Corrections and additions to the book ‘Distance-regular Graphs’

This note contains corrections for all errors in [BCN] known to us. Moreover, it provides some additional material and references. Most of the material is presented in the order of appearance in the book. There are two exceptions. A significant rewrite of Sections 3.14 and 3.15 is placed near the end. It is followed by tables of parameters for bipartite (but not antipodal) distance-regular graphs of diameter 4. We thank all correspondents who pointed out errors to us.

CHAPTER 1

ADDITION. On p. 4, line 20, add: ‘COOLSAET [859] remarks that a connected graph satisfying R1 and R2 with $\lambda = \mu = 2$ is regular.’

ADDITION. On p. 5, line −7, add: ‘For locally co-Heawood graphs, see BROUWER, FON-DER-FLAASS & SHPECTOROV [836].’

ADDITION. On p. 8, after Corollary 1.2.4, add: ‘For example, no distance-regular graph with parameters $\{28,15,9,1;1,3,15,28\}$ ($v = 256$) does exist.’

HISTORICAL CORRECTION. On p. 10, line 4, change the reference to ‘(cf. BELEVITCH [811] and WALLIS, STREET & WALLIS [1015], p. 293)’. On line 12, change the reference to ‘(RAGHAVARAO [976], cf. BELEVITCH [812], VAN LINT & SEIDEL [942])’.

HISTORICAL CORRECTION & ADDITION. On p. 11, after the proof of Proposition 1.3.2, add the following remark.

Proposition 1.3.2 is due to Delsarte (cf. [868], p. 31) in the strongly regular case, and was generalized by Hoffman to regular graphs (see Proposition 3.7.2) and by Haemers (cf. [885], §2.1) to arbitrary graphs. (See also CVETKOVIĆ, DOOB & SACHS [864], p. 115.) HOFFMAN [903] does not contain the upper bound $|C| \le 1 + M/m$ for the size of cliques C, but the lower bound $\chi \ge 1 + M/m$ for the chromatic number of a graph with largest eigenvalue M and smallest eigenvalue $-m$.
For cliques of size c in edge-regular graphs Neumaier gave the inequality $c^2(v-2k+\lambda) + c(k^2+3k-(\lambda+2)v-\lambda) - v(k-\lambda-1) \le 0$ with equality iff each vertex outside the clique has a constant number of neighbours inside. This is just the inequality $0 \le (v(k-\mu) + (v\mu + \mu - k - k^2)c - \mu c^2)c/(v-c)$ from the proof above applied to the complementary graph.

SPELLING CORRECTION. On p. 14, line −4, change ‘consist’ into ‘consists’.

ADDITION. On p. 16, strengthen Theorem 1.5.6 as follows.

1.5.6. Theorem. *Let* Δ *be a strongly regular graph with parameters* (v,k,λ,μ). *Then the two-graph* $\mathbf{D} = \mathbf{D}(\Delta)$ *is regular if and only if* $v = 2(2k-\lambda-\mu)$. *If this is the case, then* $\mathbf{D}_\infty$ *is strongly regular with parameters* $(v-1,\ 2(k-\mu),\ k+\lambda-2\mu,\ k-\mu)$, *and* $\Gamma_\infty(\mathbf{D})$ *is distance-regular with intersection array* $\{v-1, 2(k-\lambda-1), 1; 1, 2(k-\lambda-1), v-1\}$ *on* $2v$ *vertices. Conversely, if* Δ *is regular of valency* k *such that* $\mathbf{D}(\Delta)$ *is regular, then* Δ *is strongly regular with parameters* (v,k,λ,μ), *where* $\lambda = k-(v-a)/2$, $\mu = k-a/2$ *for* $a = \lambda(\mathbf{D})$, *and* k *satisfies the quadratic equation* $2k^2-(v+2a)k+(v-1)a = 0$.

Proof. $\mathbf{D}(\Delta)$ is regular when $\lambda+(v-2k+\lambda) = 2(k-\mu)$. The parameters of $\mathbf{D}_\infty$ and $\Gamma_\infty(\mathbf{D})$ follow immediately. Conversely, let $\mathbf{D} = \mathbf{D}(\Delta)$ be regular. The (∓ 1)-matrices S of all graphs in the switching class of $\mathbf{D}$ have the same spectrum - they satisfy $S^2 = (v-2a-2)S+(v-1)I$. For a regular graph Δ we have $S = J-I-2A$, so A has two eigenvalues distinct from k and hence Δ is strongly regular. The parameters λ and μ follow by counting coherent triples: $a = \lambda+\bar{\mu} = 2(k-\mu)$. The quadratic for k follows by multiplying the above equation for S by J. □

ADDITION. On p. 16, at the end of Section 1.5, add references to CAMERON [851] and SEIDEL [984].

ADDITION. On p. 20, in Theorem 1.8.1, add a reference to MULDER [953].

ADDITION. On p. 26, before the heading 'The extended bipartite double of a graph' insert the following fragment of text.

Generalizing the concept of bipartite double one may define the *conjunction* $\Gamma\wedge\Delta$ of two graphs Γ, Δ as the graph with vertices (γ,δ) for $\gamma\in\Gamma$ and $\delta\in\Delta$, and $(\gamma,\delta)\sim(\gamma',\delta')$ if and only if $\gamma\sim\gamma'$ and $\delta\sim\delta'$. It is easy to see (cf. CVETKOVIĆ & LUČIĆ [865] and SCHWENK [983]) that if Γ has spectrum $\mathbf{\Phi}$ and Δ has spectrum $\mathbf{\Psi}$, then $\Gamma\wedge\Delta$ has spectrum $\mathbf{\Phi}.\mathbf{\Psi}$. The bipartite double of Γ is the special case where $\Delta = K_2$.

ADDITION. On p. 28, at the end of §1.13, add:

Remark. MULDER [954] calls a graph *interval-regular* when it satisfies $c_i = i$ for all i. (Examples are d-cubes and folded $(2d+1)$-cubes; some irregular examples are given in MULDER [954] and in BANDELT & MULDER [807], where the case $d = 2$ is studied.) He shows that these graphs are precisely the graphs with the property that for any two vertices γ, δ the graph $I(\gamma,\delta)$ with as vertices and edges the vertices and edges occurring in geodesics between γ and δ is a $d(\gamma,\delta)$-cube.

CORRECTION. On p. 29, lines 3-4, change the sentence 'Any ... entirely' into 'If M is a maximal clique, and $\gamma,\delta\in M$, then M contains the singular line $\{\gamma,\delta\}^{\perp\perp}$'. On line 7, delete the word 'in'.

ADDITION. On p. 29, add after Lemma 1.14.1:

For arbitrary graphs one has:

1.14.2 Lemma. *In a graph* Γ *any two singular lines have at most one point in common if and only if* $e_1^\perp \subseteq e_2^\perp$ *implies* $e_1^\perp = e_2^\perp$ *for any two (intersecting) edges* e_1 *and* e_2.

Proof. Clearly, $\beta\in\{\gamma,\delta\}^{\perp\perp}$ is equivalent to $\{\gamma,\delta\}^\perp\subseteq\{\beta,\gamma\}^\perp$. Hence $\{\beta,\gamma\}^{\perp\perp} = \{\gamma,\delta\}^{\perp\perp}$ if and only if $\{\beta,\gamma\}^\perp = \{\gamma,\delta\}^\perp$. □

CORRECTION. On p. 29, Lemma 1.15.1, replace ‘Then’ by ‘If $s \geq 1$ and $t \geq 0$, then’.

CORRECTION. On p. 32, line 7 of the fourth proof, replace ‘Rad $\bar{b}$’ by ‘Rad $\bar{q}$’. Three lines later, add ‘and $\bar{q}(x+y+z) = 0$’ after ‘$x+y+z \in R$’.

ADDITION. On p. 33, change in the last line of the first remark ‘Brouwer (both unpublished; ’ into ‘BROUWER [833] (’. Add at the end of this remark: ‘The advantage of the fourth proof is that it generalizes to a classification of all polar spaces with lines of size three’.

CORRECTION. On p. 34 the proof of Proposition 1.16.1 was formulated somewhat inaccurately. An amended version of the proof follows.

