Representations of finite groups

Abstract

A micro-introduction to the theory of representations of finite groups.

1 Representations

Let G be a finite group. A *linear representation* of G is a homomorphism $\rho : G \to GL(V)$ where $GL(V)$ is the group of invertable linear transformations of the vector space V. We shall restrict ourselves to finite-dimensional V. The dimension $\dim V = n$ is called the *degree* of the representation. In order to make life easy, we only consider vector spaces over $\mathbf{C}$, the field of complex numbers.

(The theory is easy for finite groups because we can average over the group to get something that is invariant for the group action. In the averaging process we divide by the order of the group, and the theory (of modular representations) is more difficult when the characteristic of the field divides the order of G. For Schur's Lemma we need an eigenvalue, and life is a bit easier for algebraically closed fields.)

Two representations $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ are called *equivalent* when they are not really different: V_1 and V_2 have the same dimension, and for a suitable choice of bases in V_1 and V_2 the matrices of $\rho_1(g)$ and $\rho_2(g)$ are the same, for all $g \in G$. (Equivalently, ρ_1 and ρ_2 are equivalent when there is a linear isomorphism $f : V_1 \to V_2$ such that $f\rho_1(g) = \rho_2(g)f$ for all $g \in G$.)

A subspace W of V is called $\rho(G)$-invariant if $\rho(g)W \subseteq W$ for all $g \in G$. The first example of averaging is to get a $\rho(G)$-invariant complement of a $\rho(G)$-invariant subspace.

Theorem 1.1 (Maschke) *Let G be a finite group, let V be a vector space over the field F, and let $\rho : G \to GL(V)$ be a linear representation of G on V. If the subspace W of V is $\rho(G)$-invariant and $|G|$ is nonzero in F, then there is a $\rho(G)$-invariant subspace U of V such that $V = U \oplus W$.*

Proof: Let U_0 be a complement of W in V, so that $V = U_0 \oplus W$. Let P_0 be the projection onto W along U_0, that is, $P_0 v = w$ when $v = u_0 + w$ with $u_0 \in U_0$ and $w \in W$. Put

$$P = \frac{1}{|G|} \sum_{g \in G} \rho(g) P_0 \rho(g)^{-1}.$$

Then $P\rho(g) = \rho(g)P$ for all $g \in G$, and P is a projection onto W. Now the kernel U of P is $\rho(G)$-invariant. □

1.1 Direct sum

Given two representations $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$, their *direct sum* $\rho_1 \oplus \rho_2$ is the representation $\rho : G \to GL(V)$, where $V = V_1 \oplus V_2$, defined by $\rho(g)v = \rho_1(g)v_1 + \rho_2(g)v_2$ for $v = v_1 + v_2$ with $v_1 \in V_1$ and $v_2 \in V_2$. In matrix form, this says that the matrix $R(g)$ of $\rho(g)$ is given by

$$R(g) = \begin{pmatrix} R_1(g) & 0 \\ 0 & R_2(g) \end{pmatrix}.$$

The degree of a direct sum is the sum of the degrees of the summands. A representation ρ is called *irreducible* if it is not the direct sum $\rho_1 \oplus \rho_2$ of two representations of nonzero degree.

By induction on $\dim V$ we see that each representation is the direct sum $\rho_1 \oplus \cdots \oplus \rho_m$ of (zero or more) irreducible representations.

1.2 Tensor product

Given two representations $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$, their *tensor product* $\rho_1 \otimes \rho_2$ is the representation $\rho : G \to GL(V)$, where $V = V_1 \otimes V_2$, defined by $\rho(g)v = \rho_1(g)v_1 \otimes \rho_2(g)v_2$ for $v = v_1 \otimes v_2$ with $v_1 \in V_1$ and $v_2 \in V_2$. In matrix form, this says that the matrix $R(g)$ of $\rho(g)$ is $R(g) = R_1(g) \otimes R_2(g)$. The degree of a tensor product is the product of the degrees of the factors.

