Regular differential forms

1 Regular differential forms - the affine case

Let X be an affine algebraic variety with coordinate ring $k[X]$. The $k[X]$-module $\Omega[X]$ of regular differential forms is generated by elements df ($f \in k[X]$) with relations

$$d(f+g) = df + dg$$
$$d(fg) = fdg + gdf$$
$$d\alpha = 0 \quad (\alpha \in k)$$

So, elements of $\Omega[X]$ are sums of terms gdf with $f, g \in k[X]$.

Example The affine parabola $y = x^2$ has $dy = 2xdx$ and, using this, all occurrences of y and dy can be eliminated. The regular differential forms are $\omega = g(x)dx$ with $g \in k[x]$.

Example The affine cubic curve $y^2 = x^3 + x$ has $2ydy = (3x^2+1)dx$. An example of a regular differential form is

$$\omega = \frac{dx}{2y} = \frac{dy}{3x^2+1}.$$

Is it really OK to have fractions? According to the definition we should have

$$\omega = fdx + gdy \ \text{ with } \ f, g \in k[X].$$

Here

$$2y\omega = dx$$
$$(3x^2+1)\omega = dy$$

so we need $2yf + (3x^2+1)g = 1$ for certain $f, g \in k[X]$. By the Nullstellensatz that means $X \cap V(2y, 3x^2+1) = \emptyset$, and this is true: the curve is nonsingular. Of course we can also compute explicitly: take

$$f(x,y) = -\frac{9}{4}xy \ \text{ and } \ g(x,y) = \frac{3}{2}x^2 + 1.$$

Then $2yf + (3x^2+1)g = -\frac{9}{2}x(x^3+x) + (\frac{3}{2}x^2+1)(3x^2+1) = 1$. Hence

$$\omega = \frac{dx}{2y} = \frac{dy}{3x^2+1} = -\frac{9}{4}xydx + (\frac{3}{2}x^2+1)dy.$$

2 Regular differential forms - the projective case

If X is a projective variety, it has a covering with affine pieces. Now a regular differential form is one that is regular in each piece.

Example Take the projective line $\mathbf{P}^1$. It has projective coordinates (X, Y). It is covered by the two affine pieces $A_1 = \mathbf{P}^1 \setminus \{(1,0)\}$ and $A_2 = \mathbf{P}^1 \setminus \{(0,1)\}$. In A_1 the projective coordinates can be chosen as $(X, 1)$, and in A_2 the projective coordinates can be chosen as $(1, Y)$. In $A_1 \cap A_2$ the projective point (X, Y) corresponds to $(X/Y, 1)$ in A_1 and to $(1, Y/X)$ in A_2, so the Y of A_2 is the $1/X$ of A_1.

Suppose we have a regular differential form on $\mathbf{P}^1$. Restricted to A_1 it looks like $f(X)dX$. Restricted to A_2 it looks like $g(Y)dY$. And both forms agree on $A_1 \cap A_2$. That is, $f(X)dX = g(Y)dY = g(\frac{1}{X})d(\frac{1}{X}) = g(\frac{1}{X}).\frac{-1}{X^2}.dX$ but that is impossible: there are no polynomials $f(X)$ and $g(X)$ such that $f(X) = g(\frac{1}{X}).\frac{-1}{X^2}$. It follows that there are no regular differential forms on $\mathbf{P}^1$.

Example Take the projective curve $Y^2Z = X^3 + XZ^2$. The projective plane $\mathbf{P}^2$ is covered by three affine pieces: A_1 is the part with $Z \neq 0$ and coordinates $(X, Y, 1)$, A_2 is the part with $Y \neq 0$ and coordinates $(U, 1, V)$, A_3 is the part with $X \neq 0$ and coordinates $(1, S, T)$, where on $A_1 \cap A_2$ we have $U = X/Y$, $V = 1/Y$, and on $A_1 \cap A_3$ we have $S = Y/X$, $T = 1/X$, and on $A_2 \cap A_3$ we have $S = 1/U$, $T = V/U$. In our case (where $Y^2Z = X^3 + XZ^2$) the part A_3 is superfluous, since already A_1 and A_2 cover the curve. (In the projective plane the only point not covered by $A_1 \cup A_2$ is $(1, 0, 0)$, but that does not lie on our curve.)

Claim:

$$\omega = \frac{dX}{2Y} = \frac{dY}{3X^2 + 1} = \frac{dU}{2UV - 1} = \frac{-dV}{3U^2 + V^2}$$

is a regular differential form.

Check: In A_1 we have the equation $Y^2 = X^3 + X$ and we already saw that $\frac{dX}{2Y} = \frac{dY}{3X^2+1}$ is a regular differential form on that affine piece. In A_2 we have the equation $V = U^3 + UV^2$ and in the same way we see that $\frac{dU}{2UV-1} = \frac{-dV}{3U^2+V^2}$ is a regular differential form on that affine piece. Finally, in the intersection $A_1 \cap A_2$ we have

$$\omega = \frac{dY}{3X^2 + 1} = \frac{1}{3(\frac{U}{V})^2 + 1}.\frac{-1}{V^2}.dV = \frac{-dV}{3U^2 + V^2}.$$

Thus ω is a regular differential form as claimed.

3 An algebraic definition of the genus

So far we saw that there are no regular differential forms on the projective line $\mathbf{P}^1$ and we found one such form for the elliptic curve $Y^2 = X^3 + X$.

Theorem 3.1 *Let X be a nonsingular projective curve. Then* $\dim_k \Omega[X] = g$.

Earlier the genus g was defined as the number of holes in the two-dimensional real surface that is the one-dimensional complex curve. This theorem can be taken as definition when k is not the field of complex numbers.

Description of the regular differential forms

If the nonsingular projective curve X is given by the equation $f(X,Y) = 0$ for some polynomial f of degree d, then $f_X dX + f_Y dY = 0$ (where f_X and f_Y are the derivatives of f w.r.t. X and Y), and the regular differential forms look like

$$\omega = \frac{gdX}{f_Y} = \frac{-gdY}{f_X}$$

for some polynomial $g(X,Y)$ of degree at most $d-3$.

(Nonsingularity implies that f_X and f_Y do not vanish simultaneously on the curve, so that ω can be written without fractions.)

(Why $d-3$? Consider the change from $(X,Y,1)$ to $(U,1,V)$, with $Y = \frac{1}{V}$ and $dY = -\frac{1}{V^2}dV$. If $f(X,Y)$ has degree d and $f_X(X,Y)$ has degree $d-1$ and $g(X,Y)$ has degree e, then

$$\frac{-g(X,Y)dY}{f_X} = \frac{g(\frac{U}{V},\frac{1}{V})}{f_X(\frac{U}{V},\frac{1}{V})}.\frac{1}{V^2}.dV$$

behaves like $V^{-e}.V^{-2}.V^{d-1}$ near $V=0$, and since the value must be well-defined for $V=0$ we must have $e \le d-3$.)

Since a polynomial of degree at most d has $(d+1)+d+...+1 = \frac{1}{2}(d+2)(d+1)$ coefficients, we find $g = \frac{1}{2}(d-1)(d-2)$ for a nonsingular curve of degree d.

4 Rational differential forms

Rational differential forms are sums of terms gdf where now $f,g \in k(X)$. For a nonsingular curve X, the set $\Omega(X)$ of rational differential forms is a 1-dimensional vector space over $k(X)$.