... For any set $E \subseteq \Gamma(\alpha) \cap \Gamma(\beta)$ define $E_\alpha := \{\xi \in \Gamma(\alpha) \cap \Gamma_2(\beta) \mid \{\alpha,\beta,\xi\}^\perp = E\}$ and $E_\beta := \{\xi \in \Gamma(\beta) \cap \Gamma_2(\alpha) \mid \{\alpha,\beta,\xi\}^\perp = E\}$. Then, for $\omega = \alpha$ and $\omega = \beta$, we have

$$k_\Gamma(\omega) = \mu + b_2(\alpha,\beta) + \sum_E |E_\omega|, \quad k_\Delta(\omega) = \mu(\Delta) + \sum_{E \ni \gamma} |E_\omega|.$$

Since $k_\Gamma(\alpha) = k_\Gamma(\beta)$ but $k_\Delta(\alpha) \neq k_\Delta(\beta)$, there must be a proper subset E of $\Gamma(\alpha) \cap \Gamma(\beta)$ such that $|E_\alpha| \neq |E_\beta|$. Now, if $\xi \in E_\alpha$, ...

ADDITION. On p. 36, in Remark (iii), add a reference to IVANOV & SHPECTOROV [915].

CORRECTION. On p. 37, Theorem 1.16.5 (iii), replace ‘§12.1’ by ‘§12.2’.

ADDITION. On p. 38, line 6, add after [639]: ‘and GARDINER [876] and Section 7.3 of BUCKLEY & HARARY [844]’.

HISTORICAL ADDITION. On p. 40, in the attribution of 1.17.2, add ‘; J.A. Bondy, cf. BIGGS [814]’.

ADDITION. On p. 40, Remarks, add:

(iii) GARDINER [876] further investigates distance regular graphs Δ with intersection array $\{k, k-1-\lambda, 1; 1, 1, k\}$ and finds the following (in the above notation):
a) $c \geq l-1$ with equality only for $c = 2$, i.e., for the line graph of the Petersen graph. [Indeed, if C is a complete subgraph of $\Delta_2(\delta)$, then each point of C has a neighbour on a different singular line on δ, so that $|C| \leq c$. Choosing $C = L \cap \Delta_2(\delta)$ for a singular line L containing an antipode of δ shows $c \geq l-1$. If $c > 2$ we can find a singular line contained in $\Delta_2(\delta)$, so that $c \geq l$.]
b) If $c = l$, then $c = l = 2$ and Δ is a hexagon. [This follows from the standard feasibility conditions.]
c) If $l = 4$, then $a = 41$. (No such graph is known.)

CORRECTION. On p.40, Theorem 1.17.4, change ‘b), c) or d)’ into ‘b), c), d) or e)’. In the proof, change ‘and the remark after the examples we have b), c) or d)’ into ‘we have e)’. [Indeed, as A. Munemasa points out, the graph with $(v,a,l) = (21,4,3)$ obtained from a non-unitary polarity of $PG(2,4)$ is an example of e), but not of b), c) or d).]

CORRECTION. The proof of Proposition 1.17.7 given in BLOKHUIS & BROUWER [821] contains a mistake (as was pointed out by D. Fon-der-Flaass). Add on p. 42 in the first sentence of Proposition 1.17.7 a third possibility:

> *, or B has precisely two components: one a line and the other a distance-regular graph Δ with intersection array $\{c-1, c-l, l; 1, 1, c-l\}$ on c^2 vertices (so that $|A| = cl$ and $|B| = l + c^2$), where $c = b - l + 1$.*

In this additional case the graph Δ_3 is the block graph of a transversal design $TD[l,c]$. There are two surviving parameter sets on fewer than 49000 points, belonging to $(l,c,v) = (5,21,551)$, $(4,40,1764)$. No examples are known.

CHAPTER 2

CORRECTION & ADDITION. On p. 44 in the middle, in the line containing the reference to MATHON [946], change 'both by his and by the above' into 'by his'. On p. 44 at the end of §2.1, add: 'Similar inequalities were given in HOBART [902].'

CORRECTION & ADDITION. Proposition 2.2.2 as stated is correct only if all eigenvalues of L_i are distinct. (And the double use of i is somewhat confusing; replace 'L_i' by 'L_j'.) If not all eigenvalues are distinct, we can say the following:

2.2.2a. Proposition *Let λ be an eigenvalue of L_j and put $H := \{h \,|\, P_{hj} = \lambda\}$. Let u be determined by $L_j u = \lambda u$, $u_0 = 1$, $u^\top \Delta_n u$ minimal. Then the multiplicity f of λ as an eigenvalue of A_j equals $f = \sum_{h \in H} f_h = n / u^\top \Delta_n u$.*

Proof. We have $u = \sum_{h \in H} \alpha_h Q e_h$ for certain constants α_h determined by $\sum \alpha_h f_h = 1$ and $n \sum \alpha_h^2 f_h$ minimal. But the minimum occurs when all α_h are equal (to α, say), and we find $\alpha = 1/\sum f_h$, and the minimum equals $u^\top \Delta_n u = n/\sum f_h$. ■

ADDITION. At the end of §2.2, add:

C. Automorphisms

2.2.9. Proposition (G. Higman, cf. CAMERON [852], Proposition 5.4). *Let σ be an automorphism of the association scheme $(X, \mathbf{R})$ and put $s_i := \#\{x \,|\, (x, \sigma x) \in R_i\}$. Then for each j the number $n^{-1} \sum_i s_i Q_{ij}$ is an algebraic integer.*

Proof. Let σ have permutation matrix S. Then S commutes with all A_i, and $s_i = \mathrm{tr}\, SA_i$. It follows that the number mentioned equals $\mathrm{tr}\, SE_j$, and hence is a sum of eigenvalues of S. ■

ADDITION. On p. 51, second sentence of §2.4, add after [178]: 'and ASCHBACHER [804]'.

ADDITION. On p. 54, at the end of §2.4, add:

Merging classes

Let $\mathbf{X} = (X, \{R_0, \cdots, R_d\})$ be an association scheme, and let Π be a partition of $\{1, \cdots, d\}$. If $\mathbf{X}_\Pi := (X, \{R_0\} \cup \{\bigcup_{j \in J} R_j \mid J \in \Pi\})$ again is an association scheme, then it is called a *fusion scheme* of $\mathbf{X}$. For some examples, see §4.2F.

If $\mathbf{X}$ has the property that each partition Π of $\{1, \cdots, d\}$ yields a fusion scheme, then $\mathbf{X}$ is called *amorphous*. A.V. IVANOV [919] shows that in this case each (X, R_j) $(1 \le j \le d)$ is a strongly regular graph, and either all are of Latin square type, or all are of negative Latin square type. IKUTA, ITO & MUNEMASA [910] give a characterization of amorphous schemes in terms of their group of 'pseudo automorphisms' (i.e., automorphisms of the Bose-Mesner algebra $\mathbf{A}$). BAUMERT, MILLS & WARD [809] determine when a cyclotomic scheme is amorphous.

GOLDBACH & CLAASEN [882] and SONG [991] study imprimitive non-symmetric association schemes.

In [BCN] the concept of 'tight design' was mentioned occasionally, but no definition was given. Now 'tight design' is dual to 'perfect code', and thinking about the right setup for tight designs in §2.8 also led to a more elegant version of Proposition 2.5.3.

2.5.3A. Proposition. *Let b, c be two real vectors indexed by* $\{0, \cdots, d\}$. *Then*

$$\sum_{i=0}^{d} \frac{(bQ)_i (cQ)_i}{f_i} = n \sum_{j=0}^{d} \frac{b_j c_j}{n_j}.$$

Now assume moreover that $b \ge 0$, $c \ge 0$, $bQ \ge 0$, $cQ \ge 0$ *and* $b_0 = c_0 = 1$.

(i) *If* $b_j c_j = 0$ *for all* $j \ne 0$, *then* $(bQ)_0 (cQ)_0 \le n$, *with equality if and only if* $(bQ)_i (cQ)_i = 0$ *for all* $i \ne 0$.

