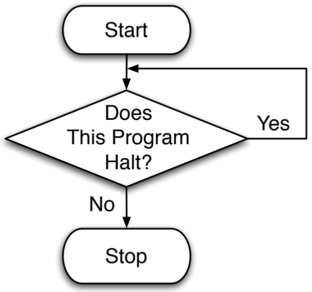
Infinity in Informatics: Endless Loops

Symposium 'Blik op ∞neindig'
18 November 2009 / *De Leidsche Flesch* / Leiden



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Motivations for Working on Scientific Problems

- Direct application in real life
- Foundation for working on other problems
- Aid to acquisition of knowledge and development of skills
- Fun and enjoyment

Informatics

What is Informatics?

Also known as Computing Science (UK) and Computer Science (US)

Branch of Mathematics?

Study of algorithms and related concepts

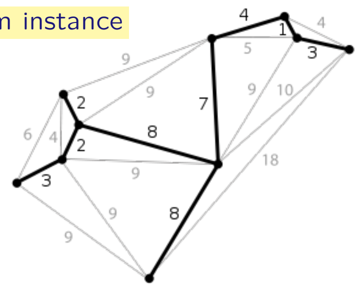
Actual infinity (finished) versus potential infinity (process)

Computational Tasks

Given an input determine an output satisfying the task specification

A valid task input is also called problem instance

Not just computations on *numbers*:



- Find a word in a (sorted) list
- Sort a list of words
- Find shortest path in a road network
- Find an *Euler* circuit or a *Hamiltonian* circuit in a graph
- Schedule jobs on machines to minimize processing time

The D-L-F Counting Problems

Given is a sequence of N characters in memory (an array A)

Efficiently count the number of occurrences of the character `L` when

1. Only characters `D` and `L` occur, no `L` left of `D`

   `D D D D D L L L L L L`

2. Only characters `D`, `L`, and `F` occur, no `L` left of `D`, no `F` left of `L` or `D`

   `D D D D D L L L L L L F F F F`

3. Only characters `D` and `L` occur, no `D` between `L`'s

   `D D D D D L L L L L L D D D D`

Algorithms

An algorithm is an *effective* solution method for a computational task:

- Given any valid input (problem instance)
- the algorithm must exactly prescribe what steps to perform,
- this must terminate after a finite number of steps, and
- produce a correct output (satisfying the task specification)

Input and output can be *abstract* entities, possibly infinitely many

One does not need to understand *why an algorithm works* and *how it was discovered* when applying it to solve given problem instances

(Computer) Programs

A mathematically precise algorithm can be executed by an automaton

A program is a sequence of instructions executable by a computer

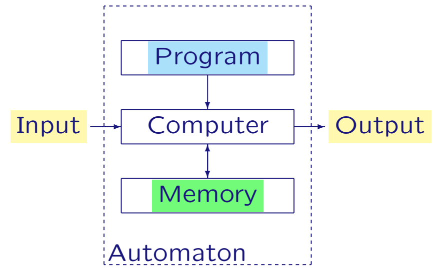
Many *instruction sets* or *programming languages* are possible

All *sufficiently rich* programming languages are universal and, hence, equivalent in terms of computational expressibility

- Not every program is an algorithm: programs need not terminate
- Every algorithm can be expressed by a suitable program*

*If the programming language is universal (Church–Turing Thesis)

Computer as Universal Automaton



Programming = rewriting algorithms into programs

Universal programming language:

Integer variables with statements: `inc(v)` `dec(v)` `while v do ... od`

Central Questions of Informatics

- What tasks are (not) algorithmically solvable?
- What tasks are efficiently solvable?
- What bounds are there on the efficiency for solving certain tasks?
- How to design efficient algorithms?
- How to reason about the correctness and efficiency of algorithms?

Termination of Computations: A Recent Success

- Start with finite sequence over $\{a, b, c\}$

  $bbaa$

- Repeatedly replace subsequences:

$$\begin{array}{rcl} aa & \to & bc \\ bb & \to & ac \\ cc & \to & ab \end{array}$$

Example:

$bb\underline{aa} \to b\underline{bb}c \to ba\underline{cc} \to b\underline{aa}b \to \underline{bb}cb \to a\underline{cc}b \to aa\underline{bb} \to a\underline{aa}c \to ab\underline{cc} \to abab$

Does this terminate for every start sequence?