1.3 Schur's Lemma

Theorem 1.2 (Schur) *Let $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ be irreducible representations. Let $f : V_1 \to V_2$ be a linear map with $\rho_2(g)f = f\rho_1(g)$ for all $g \in G$.*

(i) If ρ_1 and ρ_2 are not equivalent, then $f = 0$.

(ii) If $V_1 = V_2 = V$ and $\rho_1 = \rho_2$, then $f = \lambda I$, where I is the identity on V.

Proof: (i) We may suppose $f \neq 0$. The subspace $W_1 = \ker f$ of V_1 is $\rho_1(G)$-invariant, so $W_1 = 0$ or $W_1 = V_1$, but $f \neq 0$, so $W_1 = 0$. The subspace $W_2 = \operatorname{im} f$ of V_2 is $\rho_2(G)$-invariant, so $W_2 = 0$ or $W_2 = V_2$, but $f \neq 0$, so $W_2 = V_2$. Now f is an isomorphism, and ρ_1 and ρ_2 are equivalent.

(ii) Let λ be an eigenvalue of f (there is one, since $\mathbf{C}$ is algebraically closed). Now apply the previous argument to $f - \lambda I$ instead of f. Since $f - \lambda I$ is not an isomorphism, is must be 0. □

By averaging over G we can turn a linear map f into one that satisfies the hypothesis of Theorem 1.2. This yields:

Corollary 1.3 *Let $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ be irreducible representations. Let $f : V_1 \to V_2$ be linear and put $\tilde{f} = \frac{1}{|G|} \sum_{g \in G} \rho_2(g)^{-1} f \rho_1(g)$.*

(i) If ρ_1 and ρ_2 are not equivalent, then $\tilde{f} = 0$.

(ii) If $V_1 = V_2 = V$ and $\rho_1 = \rho_2$, then $\tilde{f} = \lambda I$, where $\lambda = \frac{1}{n} \operatorname{tr} f$.

Proof: In case (ii), $\operatorname{tr} \tilde{f} = \operatorname{tr} f$ and $\operatorname{tr} I = n$. □

Now choose bases, and let $f = E_{jk}$ be the linear map with a matrix that is zero everywhere except at the (j,k)-entry where it is 1. We obtain orthogonality relations for the matrix entries of representations.

Corollary 1.4 *Let $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ be irreducible representations.*

(i) If ρ_1 and ρ_2 are not equivalent, then

$$\frac{1}{|G|} \sum_{g \in G} R_2(g)^{-1}_{ij} R_1(g)_{kl} = 0 \quad \text{for all } i, j, k, l$$

where $R_i(g)$ denotes the matrix of $\rho_i(g)$ for $i = 1, 2$ and $g \in G$.

(ii) If $V_1 = V_2 = V$ and $\rho_1 = \rho_2 = \rho$, then

$$\frac{1}{|G|} \sum_{g \in G} R(g)^{-1}_{ij} R(g)_{kl} = \begin{cases} 1/n & \text{if } i = l \text{ and } j = k \\ 0 & \text{otherwise} \end{cases}$$

where $R(g)$ denotes the matrix of $\rho(g)$ for $g \in G$.

Proof: Apply the previous corollary, and take the (i, l)-matrix entry of $\tilde{f}$ where $f = E_{jk}$. In case (ii), $\operatorname{tr} f$ equals 0 if $j \neq k$ and 1 if $j = k$. □

2 Characters

Given a representation $\rho : G \to GL(V)$, let its *character* be the map $\chi : G \to \mathbf{C}$ defined by $\chi(g) = \operatorname{tr} \rho(g)$. It will turn out that ρ is determined up to equivalence by its character χ.

Lemma 2.1 *Let $\chi = \chi_\rho$ denote the character of ρ. Then*

(i) $\chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2}$,

(ii) $\chi_{\rho_1 \otimes \rho_2} = \chi_{\rho_1} \chi_{\rho_2}$,

(iii) $\chi_\rho(1) = n$ if ρ has degree n,

(iv) $\chi(g^{-1}) = \overline{\chi(g)}$ for all $g \in G$,

(v) $\chi(h^{-1}gh) = \chi(g)$ for all $g, h \in G$.