(ii) *If* $(bQ)_i (cQ)_i = 0$ *for all* $i \ne 0$, *then* $(bQ)_0 (cQ)_0 \ge n$, *with equality if and only if* $b_j c_j = 0$ *for all* $j \ne 0$.

Proof.

$$\sum_i f_i^{-1} (bQ)_i (cQ)_i = \sum_{i,\,j,\,k} f_i^{-1} b_j Q_{ji} c_k Q_{ki} = \sum_{i,\,j,\,k} n_k^{-1} b_j c_k Q_{ji} P_{ik} = n \sum_j n_j^{-1} b_j c_j.$$

Parts (i) and (ii) follow immediately. □

(Note that if b and c are the inner distributions of sets Y and Z, then $(bQ)_0 = |Y|$, $(cQ)_0 = |Z|$, and we obtain Proposition 2.5.3 from part (i).)

Now add at the end of §2.8:

Tight designs

The following proposition is a dual to Lloyd's Theorem 2.5.4 and MacWilliams' Inequality 2.5.5 (i).

2.8.4. Proposition. *Let Y be a t-design in a Q-polynomial association scheme* $(X, \mathbf{R})$, *and put* $s = [t/2]$. *Then*

(i) $|Y| \ge \sum_{i=0}^{s} f_i$, *and*

(ii) *at least s nonidentity relations occur between the elements of Y.*

(iii) *Equivalent are:* (a) *equality holds in (i);* (b) *equality holds in (ii);* (c) $\sum_{i=0}^{s} Q_{ji} = 0$ *for the nonidentity relations j occurring in Y.*

Proof. (i) Put $I = \{0, 1, \cdots, s\}$ and $m = \sum_{i \in I} f_i$ and $c_j = m^{-2} n_j (\sum_{i \in I} Q_{ji})^2$. Then $c_0 = 1$ and $c_j \ge 0$ and

$$(cQ)_k = m^{-2} \sum_{g,h \in I} n_j Q_{jg} Q_{jh} Q_{jk} = nm^{-2} f_k \sum_{g,h \in I} q^k_{gh} \geq 0,$$

so that in particular $(cQ)_0 = nm^{-1}$.
Since $(X, \mathbf{R})$ is Q-polynomial, we have $(cQ)_k = 0$ for $k > 2s$. Now if we let b be the inner distribution of Y, and apply 2.5.3A (ii), we find (i), and the equivalence of (iiia) and (iiic).

(ii) We have $Q_{jk} = q_k(z_j)$ for certain numbers z_j and polynomials $q_k(z)$ of degree k. Now assume that $b_j \neq 0$ for at most s nonzero values of j, and let $f(z) = \sum_{k=0}^{s} c_k q_k(z)$ be a polynomial of degree at most s such that $f(z_j) = 0$ whenever $b_j \neq 0$ and $j \neq 0$. We may take $c_0 = 1$. We have $f(z_j) = \sum_{k=0}^{s} c_k Q_{jk}$, and in particular

$$f(z_0) = \sum_{k=0}^{s} c_k f_k.$$

Also

$$f(z_0) = \sum_j f(z_j) b_j = \sum_{k=0}^{s} c_k (bQ)_k = |Y|.$$

Finally,

$$f(z_0)^2 = \sum_j f(z_j)^2 b_j = \sum_j \sum_{k,l=0}^{s} b_j c_k c_l Q_{jk} Q_{jl} = \sum_{i,j} \sum_{k,l=0}^{s} b_j c_k c_l q^i_{kl} Q_{ji} =$$
$$= \sum_j \sum_{k,l=0}^{s} c_k c_l q^i_{kl} (bQ)_i = \sum_{k=0}^{s} c_k^2 f_k |Y|.$$

Combining these three equations and using (i), we find $\sum_{k=0}^{s} (c_k - 1)^2 f_k \leq 0$, so that $c_k = 1$ for all k. This proves (ii) and (iiib) $\Rightarrow$ (iiia). But clearly (iiic) implies that at most s nonidentity relations occur, so (iiic) $\Rightarrow$ (iiib), finishing the proof. □

A t-design in a Q-polynomial association scheme $(X, \mathbf{R})$ is called *tight* when it satisfies the three equivalent conditions of Proposition 2.8.4.

The concept of t-design in a cometric association scheme has been generalized by DELSARTE, GOETHALS & SEIDEL [869] to spherical designs in Euclidean spaces, cf. §2.11. The concept of Delsarte spaces (NEUMAIER [959], BLOKHUIS [819]) provides a common setting for designs in Q-polynomial schemes, spherical designs and designs in real, complex, quaternion and octonion projective spaces (HOGGAR [905, 906], BANNAI & HOGGAR [808]). All results on cometric schemes generalize to Delsarte spaces. For a recent survey, see HOGGAR [907].

CORRECTION. On p. 56, line 8, change ‘theorem’ into ‘proposition’.

ADDITION. The McFarland difference set (MCFARLAND [948]) provides a counterexample to the conjecture on p. 68. [Indeed, a difference set with multiplier −1 gives a translation strongly regular graph. The McFarland difference set with parameters (4000, 775, 150) in $\mathbf{Z}_5^3 \times \mathbf{Z}_2^5$ has multiplier −1 and gives rise to strongly regular graphs with parameters $(v, k, \lambda, \mu) = (4000, 774, 148, 150)$ and (4000, 775, 150, 150); their duals have parameters (4000, 1935, 910, 960) and (4000, 1984, 1008, 960). For difference sets with multiplier −1, see, e.g., JUNGNICKEL [925], GHINELLI-SMIT [987], POTT [974], MA [944], ARASU, JUNGNICKEL & POTT [802].]

CHAPTER 3

CORRECTION. On p. 90-91, Theorem 3.6.1 and its proof were formulated somewhat imprecisely. A polished version follows.

3.6.1 Theorem. *Let Γ be a graph on v vertices with a (p,q,r)-representation in $\mathbb{R}^n$, $p \neq q, r$. Then Γ has at most $\frac{1}{2}n(n+3)$ vertices. Moreover, if Γ is a disjoint union of cliques, then $v \le n+1$, except when some component of Γ has size $s := (p-q)/(r-q)$ (so that s is an integer > 1); in this case $v \le n+t-\delta \le 2(n-\delta)$, where t is the number of maximal cliques of size s, and $\delta = 1$ if $r \neq 0$, $\delta = 0$ if $r = 0$.*

Proof. ... Since $p \neq q$, a straightforward calculation shows that $\mathrm{rk}\,\overline{G} \ge r-1$ unless Γ has components of size $s = (p-q)/(r-q)$; in this case $\mathrm{rk}\,\overline{G} = r+\delta-t$, where t is the number of components of size s. Now ... □

[The text of the proof is ugly as well; the variable r occurs in two rôles; call the number of cliques m instead of r. Also, in line 7 on p. 91, replace 'matrix G' by 'space $\mathbb{R}^n$'.]

SIMPLIFICATION. On p. 92-93, change the proof of Proposition 3.7.3 into:

Proof. Let $\overline{\alpha}$ denote the image of a vertex α under the given representation. Now define a new representation by projection onto $\overline{\gamma}^{\perp}$, i.e., by sending (in case (i)) $\delta \in \Gamma(\gamma)$ to $\overline{\delta} - \frac{q}{p}\overline{\gamma}$, and (in cases (ii), (iii)) $\delta \notin \gamma^{\perp}$ to $\overline{\delta} - \frac{r}{p}\overline{\gamma}$. □

CORRECTION. On p. 104, line 7, the parameters should be (16,5,0,2). Similarly, in the index on p. 480, line 11, this parameter set should be listed as {5,4; 1,2}.

CORRECTION. On p. 106, line −10, the reference should have been to HOFFMAN [904].

ADDITION. On p. 110, before Theorem 3.12.5, add a reference to BEINEKE [810] for characterizations of line graphs by forbidden subgraphs.

ADDITION. On p. 111, after line 5, add a reference to WOO & NEUMAIER [1019] for a characterization of graphs with large minimum valency and smallest eigenvalue at lest $-1-\sqrt{2}$.

CORRECTION. On p. 114, line 3, change '3.10' into '3.11'.

ADDITION. On p. 114 add to the list of examples of amply regular code graphs the polygons, the dodecahedron and the doubled Odd graphs.