Termination of Computations: Famous Open Problems

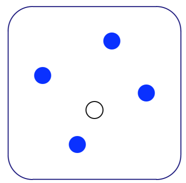
- If N even $\to$ $N/2$, if N odd $\to$ $3N+1$ (Collatz)

  $3 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1 \to 4 \to \cdots$

- While N is not a palindrome, add it to its reverse

  $152 \to 152 + 251 = 403 \to 403 + 304 = 707$ $\qquad$ $196 \to ?$

Marble Game 1

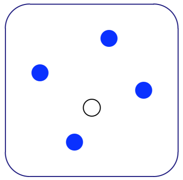


Each time take a marble:

- Remove it

Does this terminate? After how many steps?

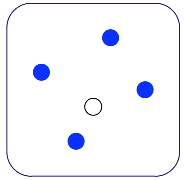
Marble Game 2



Repeatedly take a marble:

- If blue, then remove
- If white, then replace by one blue marble

Marble Game 3



Repeatedly take a marble:

- If blue, then remove it
- If white, then replace by *arbitrary* number of blue marbles

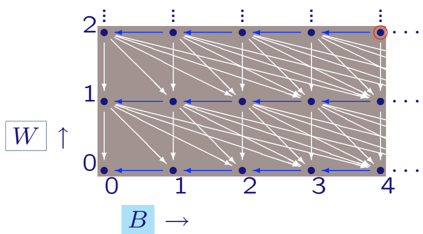
Marble Game Analysis

W white marbles B blue marbles

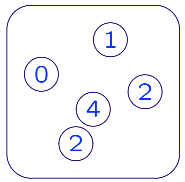
Game 1. Terminates after $f_1(W, B) = W + B$ steps

Game 2. Terminates after $f_2(W, B) = 2 * W + B$ steps

Game 3. Terminates after ten hoogste $f_3(W, B) = \omega * W + B$ steps



Lotto Ball Game



Repeatedly take an $\mathbb{N}$-numbered lotto ball:

- Replace by *arbitrary* number of balls with smaller numbers

  i.e. (n) replaced by $(<n)$ $(<n)$ $\cdots$ $(<n)$

How Large Is the Set of All Programs?

Every program is a text over some suitable enumerable alphabet $\mathcal{A}$

Not every text over $\mathcal{A}$ is program: it must be **syntactically correct** according to the rules of the **programming language**

A **compiler** checks the syntactical correctness of a text as a program (but not *semantical* correctness: whether the text is an *algorithm*)

Construct an enumeration* of all programs from an enumeration of all texts over $\mathcal{A}$ by deleting all texts that are not syntactically correct:

$$P_0,\ P_1,\ P_2,\ \ldots,\ P_i,\ \ldots$$

where P_i denotes the i-th program: $|\text{Programs}| = |\mathbb{N}|$

Every algorithm† appears in the sequence, not every P_i is an algorithm

*Each programming language gives rise to its own enumeration

†Provided the programming language is sufficiently expressive

How Large Is the Set of All Computational Tasks?

For $M \subseteq \mathbb{N}$, **decision problem** $(\mathbb{N}, M)$ is defined by

Input:	a natural number $n \in \mathbb{N}$
Output:	`YES` if $n \in M$
	`NO` if $n \notin M$

N.B. M is not an input of the problem, but a 'built-in' constant

Example: for *primality testing* take $M := \{2, 3, 5, 7, 11, 13, 17, 19, \ldots\}$

$|\text{Tasks}| \geq |\mathcal{P}(\mathbb{N})| = |\mathbb{N} \to \{\texttt{NO}, \texttt{YES}\}| > |\mathbb{N}|$

Intermezzo: The Power Set Consisting of All Subsets

For a set S, its **power set** $\mathcal{P}(S)$ consists of all subsets of S:

$$\mathcal{P}(S) = \{T \mid T \subseteq S\}$$

E.g. $\mathcal{P}(\{0,1\}) = \{\varnothing, \{0\}, \{1\}, \{0,1\}\}$

Attempt to match S and $\mathcal{P}(S)$:

