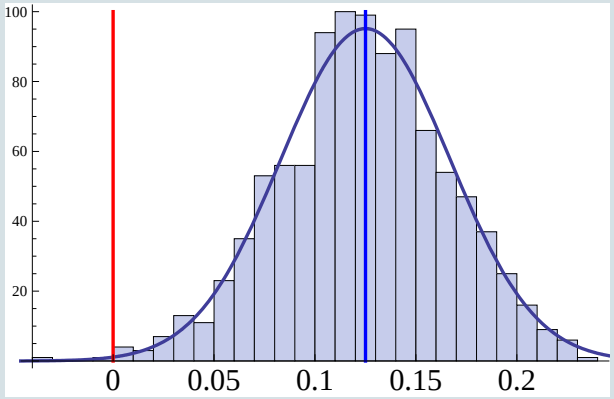
Expected payoff for Alice: +0,125 or $6\frac{1}{4}$% of 2

Moreover, this is independent of Bob's choice.

# The Effect of Variance

Expected payoff $\mu = 0,125$ with standard deviation of $\sigma = 1,884$

Three times 1000 experiments of N repetitions each

| N = 20 | N = 200 | N = 2000 |
|---|---|---|
|  |  |  |
| $\sigma = 0,42$ | $\sigma = 0,13$ | $\sigma = 0,04$ |
| 38% chance to lose | 17% chance to lose | 0,1% chance to lose |