CORRECTION. On p. 116 the conclusion of Theorem 3.15.1 in case $\mu = 4$ is incomplete: there is another family of regular root graphs with $\mu = 4$. See BROUWER & KOOLEN [840].

CORRECTION. On p. 120 the proof of Proposition 3.15.2 is incomplete in case $\mu = 2$, $\lambda > 0$, since it is not immediately clear why Γ should be locally connected. However, if $\Gamma(\gamma)$ is disconnected, then it must contain a $(\lambda+2)$-clique, and it is not difficult to check that this clique will be a direct factor of Γ. But then $\mathbb{L}(\Gamma)$ is decomposable, contradiction.

CORRECTION. On p. 125 the proof in Case 3 is incorrect. Insert in the line before Case 1 a new condition S9, and change the text in Case 3 as follows.

S9. If Δ_0 strictly contains an n-gon, then Δ_0 has girth 5, and $n = 5$ or $3|n$.

Indeed, if P is an n-gon in Δ_0 with $n \geq 6$ and a a vertex at distance 1 to P, then S8 implies that a is adjacent to every third vertex of P, so that $3|n$.

CASE 3. Δ_0 *has girth* 5. Then Δ_0 is not bipartite, and since $L(\Delta_0)$ must be regular of valency λ, Δ_0 is regular of valency $\frac{1}{2}\lambda+1$. If $\lambda = 2$ then Δ_0 is the pentagon and Γ is locally a disjoint union of pentagons. Now the set of all $\delta \in \Gamma$ such that all edges in $S(\delta)$ lie in Δ_0 is easily seen to be an icosahedron, and by our assumptions Γ itself is an icosahedron. If $\lambda > 2$, then for every vertex i of a pentagon $1 \sim 2 \sim 3 \sim 4 \sim 5 \sim 1$ of Δ_0, the set $N(i)$ of neighbours of i outside this pentagon is a $(\frac{1}{2}\lambda-1)$-coclique, and by S8, $N(i) \cup N(i+2)$ (i mod 5) is a complete bipartite graph. Since there are no quadrangles, this forces $\lambda = 4$, and Δ_0 is the Petersen graph, so that Γ is locally a disjoint union of line graphs of the Petersen graph. Now every $K_1 + K_{1,2}$ of the Petersen graph is in some path of length 4, and S7 shows that every 4-set containing two intersecting edges of Δ_0 is special. But there are 8 such 4-sets through an edge ab of Δ_0, and they determine 8 neighbours of the point δ of Γ represented by ab, forming an 8-gon in $\Gamma(\delta)$. But $3 \nmid 8$, contradicting S9. □

CHAPTER 4

ADDITION. On p. 135, at the end of §4.1D, add: 'LAMBECK [934] observes that the specialization $a_d \leq p^d_{d,d-1} + p^d_{d,d}$ whenever $p^d_{d,d} \neq 0$ rules out the intersection array $\{39,32,20,2;1,4,16,30\}$ ($v = 768$).'

ADDITION. On p. 136, line −11, insert the sentence: 'PAULUS [971] found conference graphs on 26 vertices with trivial group of automorphisms.'

CORRECTION. On p. 136, part (ii) of Theorem 4.1.10 is incorrect. In fact the stabilizer of a single antipodal class is 2-transitive on that class, but need not stabilize any other classes, or be normal in G. On the other hand, the stabilizer N of all antipodal classes is normal in G, but may be reduced to the identity.

ADDITION. Delorme's preprint has now appeared. On p. 138, add a reference to DELORME [867].

CORRECTION. On p. 138, line –2, replace 'and $c'_{i+1} > c'_i$' by 'and $c'_{i+1} > c''_i$'.

ADDITION. On p. 139 add before the remarks, after '$d > 4$.':

For $d = 4$ there is a sporadic example on $100+280$ vertices belonging to the Hall-Janko group (cf. BAGCHI [805]).

Add at the end of §4.2G:

(iii) NOMURA [967] obtained some parameter conditions using distance distribution diagrams around an edge.

(iv) SUZUKI [1000] classifies the distance-biregular graphs of girth divisible by 4 with some trivalent vertices.

Add at the end of §4.2H a reference to HILANO [897].

ADDITION. On p. 143, Corollary 4.2.6 can be sharpened somewhat (following Godsil & Hensel).

4.2.6. Corollary. *In particular, in case* $d = 1$, $\tilde{d} = 3$, *the two new eigenvalues* θ *are the two roots of* $\theta^2 - (k - 1 - r\mu)\theta - k = 0$, *(with* $\mu = \mu(\tilde{\Gamma})$*), and have multiplicity* $m(\theta) = \dfrac{k(k+1)(r-1)}{k+\theta^2}$. *Moreover,* $k\mu$ *is even, and if* $k \neq r\mu + 1$, *then* θ *is an integer.*

Proof. We find that $(r-1)k\mu$ is even by counting edges in $\tilde{\Gamma}(\infty)$ for some $\infty \in \tilde{\Gamma}$. Suppose $r \equiv k \equiv \mu \equiv 1$ (mod 2). Then the eigenvalues θ are odd integers, and their sum is also odd, contradiction. □

Almost all of the above is already contained in BIGGS & GARDINER [816]; the parity condition in 4.2.6 is from GODSIL & HENSEL [879]. GODSIL, JURIŠIĆ & SCHADE [880] discuss the case $\tilde{d} \in \{4,5\}$, and in particular find that when $\tilde{d} = 5$ at least one of the three new eigenvalues is integral.

ADDITION. On p. 145, after Proposition 4.2.11 add: ‘JURIŠIĆ [928] notes that if E is any distance-regular graph that is both bipartite and antipodal and has odd diameter, then it is a 2-cover (by the array given on p. 142) and hence 4.2.11 applies.

ADDITION. On p. 146, line 2, add: ‘The Odd graphs O_{d+1} are the only generalized Odd graphs with $c_2 = 1$, $c_3 = c_4 = 2$ [Koolen].’

SPELLING CORRECTION. On p. 146, line –4, replace ‘the’ by ‘then’.

ADDITION. On p. 147, bottom, add: ‘SUZUKI [1001] shows that case (iii) of the above proposition does not occur at all.’

LAYOUT CORRECTION. On p. 149 the page heading lost its white space.

CORRECTION. On p. 151, line 6, the intersection array should be $\{231,160,6;1,48,210\}$.

ADDITION. On p. 151, after the Problem, add:

(This question was posed for bipartite Γ by VETCHÝ [1014] who gave partial results. But from the above and Corollary 5.4.2 one sees that the only such bipartite Γ with $d \geq 4$ are those of examples (i) and (ii) above; for $d = 3$ one finds precisely the graphs $\overline{2 \times m}$.)

ADDITION. On p. 151, at the end of §4.2, add:

Nowadays an association scheme obtained by merging classes is usually called a *fusion scheme*, cf. §2.4A.

ADDITION. On p. 152, part (iv) of Proposition 4.3.2 may be deleted, since this condition is never fulfilled (E.W. Lambeck, cf. an addition to p. 174). Partly generalizing Proposition 5.4.3 we have a (rather weak) sufficient condition for the existence of singular lines:

(v) $c_{i-1} + b_i > b_1$

Proof. (In case (v).) Consider a triangle $\alpha\beta\gamma$. If there exists a vertex $\delta \in \Gamma_2(\alpha) \cap \beta^\perp \cap \gamma^\perp$ then there are $p^2_{i-1,i+1} + p^2_{i-2,i}$ vertices ε with distances $i-1, i, i, i+1$ or $i, i-1, i-1, i-2$ to $\alpha, \beta, \gamma, \delta$, respectively. On the other hand, there are $p^1_{i,i-1}$ vertices with distances $i-1, i, i$ or $i-1, i, i-1$ to α, β, γ and the number of vertices with distances $i-1, i, i-1$ equals that with distances $i, i-1, i-1$ to α, β, γ (both counts equal $p^1_{i-1,i}$ minus the number of vertices with distances $i, i, i-1$), so we find $p^2_{i-1,i+1} + p^2_{i-2,i} \le p^1_{i,i-1}$, i.e., $b_i + c_{i-1} \le b_1$. ■

ADDITION. On p. 152, add a Proposition 4.3.5 stating that if some singular line in Γ has size $\lambda+2$, then $\lambda + 2 \le 1 + k/(-\theta_d)$ (because of 4.4.6 (i)). This rules out the intersection array $\{45, 40, 11, 1; 1, 1, 40, 45\}$ ($v = 2352, \theta_4 = -11$).

