Graphs and groups

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Graphs from groups

Let G be a group, H a subgroup, and X some subset of G. We can define a graph $\Gamma = \Gamma(G, H, X)$ by taking as vertex set the collection $\{gH \mid g \in G\}$ of left cosets of H, and letting two vertices g_1H and g_2H be adjacent when $Hg_2^{-1}g_1H \subseteq HXH$.

(Now adjacency is well-defined: it does not depend on the chosen representative g of the coset gH.)

The group G acts transitively on $\Gamma(G, H, X)$ via left multiplication: $g \in G$ sends the coset aH to gaH. Indeed, this map preserves adjacency.

Conversely, if Γ is an arbitrary graph with transitive group of automorphisms G, then we can fix some vertex p of Γ and let H be the stabilizer of p in G. Now, for some suitable X, Γ is of the form $\Gamma(G, H, X)$.

Primitive actions

Let G be a permutation group, acting on a set Ω. The permutation action is called *primitive* when there is no nontrivial partition of Ω that is preserved by G. (Trivial partitions are the partition into singletons and the partition into one big piece.)

If H is not maximal, say $H < B < G$ then $\{\{gbH \mid b \in B\} \mid g \in G\}$ is a nontrivial partition of the vertex set of $\Gamma(G, H, X)$, and the representation of G on the vertices of $\Gamma(G, H, X)$ is not primitive.

Conversely, a primitive permutation action of a transitive permutation group belongs to a maximal subgroup. (Indeed, let us call the parts of a nontrivial partition preserved by G *blocks*. If B is a block, then for $b_1, b_2 \in B$ we see that $b_2^{-1}b_1B = B$ so that B is a coset of a subgroup.)

Imprimitive actions are precisely the actions for which one can pick a nonempty X such that the graph $\Gamma(G, H, X)$ is disconnected.

Several subgroups

If G has several maximal subgroups, and one is chosen as vertex stabilizer H, then the other subgroups have some permutation action on the set of cosets G/H and hence on the graph $\Gamma(G, H, X)$. It is a game to recognize all maximal subgroups from substructures of the graph Γ.

Petersen

An example. Let G be the symmetric group S_5 of order 120. It has three maximal subgroups, namely S_4 and $AGL(1,5)$ and $2 \times S_3$, where $AGL(1,5)$ denotes the affine group of transformations $x \mapsto ax+b$, with $a,b,x \in \mathbf{F}_5$, $a \neq 0$. These subgroups have orders 24 and 20 and 12, and hence have index 5 and 6 and 10, respectively.

Thus, one obtains a graph on 10 vertices if one chooses $H = 2 \times S_3$ (whether one gets Petersen or its complement, the triangular graph $T(5)$, or the complete graph, or the cocomplete graph, depends on the choice of X).

Choose X, and $H = 2 \times S_3$, so as to obtain the Petersen graph Γ. What substructures correspond to the maximal subgroups of G? For $2 \times S_3$ one finds a vertex. For S_4 one finds a 4-coclique. For the third group $AGL(1,5)$ one finds a partition of the graph into two pentagons.

(Indeed, the vertices of Petersen are the pairs from a 5-set, adjacent when disjoint. Now S_4 preserves one symbol, so has orbits of lengths $4+6$ on the 10 vertices, and the orbit of length 4 is a 4-cocliqe. And $AGL(1,5)$ has orbits of lengths $5+10$ on the 15 edges of the Petersen graph, where the orbit of length 5 is a matching that splits the graph into two pentagons.)

Permutation character

Recall that if G acts as a group of permutations on a set Ω, then the corresponding *permutation character* is the function $\pi : G \to \mathbf{N}$ defined by $\pi(g) = \#\{\omega \in \Omega \mid g\omega = \omega\}$, the number of fixpoints of g in this action.

The inproduct $\langle 1, \pi \rangle$ is the number of orbits of G.

If G is transitive, then the inproduct $\langle \pi, \pi \rangle$ is the *rank* of the action, the number of orbits of a point stabilizer.

If H_1 and H_2 are two subgroups of G, and the permutation actions of G on G/H_1 and G/H_2 have permutation characters π_1 and π_2, respectively, then $\langle \pi_1, \pi_2 \rangle$ is the number of cosets gH_1 fixed by H_2 (and the number of cosets gH_2 fixed by H_1).

Results like these are proved using Frobenius reciprocity, see the notes on representations of finite groups.

Now S_5 has character table

	1	(12)(34)	(123)	(12345)	(12)	(1234)	(12)(345)
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1	-1
χ_3	4	0	1	-1	2	0	-1
χ_4	4	0	1	-1	-2	0	1
χ_5	5	1	-1	0	1	-1	1
χ_6	5	1	-1	0	-1	1	-1
χ_7	6	-2	0	1	0	0	0

and the three subgroups S_4, $AGL(1,5)$, $2 \times S_3$ have permutation characters $\pi_1 = \chi_1 + \chi_3$ and $\pi_2 = \chi_1 + \chi_6$ and $\pi_3 = \chi_1 + \chi_3 + \chi_5$, respectively.

The fact that $(\pi_1, \pi_1) = (\pi_2, \pi_2) = 2$ means that these are 2-transitive representations: one does not obtain a graph other than the complete or co-complete graphs, the only two relations are being equal or distinct. The fact that $(\pi_3, \pi_3) = 3$ means that the stabilizer of a vertex in the Petersen graph has three orbits: that vertex, its neighbours, and its nonneighbours, so that the Petersen graph is strongly regular.

The fact that $(\pi_2, \pi_3) = 1$ means that a vertex stabilizer of Petersen still is transitive on the 6 objects permuted by $AGL(1, 5)$. Thus, these objects are not described by special subsets of Petersen but by something homogeneous—in this case a partition into two pentagons.

If one restricts to the alternating group A_5, then χ_5 and χ_6 coincide, and we do get special subsets, namely two orbits of six pentagons.