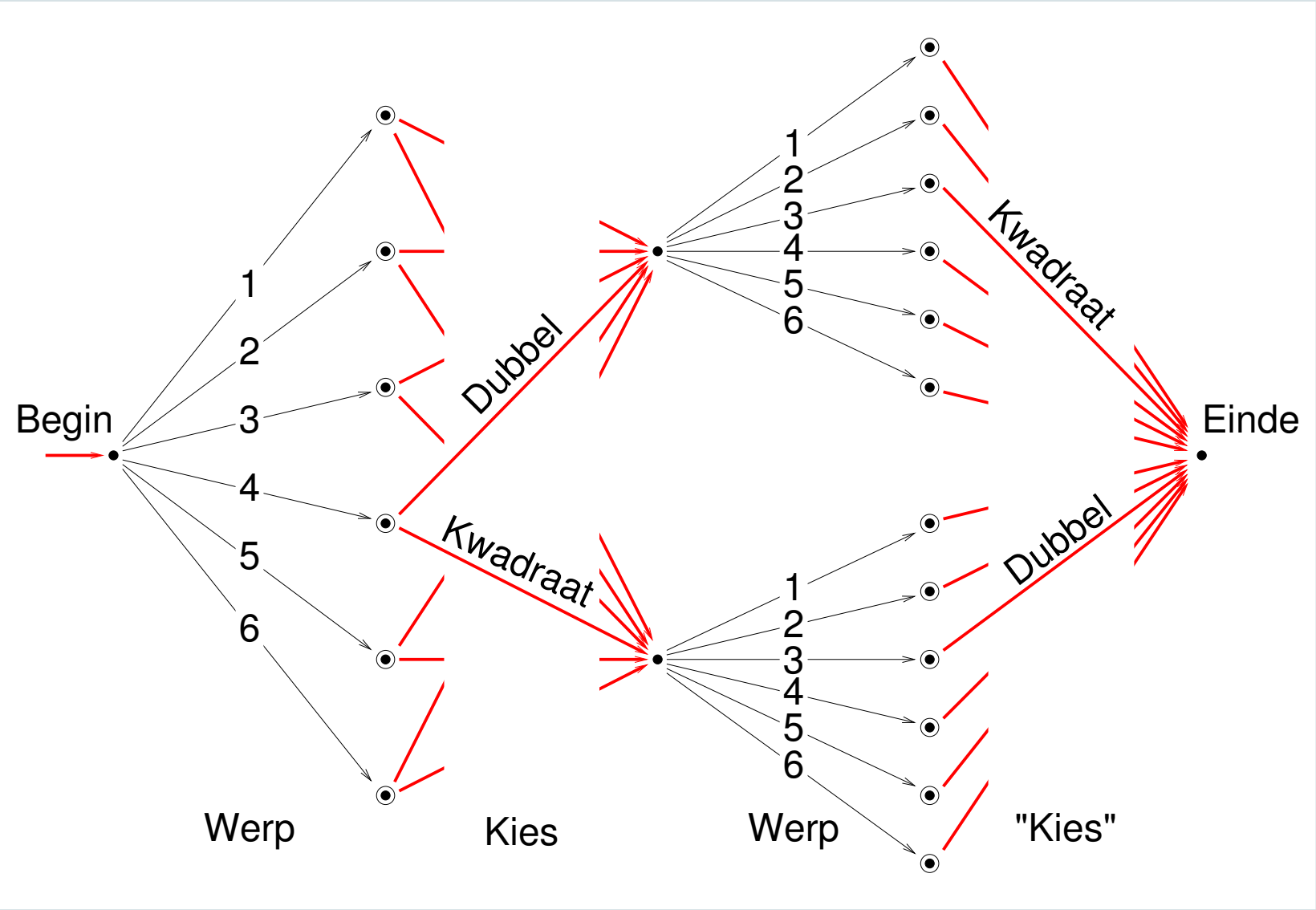
# Third Game: Choosing Categories

| Category | Score |
|---|---|
| *Double* (value + value) | . . . |
| *Square* (value × value) | . . . |
| *Total* | . . . |

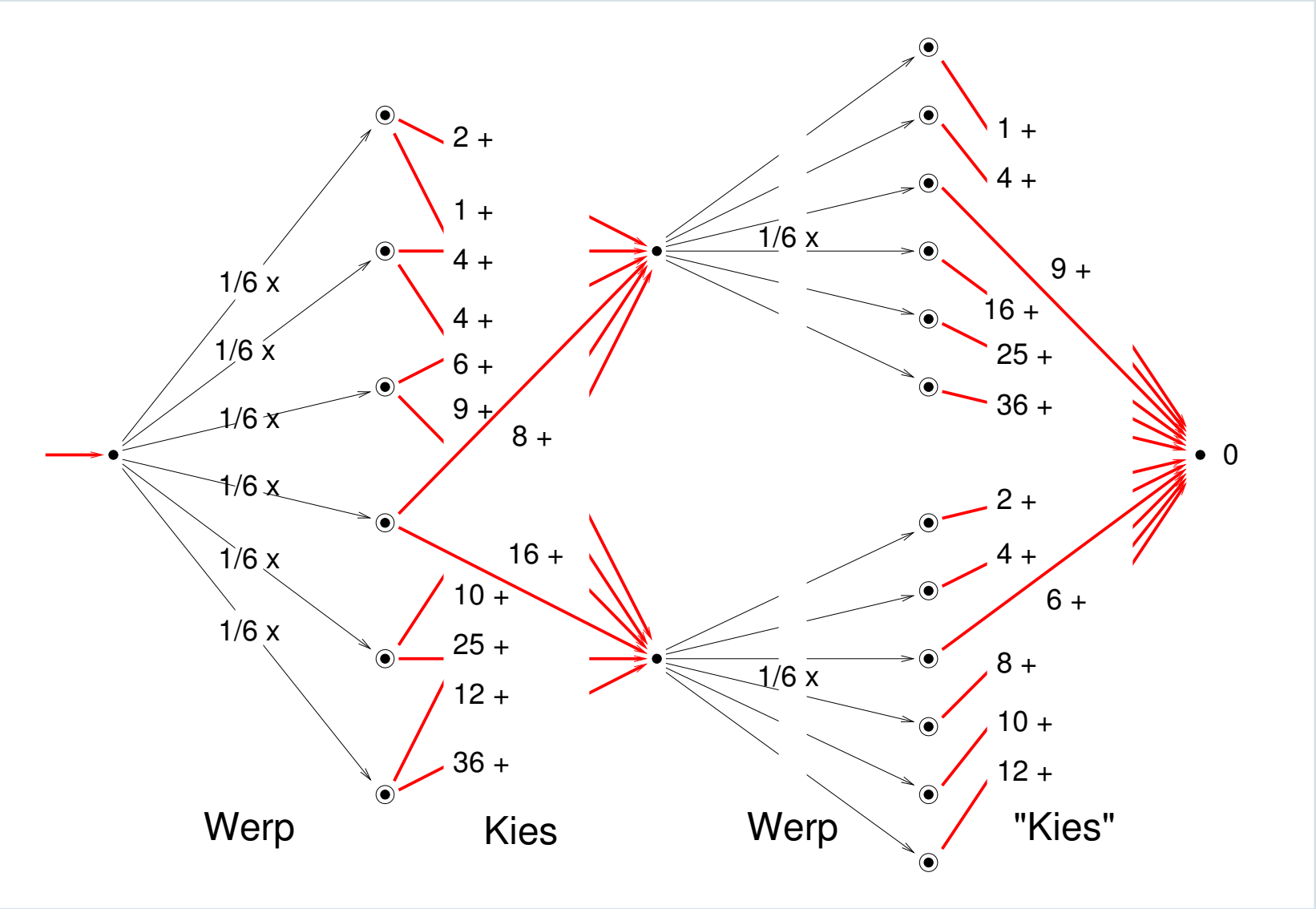
One player: Roll dice, choose *unscored* category, repeat.