Add a subsection:

Big cliques

As Godsil [878] remarked, the standard claw-and-clique arguments due to Bose can sometimes be applied to distance-regular graphs to conclude that these are geometric, i.e., the collinearity graph of a partial linear space.

Lemma. *If Γ contains an s-claw, then* $k \ge s(\lambda+1) - \binom{s}{2}(\mu - 1)$. ■

[Godsil also shows, following Metsch, that if Γ contains an s-claw and $\lambda \ge (2m-1)(\mu-1)$, then $k \ge s(\lambda+1) - s(s-1)^2$.]

Proposition. *Let Γ be a distance-regular graph with smallest eigenvalue $-m$. If Γ does not have s-claws for $s > m$ and $\lambda \ge (2m-1)(\mu-1)$, then Γ is the collinearity graph of a partial linear space with lines of size $1 + k/m$ and nexus $e = 1 + k/m - b_1/(m-1)$. In particular, m, k/m and $b_1/(m-1)$ are integers.* ■

This proposition rules out the intersection arrays $\{140, 104, 1; 1, 4, 140\}$ ($v = 3807$) and $\{104, 78, 1; 1, 3, 104\}$ ($v = 2835$) - the latter because the resulting partial linear space must be a $GQ(26, 4)$ minus a spread (cf. Proposition 12.5.2), but no $GQ(26, 4)$ exists.

ADDITION. On p. 153, after Proposition 4.3.6, add: ‘The situation of this proposition is also discussed in Matsumoto [947].’

ADDITION. On p. 153, after Corollary 4.3.7, add:

In the terminology of Chapter 11, this corollary implies that the codes $\pi^{-1}(\gamma)$ are uniformly regular, and then Theorem 11.1.6 says that if Γ is distance-regular, then these codes are completely regular. (Similar observations were also made by Rifà & Huguet [979].)

CORRECTION. On p. 154, the last line of Corollary 4.3.8, replace ‘halved’ by ‘folded’.

ADDITION. On p. 155 at the end of §4.3B, add:

In fact, NOMURA [966] shows that if $\lambda \neq 2$ then no condition on a_3 is necessary, i.e., if $\mu = 2$, $c_3 = 3$, $\lambda \neq 2$ and $a_2 = 2\lambda$, then we have a quotient of a Hamming graph. NOMURA [964] shows that if $\mu = 2$, $c_3 = 3$ and $d(\gamma,\delta) = 3$, then $C(\gamma,\delta)$ is either a cube or the direct product $K_2 \times K_{1,1,2}$, and the latter can occur only for $\lambda = 2$. COOLSAET [859] shows that if $\lambda = \mu = 2$, $a_2 = 4$, then Γ is locally a union of triangles, hexagons and heptagons. Furthermore, that a vertex together with a triangle, hexagon or heptagon in its neighbourhood is contained in a unique K_4, Shrikhande or Klein subgraph of Γ, respectively, where the Klein graph is the unique distance-regular graph with intersection array $\{7,4,1;1,2,7\}$ (cf. p. 386, Remark (iv)). He concludes that there is no distance-regular graph with intersection array $\{13,10,7;1,2,7\}$.

ADDITION. On p. 157 add a new Section 4.3E.

E. Pappus graphs

4.3.16. Proposition (KOOLEN [930]). *Let Γ be a graph with $a_1 = a_2 = a_3 = 0$ and $c_2 = 1$, $c_3 = 2$ and $c_4 = 3$. Then any pair of vertices at mutual distance 4 determines a unique geodetically closed Pappus subgraph (with intersection array $\{3,2,2,1;1,1,2,3\}$ on 18 points).*

Proof. Let $d(\gamma,\delta) = 4$. Then $C(\gamma,\delta) \setminus \{\gamma,\delta\}$ induces a 12-gon $(\alpha_0\alpha_1 \cdots \alpha_{11}\alpha_0)$ with $\gamma \sim \alpha_{4t+1}$ and $\delta \sim \alpha_{4t+3}$ $(t = 0,1,2)$. We have $d(\alpha_0,\alpha_3) = 3$ and $\Gamma(\alpha_3) \cap \Gamma_2(\alpha_0) = \{\delta,\alpha_2\}$, so $d(\alpha_0,\alpha_4) = 4$. We have $d(\alpha_5,\delta) = 3$ and $\Gamma(\delta) \cap \Gamma_2(\alpha_5) = \{\alpha_3,\alpha_7\}$, so $d(\alpha_5,\alpha_{11}) = 4$. In the 12-gon $C(\alpha_5,\alpha_{11}) \setminus \{\alpha_5,\alpha_{11}\}$ we see $\alpha_0 \sim \beta_0 \sim \alpha_6$ and $\alpha_4 \sim \beta_2\alpha_{10}$ for certain vertices β_0, β_2. Similarly we find $\alpha_2 \sim \beta_1\alpha_8$. In the 12-gon $C(\alpha_0,\alpha_4) \setminus \{\alpha_0,\alpha_4\}$ we see $\beta_0 \sim \varepsilon\beta_2$, and similarly we find $\beta_1 \sim \varepsilon' \sim \beta_0$. But $\{\alpha_0,\alpha_6,\varepsilon,\varepsilon'\} \subseteq \Gamma(\beta_0) \cap \Gamma_3(\alpha_3)$, so $\varepsilon = \varepsilon'$ and we have found a Pappus subgraph on 18 vertices. □

4.3.17. Corollary. *The numbers $\frac{1}{4}b_1b_2b_3$ and $\frac{1}{36}vk_4$ are integral.* □

This rules out the existence of distance-regular graphs with intersection array $\{7,6,6,5,4,3;1,1,2,3,4,7\}$ $(v = 686,\ k_4 = 210)$.

F. General

SUZUKI [1003] investigates strongly closed subgraphs of distance-regular graphs. (A subgraph Δ of Γ is strongly closed if for $\gamma,\delta \in \Gamma$, with $d(\gamma,\delta) = i$, also $\Gamma(\delta) \cap \Gamma_{\leq i}(\gamma) \subseteq \Delta$.)

ADDITION On p. 160 add to the Remarks: Yamazaki & Nomura announce (email, 940301): ‘Let Γ be a bipartite distance-regular graph with valency $k \geq 3$ and diameter $d \geq 3$. Let θ be an eigenvalue of Γ and f the multiplicity w.r.t. θ. If $k = f$, then Γ is 2-homogeneous.’ and ‘Let Γ be a bipartite 2-homogeneous distance-regular graph with valency $k \geq 3$ and diameter $d \geq 3$. Then Γ is a d-cube or the antipodal 2-cover with $d \leq 5$. These results have now been published: NOMURA [969] and YAMAZAKI [1020].’

ADDITION. On p. 161 add before the last sentence of the Remark: ‘Conversely, a completely regular clique C of Γ with covering radius $d-1$ satisfies (28) with equality.’

CORRECTION. On p. 162 replace equations (30c) and (30d) by

$$f - i \geq \min(2 + \tfrac{1}{2} b_i, b_i) \;\; \textit{if} \;\; \lambda = 1 \textit{ or } \mu = 1, \tag{30c}$$

$$f - i \geq \min(2 + \frac{\lambda}{\lambda + 1} b_i, b_i) \;\; \textit{if} \;\; \lambda > \mu = 1 \textit{ and } i \leq 1. \tag{30d}$$

ADDITION. On p. 163 add after Remark (iv):

(v) ZHU [1022] (Proposition 3.5 (i)) notices that a coconnected distance-regular graph with an eigenvalue of multiplicity $f > 2$ also satisfies $\lambda \leq \frac{1}{2}(f-2)(f+1)$. (The proof is the same as that for the inequality $b_1 \leq \frac{1}{2}(f-2)(f+1)$ above - all one uses of the subgraph Δ in the proof above is that $d(\delta, \gamma_l)$ is known for $\delta \in \Delta$ and $0 \leq l \leq i$, and that Δ has diameter at most 2 in Γ.)