Proof: Only part (iv) requires comment. Since G is finite, g has finite order, so $\rho(g)$ has finite order, and its eigenvalues are roots of unity. If $\rho(g)$ has eigenvalue ζ, then $\rho(g^{-1})$ has eigenvalue $\zeta^{-1} = \overline{\zeta}$. And the trace of $\rho(g^{-1})$ is the sum of its eigenvalues. □

2.1 Inner product

For functions $\phi, \psi : G \to \mathbf{C}$, put

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\phi(g)}\, \psi(g).$$

Theorem 2.2 *If the representation $\rho : G \to GL(V)$ is irreducible, then its character χ satisfies $\langle \chi, \chi \rangle = 1$. If the irreducible representations ρ, ρ' are inequivalent, then their characters χ, χ' satisfy $\langle \chi, \chi' \rangle = 0$.*

Proof: Since $\chi(g) = \operatorname{tr} \rho(g) = \sum_i R(g)_{ii}$ and $\overline{\chi(g)} = \chi(g^{-1})$, we have

$$\langle \chi, \chi' \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1}) \chi'(g) = \frac{1}{|G|} \sum_{g \in G} \sum_{i,j} R(g)^{-1}_{ii} R'(g)_{jj}.$$

Now apply Corollary 1.4. □

2.2 The character determines the representation

Theorem 2.3 *Let $\sigma : G \to GL(V)$ be a representation with character ϕ, and let $\sigma = \sigma_1 \oplus \cdots \oplus \sigma_m$ be a decomposition of σ into irreducible representations. Let $\rho : G \to GL(W)$ be an irreducible representation with character χ. Then the number of σ_i equivalent to ρ equals $\langle \phi, \chi \rangle$.*

Proof: Let σ_i have character ϕ_i, so that $\phi = \phi_1 + \cdots + \phi_m$. Then $\langle \phi, \chi \rangle = \langle \phi_1, \chi \rangle + \cdots + \langle \phi_m, \chi \rangle$. Now apply Theorem 2.2. □

It follows that although the decomposition into irreducible subspaces is not unique in general, the number of subspaces of given type is determined. (In fact, also the sum of all subspaces of given type is determined uniquely.)

Corollary 2.4 *Two representations are equivalent if and only if they have the same character.* □

We saw that if ρ is irreducible, then $\langle \chi, \chi \rangle = 1$. But the opposite is also true: if $\langle \chi, \chi \rangle = 1$ then ρ is irreducible. Indeed, if $\chi = \sum_i m_i \chi_i$ where the χ_i are distinct irreducible characters (that is, characters of irreducible representations), then $\langle \chi, \chi \rangle = \sum_i m_i^2$, and this equals 1 only when there is only one summand and $m_1 = 1$.

2.3 The regular representation

So far, we have not constructed any representations. Let G be a finite group. The *regular representation* of G is the representation ρ on the vector space $\mathbf{C}^G$ (with basis G) defined by $\rho(g)h = gh$ for $g, h \in G$. This representation has degree $n = |G|$, and character χ satisfying

$$\chi(g) = \operatorname{tr} \rho(g) = \begin{cases} |G| & \text{if } g = 1, \\ 0 & \text{otherwise.} \end{cases}$$

If χ_i is any irreducible character, of degree n_i, then

$$\langle \chi, \chi_i \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \chi_i(g) = \chi_i(1) = n_i,$$

so that $\chi = \sum n_i \chi_i$ and $|G| = \sum_i n_i^2$.

It follows that there are only finitely many distinct irreducible characters, all found in the character of the regular representation.