$x \in S$	$\leftrightarrow$	subset $T_x \subseteq S$: $\bullet = y \in T_x$				
		a	b	c	d	$\cdots$
a	$\leftrightarrow$	$\bullet$	$\circ$	$\circ$	$\circ$	$\cdots$
b	$\leftrightarrow$	$\circ$	$\circ$	$\bullet$	$\circ$	$\cdots$
c	$\leftrightarrow$	$\bullet$	$\bullet$	$\circ$	$\circ$	$\cdots$
d	$\leftrightarrow$	$\circ$	$\circ$	$\bullet$	$\bullet$	$\cdots$
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
?	$\leftrightarrow$	$\circ$	$\bullet$	$\bullet$	$\circ$	$\cdots$

D = complement of diagonal

Intermezzo: A Set is Smaller Than Its Power Set (Cantor)

Diagonalization method: assume matching $\{x \leftrightarrow T_x \mid x \in S, T_x \subseteq S\}$

$$\begin{array}{rcl} D & = & \{x \in S \mid x \notin T_x\} \\ x \in D & \Leftrightarrow & x \in S \text{ and } x \notin T_x \qquad \text{for all } x \\ D \subseteq S & \text{and} & D \neq T_x \end{array}$$

Hence, $D \in \mathcal{P}(S)$ is not matched with any $x \in S$

Consequently, S is smaller than $\mathcal{P}(S)$: $|S| < |\mathcal{P}(S)|$, in particular

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$$

The power set of S is isomorphic to the set of mappings $S \to \{0,1\}$

$$f : S \to \{0,1\} \text{ corresponds to } \{x \in S \mid f(x) = 1\}$$

$\mathcal{P}(\mathbb{N})$ corresponds to the set of all $\mathbb{N}$-infinite $0,1$-sequences

Not All Problems $(\mathbb{N}, M)$ Are Decidable

Algorithm A solves decision problem $(\mathbb{N}, M)$ when

for any given $n \in \mathbb{N}$, A outputs `YES` if $n \in M$ and `NO` if $n \notin M$

$(\mathbb{N}, M)$ is called **decidable** when there exists an algorithm to solve it, and **undecidable** if no such algorithm exists

Because $|\mathcal{P}(\mathbb{N})| > |\mathbb{N}|$, there are more problems $(\mathbb{N}, M)$ than algorithms

There exist $M \subseteq \mathbb{N}$ such that $(\mathbb{N}, M)$ is undecidable

Decision Problem $(\mathbb{N}, DIAG)$

Define

$\boxed{DIAG = \{ i \in \mathbb{N} \mid \text{program } P_i \text{ does } \textit{not} \text{ output } \texttt{YES} \text{ on input } i \}}$

N.B. Each way of enumerating all programs, gives rise to its own set $DIAG$

Problem $(\mathbb{N}, DIAG)$ is undecidable, because

no program P_i implements an algorithm A that solves $(\mathbb{N}, DIAG)$:

A outputs `YES` on input i

$\Leftrightarrow$

$i \in DIAG$

$\Leftrightarrow$

P_i does *not* output `YES` on input i

Problems UNIV and HALT

More interesting problems:

UNIV (the **universal problem**)

Input: a program P and a natural number $i \in \mathbb{N}$

Output: `YES`, if P outputs `YES` on input i

`NO`, if P outputs `NO` or does not halt on input i

HALT (the **halting problem**)

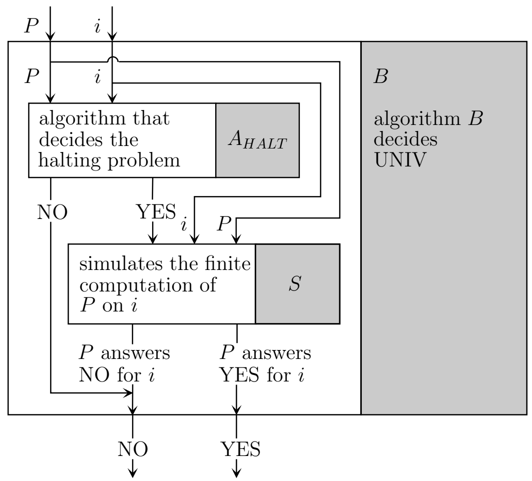
Input: a program P and a natural number $i \in \mathbb{N}$