ADDITION. On p. 165 add at the end of line 8: ‘and the Ivanov-Ivanov-Faradjev graph’.

CHAPTER 5

ADDITION. On p. 168, bottom, the question in Problem (ii) was (almost) answered affirmatively by Hiroshi Suzuki.

5.1.4. Proposition (SUZUKI [994]). *If Γ is distance-regular with $k_i = k_j$ where $i < j$ and $i + j \leq d$, then either Γ is antipodal (with $k_d = 1$), or $k_l = k_i$ for $i \leq l \leq j$. Moreover, if in the latter case we have $k_j \neq k_{j+1}$, then $p^d_{ij} = 0$ for $i + j > d+1$, so that in particular $\Gamma_d(\gamma)$ is a clique for all γ.* □

HIRAKI, SUZUKI & WAJIMA [901] show that if $j+2 \leq d$ then $k_2 = k_j$ implies $k = 2$ or $k_d = 1$.

ADDITION. On p. 170 add after Remark (v) and before Corollary 5.2.2:

(vi) The inequalities (1) imply (by induction) that

$$a_j + 2c_j \geq j(\lambda + 2), \tag{*}$$

and, as the proof shows, one only needs local assumptions (distance-regularity up to distance j) to prove this. It would be interesting to characterize the graphs with equality in (*); note that the geometric information in 5.2.1 is available for all $i \leq j$. For $j = d$, (*) is equivalent to Corollary 5.2.2 (and the equality case is characterized in Theorem 5.2.3).

ADDITION. On p. 171, add after Theorem 5.2.5:

In the case where Γ has girth $2s$, Terwilliger’s argument can be sharpened to provide the diameter bound (3c) without the restriction on a_{s-1} (Neumaier [pers. comm.]).

ADDITION. On p. 172, strengthen Theorem 5.3.2 as follows:

5.3.2. Theorem (GODSIL [877]). *There are only finitely many coconnected distance-regular graphs with an eigenvalue of fixed multiplicity $f > 2$. Any such graph has diameter $d \leq 3f - 4$ and valency $k \leq \frac{1}{2}(f-1)(f+2)$. If $\lambda > 0$, then $d \leq 2f - 2$. If $\lambda = 0$, then $k \leq f$.*

Proof. Let $f > 2$. If $d \geq 3f - 3$ (or $d \geq 2f - 1$ and $\lambda > 0$), then the lemma implies $b_{f-1} > 1$, and

Proposition 4.4.8 (iii) gives a contradiction. Hence $d \le 3f - 4$, and the same proposition gives $k = b_0 \le \frac{1}{2}(f-1)(f+2)$ (and $k \le f$ if $\lambda = 0$). ... □

ADDITION. On p. 172, add:

SUZUKI [1002] improves the diameter bound of 5.3.2 to '$2d \le 5f - 11$ when Γ is not an antipodal 2-cover' as a consequence of the following improvement to 5.3.1: *If Γ is distance-regular, not an antipodal 2-cover and not a polygon, then if $b_e = 1$ and $d \ge 2e+1$, it follows that $2d \le 5e - 8$.*

ADDITION. On p. 172, add at the end of Remark (ii):

ZHU [1022] determined the distance-regular graphs with a multiplicity $f = 4$. He finds K_5, $L(K_5)$, $\overline{K_{3,3}}$, $L(K_{3,3})$, $\overline{T(5)}$, $L(\overline{T(5)})$, the Pappus graph, the Desargues graph, the 4-cube, the dodecahedron, $K_{4\times 2}$, $\overline{2\times 5}$ and $K_{5\times m}$ for $m \ge 2$.

ADDITION. The question in the problem on p. 174 was answered affirmatively by LAMBECK [932]. [Indeed, in the proof of 4.3.2 (iv) we saw that if $b_i > 1$ then there are singular lines of size $\lambda+2$, and any singular line meeting $\Gamma_i(\gamma)$ in at least two points is contained in $\Gamma_i(\gamma) \cup \Gamma_{i+1}(\gamma)$. Viewed from a neighbour γ_1 of γ we have a contradiction.]

CORRECTION. In part (ii) of Proposition 5.5.1 (page 174) a restriction was omitted. A corrected version follows.

5.5.1. Proposition. *Let Γ be a distance-regular graph of diameter d and assume $\lambda > 0$. Then, for $i \in \{1, \cdots, d-1\}$:*

(i) $2a_i \ge \lambda + 1$,

(ii) $a_i + a_{i+1} \ge \lambda + 1$, *provided that $i < d-1$ or $a_d > 0$ or ($d > 2$ and $b_{d-1} > 1$),*

(iii) $b_i + c_{i+1} \ge \lambda + 2$.

Proof. Let γ, δ be two points of Γ at distance $i+1$, and let $\gamma = \gamma_0, \gamma_1, \cdots, \gamma_i, \gamma_{i+1} = \delta$ be a geodesic from γ to δ.
(i) ...
(ii) By Lemma 4.1.7, the size of $\Gamma_{i+1}(\gamma) \cap \Gamma_2(\gamma_i)$ is $p^i_{2,i+1} = b_i(a_i + a_{i+1} - \lambda)/\mu$ so that $a_i + a_{i+1} \ge \lambda$. Suppose that $a_i + a_{i+1} = \lambda$. Then by (i) we have $i = d-1$. Since $p^{d-1}_{2,d} = 0$, each vertex of $\Gamma(\delta) \cap \Gamma_{d-1}(\gamma)$ is adjacent to each vertex of $\Gamma(\delta) \cap \Gamma_d(\gamma)$. If $a_d > 0$, this means that $\overline{\Gamma(\delta)}$ is disconnected. But this contradicts Lemma 1.1.7. Hence $a_d = 0$. If $b_{d-1} > 1$, then let $\delta, \delta' \in \Gamma(\gamma_{d-1}) \cap \Gamma_d(\gamma)$. We find $a_2(\delta,\delta') = 0$, so by (i) we have $d = 2$.
(iii) ... □

For a strengthening of part (ii), see inequality (4) following Lemma 5.5.5. The proviso in part (ii) is necessary, as shown by the antipodal graphs. We do not know of nonantipodal cases where $a_{d-1} = \lambda$, $a_d = 0$.

CORRECTION. On p. 176, in Proposition 5.5.4 parts (i) and (ii), the condition for equality is stated incorrectly. The text should read:

5.5.4 Proposition (BROUWER & LAMBECK [841]). *Suppose the intersection array of a distance-regular graph satisfies $a_i \ne 0$, and define $a_{d+1} = 0$.*

(i) *If* $i < d$, *then* $b_i \le a_i + \dfrac{a_{i+1} b_i}{a_i}$, *and equality implies* $a_{i-1} \ge a_i \ge a_{i+1}$.

(ii) *If* $i > 1$, *then* $c_i \le a_i + \dfrac{a_{i-1} c_i}{a_i}$, *and, for* $i < d$, *equality implies* $a_{i+1} \ge a_i \ge a_{i-1}$.

(iii) *If* $1 \le i \le d$, *then* $b_i + c_i \le a_i + \dfrac{a_{i+1} b_i}{a_i} + \dfrac{a_{i-1} c_i}{a_i}$, *and equality holds if and only if for any four vertices* $\alpha, \beta, \gamma, \delta$ *of* Γ *such that* $\alpha \sim \beta$, $\gamma \sim \delta$, *and* $d(\alpha,\gamma) = i$, *the three distances* $d(\alpha,\delta)$, $d(\beta,\gamma)$, $d(\beta,\delta)$ *are not all equal. In particular, equality implies* $a_{i+1} \le a_i$ *and* $a_{i-1} \le a_i$.

Proof. ... Similarly, we find $a_{i+1} \le a_i$ in case of equality in (i), and comparing with (iii) yields $a_{i-1} \ge a_i$. For (ii) the proof is similar. □

CORRECTION. On p. 178, delete lines −8, −7 and the second inequality in 5.5.6 (i). In the second and sixth line of the proof of 5.5.6, replace '$\beta^{\perp}$' by '$\Gamma(\beta)$'.