2.4 Class functions

For $g \in G$, the *conjugacy class* $C(g)$ is the set $\{h^{-1}gh \mid h \in G\}$. Lemma 2.1(v) says that a character is a *class function*, that is, is constant on conjugacy classes. We shall see that conversely any class function is a linear combination of characters.

For a class function ϕ and a representation ρ, let $f_{\phi,\rho} = \frac{1}{|G|} \sum_{g \in G} \overline{\phi(g)} \rho(g)$, a linear transformation of the vector space V, weighted average of the $\rho(g)$.

Lemma 2.5 *If ρ is irreducible, then $f_{\phi,\rho} = \lambda I$, where $n\lambda = \operatorname{tr} f_{\phi,\rho} = \langle \phi, \chi \rangle$.*

Proof: We have

$$\rho(g)^{-1} f_{\phi,\rho}\rho(g) = \frac{1}{|G|}\sum_{h\in G} \overline{\phi(h)}\rho(g^{-1}hg) = \frac{1}{|G|}\sum_{h\in G} \overline{\phi(ghg^{-1})}\rho(h) = f_{\phi,\rho}.$$

Now by Schur's Lemma (Theorem 1.2) $f_{\phi,\rho} = \lambda I$ for some constant λ, and λ is found by taking traces. □

Theorem 2.6 *The irreducible characters form an orthonormal basis for the vector space of class functions. In particular, the number of irreducible characters equals the number of conjugacy classes of the group G.*

Proof: The 'orthonormal' part is the content of Theorem 2.2. Remains 'basis'. If ϕ is a class function orthogonal to all irreducible characters χ_i, then consider the linear transformation $f_{\phi,\rho}$ for various ρ. The above lemma says that $f_{\phi,\rho} = 0$ when ρ is irreducible. For arbitrary ρ the function $f_{\phi,\rho}$ is a direct sum of the functions f_{ϕ,ρ_j} for the irreducible constituents ρ_j of ρ, hence $f_{\phi,\rho} = 0$ for all ρ. Now let ρ be the regular representation and compute the image of $f_{\phi,\rho}$ on the basis vector 1. Since $\rho(g)1 = g$, we find $0 = f_{\phi,\rho}1 = \frac{1}{|G|}\sum_{g\in G} \overline{\phi(g)}g$, so that all coefficients $\overline{\phi(g)}$ vanish, and $\phi = 0$. □

2.5 Character table

The square matrix X with rows indexed by the irreducible characters χ_i and columns by the conjugacy classes of G and entries $X_{\chi,C} = \chi(g)$ for $g \in C$, is called the *character table* of G.

Let D be the diagonal matrix with rows and columns indexed by the conjugacy classes of G, where $D_{CC} = \frac{|C|}{|G|}$, so that $\operatorname{tr} D = 1$.

The fact that different characters are orthogonal is expressed by $XD\overline{X}^\top = I$. But if $AB = I$ then also $BA = I$, so it follows that $\overline{X}^\top X = D^{-1}$. This shows that also the columns of X are orthogonal, and that the sizes of the conjugacy classes can be seen from X.

3 Example: Sym(5)

Let G be the symmetric group Sym(5). Its character table is

	1	(12)(34)	(123)	(12345)	(12)	(1234)	(12)(345)
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	1	−1	−1	−1
χ_3	4	0	1	−1	2	0	−1
χ_4	4	0	1	−1	−2	0	1
χ_5	5	1	−1	0	1	−1	1
χ_6	5	1	−1	0	−1	1	−1
χ_7	6	−2	0	1	0	0	0

The table is square, with 7 rows and columns. The columns are labeled by representatives of the conjugacy classes. The conjugacy classes have sizes 1, 15, 20, 24, 10, 30, 20, respectively. The sum of the squares of the character degrees $1^2 + 1^2 + 4^2 + 4^2 + 5^2 + 5^2 + 6^2 = 120$ equals $|G|$.

How was this table constructed? By finding some easy characters and decomposing those into irreducibles.

1. The first is the *trivial character*, the character of the *trivial representation* that maps every $g \in G$ to the identity $I = (1)$ of order 1. This gives χ_1.