**Goal**: Maximize total score under repeated play.

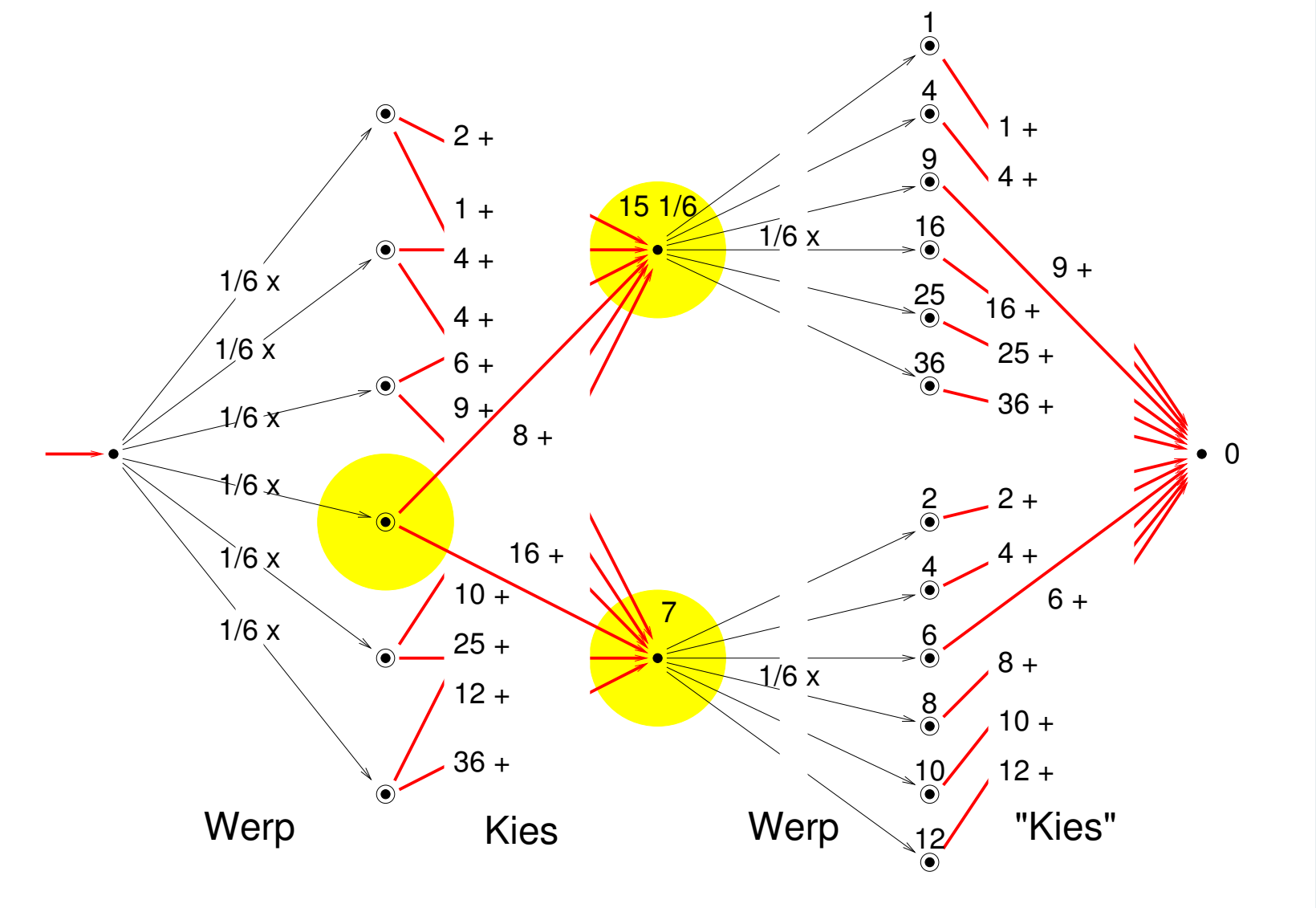
# Weighing Chances: MicroYahtzee Game Graph