ADDITION. On p. 179, add a Remark (iv):

Nomura [968] observes that if $c_s = c_{s+t}$ and $d(\gamma,\delta) = t$, then $\Gamma_s(\gamma) \cap \Gamma_{s+t}(\delta)$ is a subgraph of Γ on $p^t_{s,s+t}$ vertices, regular of valency a_s. Thus, if $v(k,g)$ denotes the minimum number of vertices of a graph of valency k and girth g (cf. §6.9), and Γ has girth g, then $p^t_{s,s+t} \ge v(a_s, g)$. Similarly, if $b_s = b_{s+t}$, then $p^t_{s,s+t} \ge v(a_{s+t}, g)$, and if $b_s = c_t$, then $p^{s+t}_{s,t} \ge v(a_t, g)$. Now $v(k,g)$ may be estimated by (21b) in §6.7, and we find restrictions for graphs with very large a_i:

Lemma. *Let* Γ *have girth* $g = 2r+1$.

(i) *If* $2a_r \ge k$ *and* $d \ge 2r$, *then* $c_{2r} > 1$.

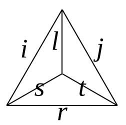
(ii) *If* $c_{r+1} = 1$, *then* $(a_r - 1)^r < (k-1)^{r-1} b_r$.

(iii) *If* $a_e = \cdots = a_{e+m} = k - 2 > 1$, *then* $m \le r-1$ *and* $m < (\frac{1}{2} + \varepsilon)e$, *where* $\varepsilon < 1/2(k-3)\log(k-3)$. ■

ADDITION. On p. 179, at the end of Section 5.5, add:

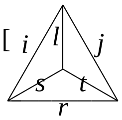
A. 4-vertex conditions

Let Γ be a distance-regular graph. Given a finite metric space $\mathbf{X} = (X,d)$, let $[\mathbf{X}]$ be the number of distance-preserving embeddings of $\mathbf{X}$ into Γ. In particular, when $|X| = 4$, we call the $[\mathbf{X}]$ 4-*vertex counts*; these satisfy

$\sum_l$ [

] $= v k_r \, p^r_{st} \, p^r_{ij}$ for all i, j, r, s, t

and, as a consequence of Theorem 2.3.2 (5),

$$\sum_{i,j,l} Q_{ia}Q_{jb}Q_{lc}[$$ 

$$] = 0 \text{ for all } r, s, t \text{ and all } a, b, c \text{ with } q_{ab}^{c} = 0.$$

These linear equations must be consistent with the nonnegativity constraints $[\mathbf{X}] \ge 0$; for particular intersection arrays this can be checked by standard linear programming software. In particular, for Q-polynomial graphs, TERWILLIGER [1006] shows that all 4-vertex counts can be expressed as linear combinations of the 4-vertex counts in which all distances are equal and at most $[d/2]$. For $d = 3$ this allows us to express all 4-vertex counts as linear functions of the number of 4-cliques, and the nonnegativity constraints give lower and upper bounds for this number. If one does this for the special case of the intersection array $\{19,12,5;1,4,15\}$ (with $v = 96$) one finds a contradiction, so that no such graph exists (NEUMAIER [963]).

CORRECTION. On p. 181, Proposition 5.6.3, we wrote in a moment of unbelievable blindness ‘if and only if’, while three counterexamples to that statement are given on the same page. This causes changes in three places:
On p.145, line −3, delete the sentence ‘The graphs ... 5.6.3.’.
On p. 180, line −2, delete ‘precisely the’.
On p. 181, delete the ‘Conversely’ part of the proof of Proposition 5.6.3, and replace its statement by:

5.6.3. Proposition. *Let Γ be distance-regular of diameter $d \ge 3$, but not antipodal. If $k = k_d(k_d - 1)$ then $k_d = a_d + 1$, i.e., Γ_d is a generalized Odd graph.*

ADDITION. On p. 181, after the example following Lemma 5.6.4, add: ‘(Such graphs are also ruled out by observing that $p_{22}^{1} = \frac{7}{2}$ is not integral.)’.

ADDITION. On p. 181, at the end of Section 5.6, insert: ‘SUZUKI [998] shows that if $\Gamma_d(\gamma)$ is not a union of cliques, and $c_r = 1$, $a_r = \lambda$, then $r < 2(k_d - 1)$. Using Ivanov’s bound it follows that d is bounded by a function of k_d for distance-regular graphs Γ for which $\Gamma_d(\gamma)$ is not a union of cliques. SUZUKI [997] extends this result and shows that d is bounded by a function of k_d for distance-regular graphs with $a_d \ne 0$ and $k > 2$. Let $h = \max\{i \mid p_{di}^{d} \ne 0\}$ be the *height* of Γ. TOMIYAMA [1012] studies graphs of height 2.’

ADDITION. On p. 182, insert before the remark:

5.7.2. Proposition (HIRAKI & SUZUKI [900]). *Let Γ be a distance-regular graph with diameter d and valency k. Suppose that for some $e \ge 1$ we have $b_i = c_{d-i}$ for $1 \le i \le e$. Then $c_i = b_{d-i}$ for $1 \le i \le e$, and if $a := a_d \ne 0$ then $k = a(a+1)$, $b_i = a^2$ for $1 \le i \le e$ and $c_i = 1$ for $1 \le i \le e+1$.* □

This implies another affirmative answer to Problem (iii) on this page, earlier answered by Lambeck (see above, on p. 174).

ADDITION. On p. 182, line −5, add to Problem (i), parenthetical remark: ‘LAMBECK [932] shows that $d = 4$ is impossible, and gives some restrictions in case $d = 3$’.

ADDITION. On p. 182, Problem (iii) is equivalent to the problem on p. 174, and was solved affirmatively by E.W. Lambeck (see above).

CORRECTION. On p. 182, Problem (iv), the first occurrence of 'b_i' should have been 'b_{d-1}'.

ADDITION. On p. 183, strengthen Proposition 5.8.1 by adding that in case of equality we have $|\{\alpha,\beta,\gamma\}^{\perp}| = \mu(k-2-2\lambda)/p$ for every triple α, β, γ of vertices at pairwise distance 2.

CORRECTION. On p. 185, third line to the right of the diagram should end in $(i+2)^{b_{i+1}}$.

ADDITION. On p. 189 the text in small print shows $v > d^2$ for $d \ge 20$. Using the same ideas one can derive a bound that is slightly better and valid for all d.

Proposition. *Let Γ be a distance-regular graph of diameter $d \ge 4$ and valency $k > 2$. Put $e = [(d-1)/3]$. Then*

$$v - 2 \ge \max(4e^2 + (3k-2)e,\ \ 13e^2 + e).$$

Proof. For $1 \le i < \frac{1}{3}d$ we have $k_{i+1} = b_i k_i / c_{i+1} > k_i$, so that $b_i > c_{i+1}$ and

$$k_{i+1} \ge \left\lceil k_i\left(1 + \frac{1}{c_{i+1}}\right)\right\rceil \ge \max\left(k_i + 2,\ k_i\left(1 + \frac{1}{k-2}\right)\right)$$

for these i. Now $k_i/c_{i+1} \ge k_{i-1}/c_i$ (since $c_i k_i = b_{i-1}k_{i-1} \ge c_{i+1}k_{i-1}$ for $d \ge 2i$), so $k_i \ge k_j + (i-j)e_j$ for $1 \le j \le i$, where $e_j = \lceil k_j/c_{j+1} \rceil$. Now Corollary 5.1.3 yields

$$\begin{aligned} v &\ge 2\sum_{l=0}^{j-1} k_l + (i-j+1)(2k_j + (i-j)e_j) + (d-1-2i)(k_j + (i-j+1)e_j) \\ &= 2\sum_{l=0}^{j-1} k_l + (d+1-2j)k_j + (i-j+1)(d-1-i-j)e_j. \end{aligned} \tag{18}$$

In particular, with $j = 1$, $i = e$, this yields

$$v - 2 \ge (d-1)k + e\,(d-2-e)\lceil k/\mu \rceil \ge 3ek + 2e\,(2e-1) = 4e^2 + (3k-2)e. \tag{19}$$