2. The second is the sign character. A permutation can be even or odd, and the sign character is 1 on even and -1 on odd permutations. This gives χ_2.

3. The third construction is that of a *permutation character*. If G acts as a group of permutations on a set Ω, we find a representation in $\mathbf{C}^\Omega$. (The regular representation is the example where G acts on itself.) Now $\chi(g) = \mathrm{tr}\,\rho(g)$ is the number of fixpoints of g.

The group Sym(5) has an obvious action on the set $S = \{1, 2, 3, 4, 5\}$, and the permutation character is $\pi = (5, 1, 2, 0, 3, 1, 0)$ with entries in the order of the columns of the table. Now $\langle \pi, \pi \rangle = 2$, so this is the sum of two irreducible characters. And $\langle \pi, \chi_1 \rangle = 1$, so χ_1 is one of them. Then $\chi_3 = \pi - \chi_1$ must be the other. This gives χ_3.

The group Sym(5) also has an action on the ten pairs from S. The corresponding permutation character is $\pi_2 = (10, 2, 1, 0, 4, 0, 1)$. Now $\langle \pi_2, \pi_2 \rangle = 3$, and $\langle \pi_2, \chi_1 \rangle = 1$, and $\langle \pi_2, \chi_3 \rangle = 1$, so π_2 decomposes into three irreducibles, namely χ_1 and χ_3 and $\chi_5 = \pi_2 - \chi_1 - \chi_3$. This gives χ_5.

4. The fourth construction is that of taking tensor products. We find irreducible characters $\chi_4 = \chi_2\chi_3$ and $\chi_6 = \chi_2\chi_5$. Now only χ_7 is left, and we can write it down using the orthogonality relations of the columns. But we can also compute the product χ_3^2 and find that it decomposes as $\chi_3^2 = \chi_1 + \chi_3 + \chi_5 + \chi_7$. That completes the table.

3.1 Alt(5)

The even permutations in Sym(5) form the alternating group Alt(5). It has character table

	1	(12)(34)	(123)	(12345)	(12354)
χ_1	1	1	1	1	1
χ_2	3	-1	0	s	t
χ_3	3	-1	0	t	s
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

where $s = (1 - \sqrt{5})/2$ and $t = (1 + \sqrt{5})/2$.

We lose the classes of odd permutations, but the class of 5-cycles now splits into two, since (12345) and (12354) are no longer conjugate. (In Sym(5) one had $(12354) = (45)(12345)(45)$, but (45) is odd.) The restrictions of the characters of Sym(5) to Alt(5) give characters again, and we find χ_1, χ_4, χ_5 and $\chi_2 + \chi_3$. (The formula for the inner product of two characters involves a factor $\frac{1}{|G|}$, so if the irreducible character χ of Sym(5) vanishes outside Alt(5), and χ' is the restriction of χ to Alt(5), then $\langle \chi', \chi' \rangle = 2$.) If x is an element of Sym(5)$\backslash$Alt(5), and χ a character of Alt(5), then also χ' defined by $\chi'(g) = \chi(x^{-1}gx)$ is a character of Alt(5). The characters χ_2 and χ_3 must be related this way, so have the same value on the classes that do not split, and all that remains is to find s

and t. The orthogonality relations give $s^2+t^2=3$ and $st=-1$ (and we already knew $s+t=1$), and this determines s,t. That completes the table.

Exercise *Construct matrices for a representation with character χ_2.*

4 Some additional material

4.1 Frobenius reciprocity

Let G be a group and H a subgroup and let $\phi : H \to \mathbf{C}$ be a class function on H. The function ϕ^G obtained by *inducing ϕ up* to G is by definition $\phi^G(g) = \frac{1}{|H|}\sum_{x\in G}\hat{\phi}(x^{-1}gx)$, where $\hat{\phi} : G \to \mathbf{C}$ is defined by $\hat{\phi}(h) = \phi(h)$ for $h \in H$, and $\hat{\phi}(x) = 0$ for $x \in G \setminus H$. Now ϕ^G is a class function on G. Conversely, if $\psi : G \to \mathbf{C}$ is a class function on G, then the restriction $\psi|_H$ of ψ to H is a class function on H.