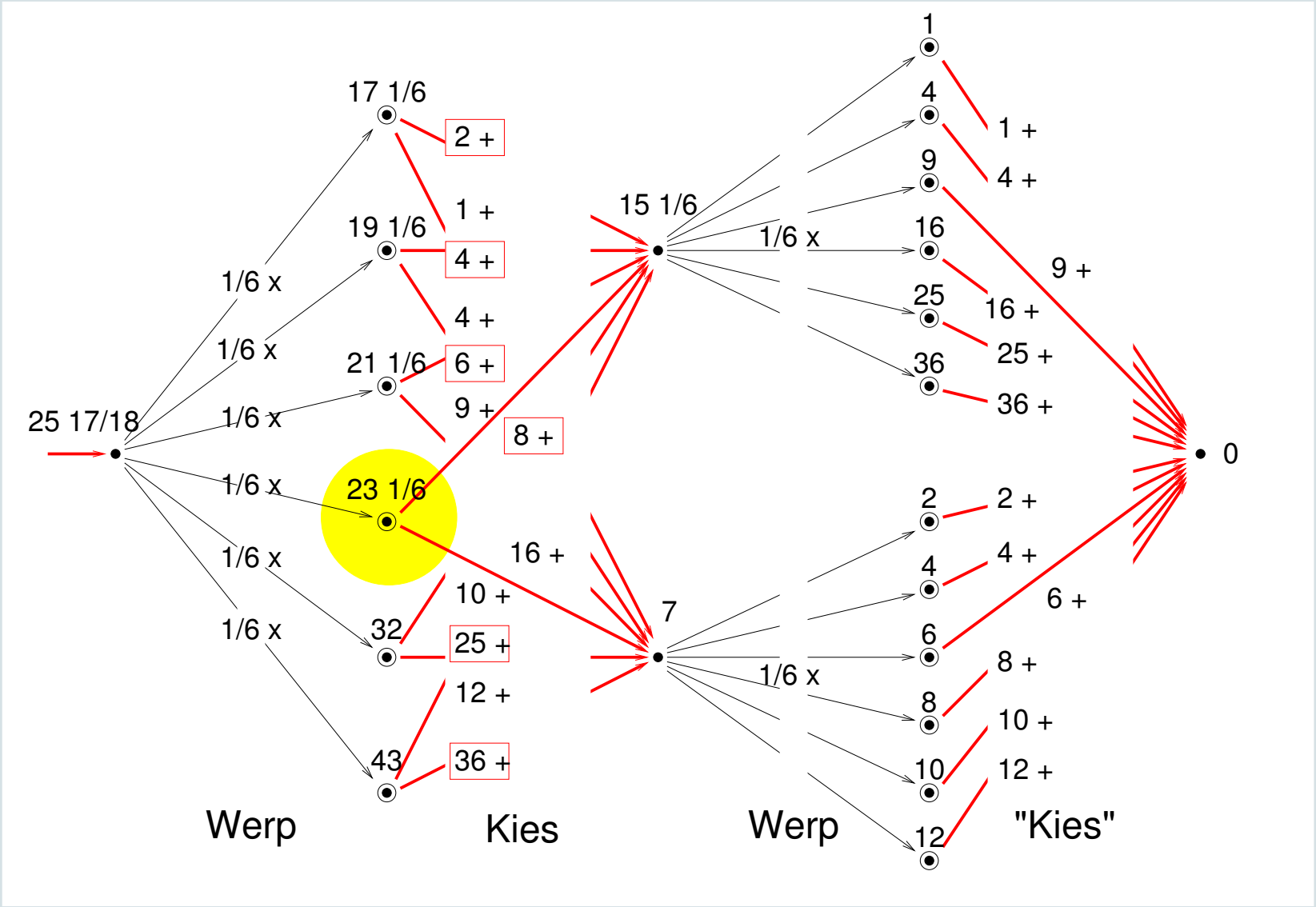
# Weighing Chances: MicroYahtzee Expected Scores