Output: `YES`, if P halts on input i

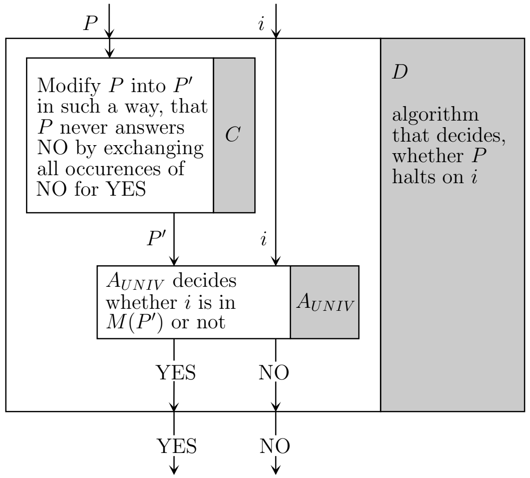
`NO`, if P does not halt on input i

N.B. Simulation of P will not work, because it need not terminate

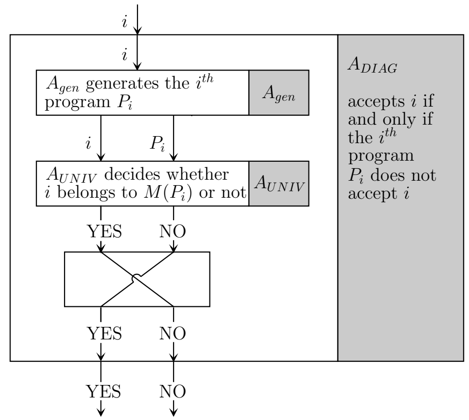
UNIV $\leq_{\text{Alg}}$ HALT by Reduction

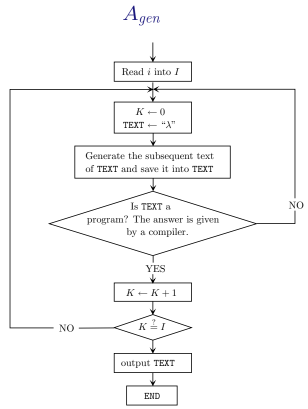


HALT $\leq_{\mathsf{Alg}}$ UNIV by Reduction



$(\mathbb{N}, DIAG) \leq_{\mathsf{Alg}}$ UNIV by Reduction

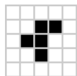




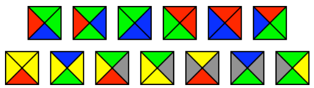
Conclusion: UNIV and HALT are also not solvable by an algorithm

Undecidability Is Not Rare

- Decide* whether a Game of Life configuration stabilizes



- Decide whether a set of Wang tiles can tile the plane



- Decide whether a Diophantine equation (multivariable polynomial equation, like $a^3 + b^3 = c^3$) has a solution in integers
- Decide whether a program has a specific non-trivial property, like whether it always halts, always outputs 0, ... [cf. Rice's Theorem]

*In each case, the algorithm needs to work for *all* possible inputs (shown in yellow). All these decision problems turn out to involve a *universal* mechanism.

(Other) Infinities in Informatics

- Infinitely many computational tasks, with infinitely many inputs
- Infinitely many algorithms and programs
- Infinitely many execution traces, finite or infinite
- Infinite (nonterminating, reactive) computations: lists, trees, ...
- Inductively defined data types, having infinitely many values
- Computations on infinite sequences (generated by other programs)
- Infinite automata

Concluding Challenge to Grapple with Infinity

Write a program

- that is not empty
- that processes no input
- that produces a listing of its own source code

After some attempts, you might get into an infinite regress
But it is doable (and instructive to discover your own solution)

Try this Challenge on-line with *Tom's JavaScript Machine* at

`www.win.tue.nl/~wstomv/edu/javascript`

Literature

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- Dieter Hofbauer and Johannes Waldmann:
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