Proposition 4.1 *Let $\phi : H \to \mathbf{C}$ be a class function on H and $\psi : G \to \mathbf{C}$ a class function on G. Then $\langle \phi^G, \psi\rangle = \langle \phi, \psi|_H\rangle$. In particular, if ϕ is a character of H, then ϕ^G is a character of G.*

Proof:

$$\begin{aligned}\langle \psi, \phi^G\rangle &= \tfrac{1}{|G|.|H|}\sum_{g,x\in G}\overline{\psi(g)}\hat{\phi}(x^{-1}gx) = \tfrac{1}{|G|.|H|}\sum_{g,x\in G}\overline{\psi(x^{-1}gx)}\hat{\phi}(g)\\ &= \tfrac{1}{|H|}\sum_{g\in H}\overline{\psi(g)}\phi(g) = \langle \psi|_H, \phi\rangle.\end{aligned}$$

Necessary and sufficient in order to be a character is that the inner product with all irreducibles is a nonnegative integer. □

Another useful identity is $(\phi.\psi|_H)^G = \phi^G.\psi$.

4.2 Permutation characters

Recall that if G acts as a group of permutations on a set Ω, then the corresponding *permutation character* is the function $\pi : G \to \mathbf{N}$ defined by $\pi(g) = \#\{\omega \in \Omega \mid g\omega = \omega\}$, the number of fixpoints of g in this action.

If G acts transitively on Ω, then the *rank* of the action is the number of orbits of a point stabilizer, or, what is the same, the number of orbits of G in the natural action on $\Omega \times \Omega$.

For a group G, let 1_G be the trivial character on G (that is identically 1).

Proposition 4.2 *Let π be the permutation character of a permutation representation of the group G on the set Ω. Let G have orbits $\Omega_1, \ldots, \Omega_m$. Let, for $1 \le i \le m$, the group H_i be the stabilizer in G of some element in Ω_i. Then $\pi = \sum_{i=1}^m (1_{H_i})^G$. In particular,*

(i) The number of orbits m of G equals $\langle 1, \pi\rangle$.

(ii) If G is transitive, then it has rank $\langle \pi, \pi\rangle$.

(iii) For $m = 2$, the number of orbits of G on $\Omega_1 \times \Omega_2$ (which equals the number of orbits of H_1 on Ω_2 and the number of orbits of H_2 on Ω_1) equals $\langle \pi_1, \pi_2\rangle$.

Proof: The permutation representation on Ω is the direct sum of the representations on the Ω_i, so we may assume that G is transitive, i.e., $m = 1$. Put $H = H_1$. Now Ω can be identified with the set of left cosets gH of H, with G acting by left multiplication. A left coset xH is fixed by multiplication by g when $gxH = xH$, i.e., when $x^{-1}gx \in H$ (and each left coset xH has $|H|$ representatives x). Now by definition $(1_H)^G(g) = \frac{1}{|H|}\#\{x \in G \mid x^{-1}gx \in H\} = \pi(g)$, so that $\pi = (1_H)^G$.

For (i), $\langle 1_G, \pi\rangle = \langle 1_H, 1_H\rangle = 1$.

For (ii): the action of G on $\Omega \times \Omega$ (via $g(a, b) = (ga, gb)$) has character π^2. Now the rank is $\langle 1, \pi^2\rangle = \langle \pi, \pi\rangle$.

For (iii): the action of G on $\Omega_1 \times \Omega_2$ (via $g(a, b) = (ga, gb)$) has character $\pi_1\pi_2$. Now the number of orbits is $\langle 1, \pi_1\pi_2\rangle = \langle \pi_1, \pi_2\rangle$. □