# Weighing Chances: MicroYahtzee Optimal Scores



# Weighing Chances: MicroYahtzee Dilemma Resolved

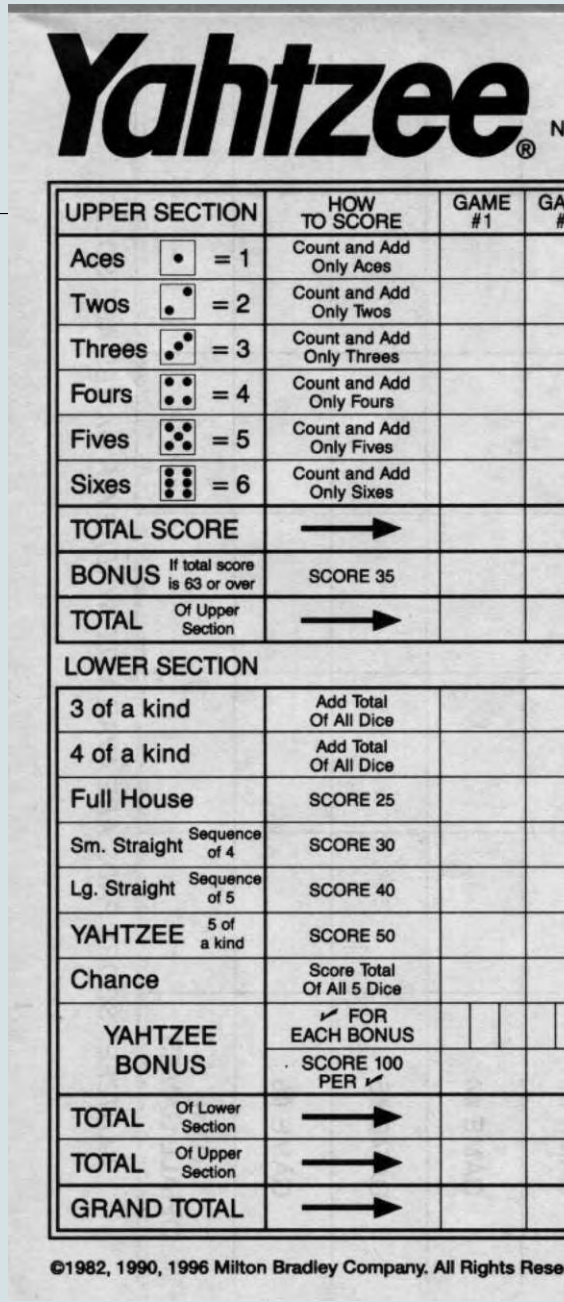


## Weighing Chances: 'Real' Yahtzee

- Score card with 13 categories and 5 dice
- Choose 39×: more than $10^9$ game states
- Expected total score under optimal play: $254, \cdots$ [Verhoeff, 1999]

  But variance is high: $\sigma = \pm 60$ (70% in 200–300)
- On-line advice and practicing:

  `www.win.tue.nl/~wstomv/misc/yahtzee`



Yahtzee®

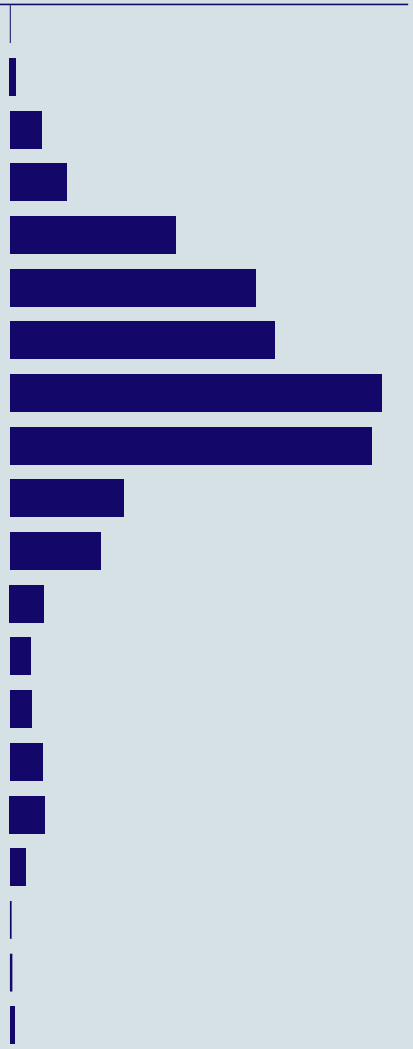
| UPPER SECTION | HOW TO SCORE | GAME #1 | GA # |
|---|---|---|---|
| Aces = 1 | Count and Add Only Aces | | |
| Twos = 2 | Count and Add Only Twos | | |
| Threes = 3 | Count and Add Only Threes | | |
| Fours = 4 | Count and Add Only Fours | | |
| Fives = 5 | Count and Add Only Fives | | |
| Sixes = 6 | Count and Add Only Sixes | | |
| TOTAL SCORE | → | | |
| BONUS If total score is 63 or over | SCORE 35 | | |
| TOTAL Of Upper Section | → | | |
| LOWER SECTION | | | |
| 3 of a kind | Add Total Of All Dice | | |
| 4 of a kind | Add Total Of All Dice | | |
| Full House | SCORE 25 | | |
| Sm. Straight Sequence of 4 | SCORE 30 | | |
| Lg. Straight Sequence of 5 | SCORE 40 | | |
| YAHTZEE 5 of a kind | SCORE 50 | | |
| Chance | Score Total Of All 5 Dice | | |
| YAHTZEE BONUS | ✓ FOR EACH BONUS | \| \| \| | |
| | SCORE 100 PER ✓ | | |
| TOTAL Of Lower Section | → | | |
| TOTAL Of Upper Section | → | | |
| GRAND TOTAL | → | | |

# Optimal Solitaire Yahtzee: Final Scores per Category

| Category | E | SD | % 0 |
|---|---|---|---|
| *Aces* | 1.88 | 1.22 | 10.84 |
| *Twos* | 5.28 | 2.00 | 1.80 |
| *Threes* | 8.57 | 2.71 | 0.95 |
| *Fours* | 12.16 | 3.29 | 0.60 |
| *Fives* | 15.69 | 3.85 | 0.50 |
| *Sixes* | 19.19 | 4.64 | 0.53 |
| *U. S. Bonus* | 23.84 | 16.31 | 31.88 |
| *Three of a Kind* | 21.66 | 5.62 | 3.26 |
| *Four of a Kind* | 13.10 | 11.07 | 36.34 |
| *Full House* | 22.59 | 7.38 | 9.63 |
| *Small Straight* | 29.46 | 3.99 | 1.80 |
| *Large Straight* | 32.71 | 15.44 | 18.22 |
| *Yahtzee* | 16.87 | 23.64 | 66.26 |
| *Chance* | 22.01 | 2.54 | 0.00 |
| *Extra Y. Bonus* | 9.58 | 34.08 | 91.76 |
| *GRAND TOTAL* | 254.59 | 59.61 | 0.00 |
| *Yahtzees Rolled* | 0.46 | 0.69 | 63.24 |
| *Jokers Applied* | 0.04 | 0.19 | 96.30 |

# Optimal Solitaire Yahtzee: Distribution of Final Score

| Score range | % | Cum.% |
|---|---|---|
| 100 – 119 | 0 % | 0 % |
| 120 – 139 | 0 % | 0 % |
| 140 – 159 | 2 % | 2 % |
| 160 – 179 | 3 % | 5 % |
| 180 – 199 | 9 % | 14 % |
| 200 – 219 | 13 % | 27 % |
| 220 – 239 | 14 % | 41 % |
| 240 – 259 | 20 % | 60 % |
| 260 – 279 | 19 % | 80 % |
| 280 – 299 | 6 % | 86 % |
| 300 – 319 | 5 % | 90 % |
| 320 – 339 | 2 % | 92 % |
| 340 – 359 | 1 % | 93 % |
| 360 – 379 | 1 % | 94 % |
| 380 – 399 | 2 % | 96 % |
| 400 – 419 | 2 % | 98 % |
| 420 – 439 | 1 % | 99 % |
| 440 – 459 | 0 % | 99 % |
| 460 – 479 | 0 % | 99 % |
| 480 – 499 | 0 % | 99 % |



Results based on **simulation** of $10^5$ games

## Optimal Solitaire Yahtzee: Game with Minimum Score

| Turn | Third Roll | Score | in Category |
|---|---|---|---|
| 1 | 1 4 4 5 5 | 1 | *Aces* |
| 2 | 1 2 3 5 5 | 2 | *Twos* |
| 3 | 1 1 2 2 6 | 0 | *Four of a Kind* |
| 4 | 1 2 2 4 6 | 0 | *Yahtzee* |
| 5 | 1 1 2 2 6 | 0 | *Threes* |
| 6 | 1 2 2 3 3 | 0 | *Fours* |
| 7 | 1 2 2 3 3 | 0 | *Fives* |
| 8 | 1 2 2 3 3 | 0 | *Full House* |
| 9 | 1 2 2 3 3 | 0 | *Sixes* |
| 10 | 1 1 2 3 3 | 0 | *Large Straight* |
| 11 | 1 1 2 2 3 | 9 | *Chance* |
| 12 | 4 5 5 6 6 | 0 | *Three of a Kind* |
| 13 | 5 6 6 6 6 | 0 | *Small Straight* |
| | | 12 | *GRAND TOTAL* |

# Three Kinds of Situations for Making Decisions

Logical combinations (complete information)

Strategic bluffing (secrets)

Weighing chances (fortune)

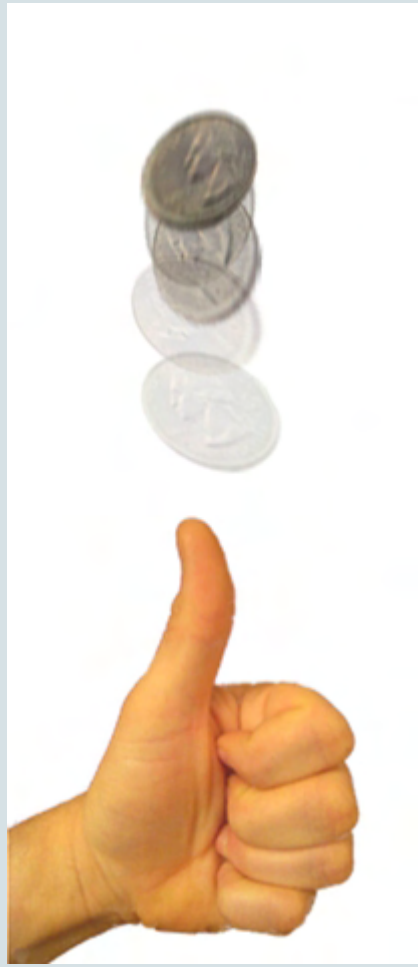
## Wise Lessons

1. 'Playful' mathematical techniques may help at making good choices in diverse situations.

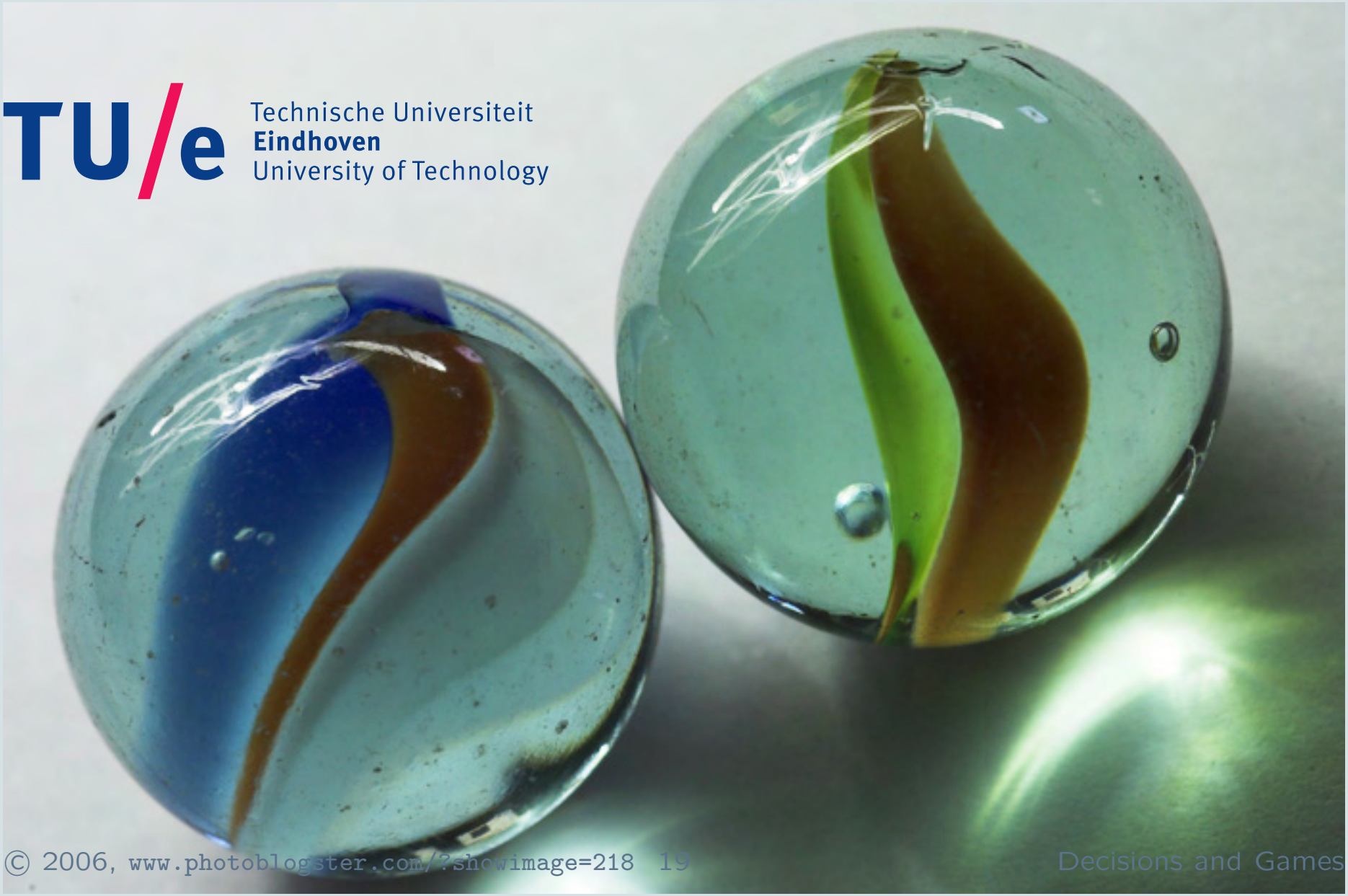
2. For repeated strategical choices it can be optimal to toss a (well-chosen) 'coin'
(cf. the so-called *Mixed Nash Strategy*, preventing predictability and exploitation).

3. For repeated tests of fortune it can be optimal to make a (well-chosen) fixed choice
(cf. the so-called *Markov Decision Processes*).

4. Large variance requires patience for increased certainty (due to the factor $1/\sqrt{N}$).



# Decide Playfully, Consult a Mathematician!



## Literature

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