This proves one of the two inequalities claimed, and the other follows when $k \ge 3e+1$, so we may assume $k \le 3e < d$, so that Γ is a Terwilliger graph. If $k/\mu > 6$ then (19) yields the desired inequality, and if $k \le 6\mu$, $\mu > 1$ we are done by Corollary 1.16.6 (ii). If $k = 3$, all possible Γ are known (see Theorem 7.5.1) and our inequality follows by inspection of the list. So, we may assume that $\mu = 1$, $k \in \{4,5,6\}$ and (hence) $e \ge 2$. Applying (18) with $j = 2$, $i = e$ yields

$$v - 2 \ge 2k + (3e-2)k_2 + 2(e-1)^2 \lceil k_2/c_3 \rceil .$$

But $c_3 < b_2 \le b_1$, so $k_2/c_3 \ge kb_1/(b_1 - 1)$ and we find the desired result unless $k = 4$, $b_1 = b_2 = 3$, $c_3 = 2$, $k_2 = 12$, $k_3 = 18$, $e \ge 11$. This final case is settled either by estimating $k_{i+1} \ge k_i(1 + \frac{1}{k-2}) \ge 8(\frac{3}{2})^i$ for $1 \le i \le e$, or by observing that (17) with $j = 2$ does not hold, so that we find a bound $d \le [2\cdot 14/3]+3 = 12$ on the diameter, contradicting $e \ge 11$. $\square$

CORRECTION. On p. 190, line –6, replace ‘u’ by ‘ζ’.

ADDITION. On p. 191, replace lines 2-3 by ‘He conjectures that in fact $j > r$ is impossible (which would yield a diameter bound $d \le kg$), cf. IVANOV [920]’.

ADDITION. On p. 192, line –5 add:

HIRAKI [899] improved the above result to: if $r > 0$ then $s \le 3$.

ADDITION. On p. 192 add at the end of the section:

HIGASHITANI & SUZUKI [895] show (using analytical methods) that the number of i for which $(c_i, a_i, b_i) = (1, k-2, 1)$ is at most $46\sqrt{k-3}$ when $k \ge 5$. HIRAKI [898] shows (using circuit chasing) that if $\lambda > 0$ and $(c_i, a_i, b_i) = (1, \lambda, b_1)$ for $1 \le i < r$, and $(c_i, a_i, b_i) = (2, 2\lambda, b_r)$ for $r \le i < r + j$, then $j \le 1$ (generalizing Gardiner’s result from $r = 2$ to arbitrary r).

SUZUKI [1000] shows (using circuit chasing) that if $(c_{2j+1}, a_{2j+1}, b_{2j+1}) = (1, 0, k-1)$ and $(c_{2j+2}, a_{2j+2}, b_{2j+2}) = (2, 0, k-2)$, then $c_{2j+3} > 2$.

CHAPTER 6

CORRECTION. On p. 195 in Table 6.2, change ‘9.4.5’ into ‘9.4.6, 9.4C’.

CORRECTION. On p. 195, the statement and proof of Proposition 6.1.2 are incorrect.

6.1.2. Proposition. *The intersection numbers of every distance-regular graph of diameter $d \ge 3$ with classical parameters satisfy*

$$c_{i+1} > ci \quad \text{for } i = 2, \cdots, d-1;$$

in particular the geometric girth of such a graph cannot exceed 6.

Proof. ... therefore $\left[{i \atop 1}\right] = \pm 1$ which forces $i = 1$. □

Note that the generalized hexagons have geometric girth 6.

CORRECTION. On p. 200-201 the (trivial) case of generalized 2-gons is not handled correctly. On p. 200 line 4 of §6.5, replace ‘partial linear space’ by ‘point-line geometry’; on line –13 replace ‘linear space’ by ‘a geometry’; on line –12 replace ‘the partial linear space’ by ‘it’; on the last line of p. 200 and the first line of p. 201 insert ‘2,’.

ADDITION. On p. 204, add: ‘There are unique generalized quadrangles of orders $(s, 1)$, $(1, t)$, (2,4), (4,2), (3,5), (5,3), (3,9) and (9,3). No generalized quadrangle of order (3,6) exists. For orders (3,5), (3,6) and (3,9), see DIXMIER & ZARA [870] and PAYNE & THAS [972].’

CORRECTION. On p. 205, something went wrong in the last three lines of Table 6.6. These should have been:

	name	§	n	k	λ	c_i
(R4)	Folded cubes (P1)	9.2D	n	n	0	$c_i = i\ (i < d),\ c_d = \gamma d$
(R5)	Hamming graphs (C8)	9.2	$2d$	sd	$s-1$	$c_i = i$
(R6)	Dual polar graphs (C3)	9.4	$2d$	$q^e\left[{d \atop 1}\right]_q$	$q^e - 1$	$c_i = \left[{i \atop 1}\right]_q$
(R6a)	Pseudo $D_m(q)$ graphs (C3a)	9.4.6, 9.4C	$2d$	$\left[{d \atop 1}\right]_q$	0	$c_i = \left[{i \atop 1}\right]_q$

CORRECTION. On p. 206, in Table 6.7, line (N2), the group should be $3^6.2.M_{12}$. In the statement of Theorem 6.6.1, replace '$\mu = 1$' by '$\mu \le 2$'. In the subsequent discussion replace '$\mu > 1$' by '$\mu > 2$'.

ADDITION. On p. 206, at the end of Section 6.6, add:

Mathon (pers. comm.) showed that the Krein condition $q^2_{33} \ge 0$ for regular near hexagons is equivalent to $t \le s^3 + (\mu - 1)(s^2 - s + 1)$ if $s > 1$. (For this, and more information on Krein conditions for near polygons, cf. Brouwer & Wilbrink [843].) A generalization was given in Neumaier [960].

ADDITION. On p. 207, add after line 10: 'Cameron [852], Proposition 5.4, shows that no example with $k = 57$ can be vertex-transitive, a result due to G. Higman (cf. Proposition 2.2.9).'

ADDITION. On p. 207, add in line 18: 'Jørgensen [922] shows that no cubic graph of diameter $d \ge 4$ has defect 2.'

ADDITION. On p. 208, replace lines −4, −3 by:

and Roos, van Zanten & Coster [981] and Damerell, Roos & van Zanten [866], where it is shown that the eigenvalues are at most quadratic, and that this array is not feasible for $d \ge 4$ when $st > 1$).

ADDITION. On p. 209, line 1, insert:

Van Zanten [1021] gives another proof of the consequence of 6.8.2 (ii) that all eigenvalues of Γ are at most quadratic over $\mathbb{Q}$.

CORRECTION. On p. 210, in the description of the Robertson-Wegner graph it is stated that the full group has order 20 with three point orbits. Gordon Royle pointed out that in fact the full group has order 120 and has two point orbits (of sizes 10, 20). Of course this means that there is a much nicer description, and indeed there is: The 30 vertices are the 20 vertices of the dodecahedron and the 10 4-subsets of the dodecahedron that have all internal distances 3; the adjacencies are the obvious ones: the dodecahedron is an induced subgraph of valency 3, each 4-subset is adjacent to its 4 elements and to the antipodal 4-subset. This description shows the (full) group $2 \times \mathrm{Alt}(5)$.

It is also stated that the subgraph induced by 3 pentagons and 3 pentagrams in the Hoffman-Singleton graph has automorphism group of order 10. Gordon Royle points out that the order is 20 (with orbit sizes {5,5,10,10}).

Part of the confusion is caused by the fact that the pictures drawn in Wong [cages, a survey] do not correspond to the descriptions given. Quoting Royle [email, 951222]: the Robertson-Wegner graph is Wong's figure 4 (not 5); the 3 pentagons + 3 pentagraphs graph is Wong's figure 5 (not 4). Wong's figure 6 is a (5,5)-cage constructed by Foster, with group of order 30 and orbit sizes {15,15}. It is rumoured that there is precisely one further (5,5)-cage.

CORRECTION. On p. 210, in the discussion of the (7,6)-cage, change 'Apparently ... $L_2(7)$' into 'However, as Baker [pers. comm.] (see also BAKER [806], p. 96) pointed out, the elliptic semiplane is self-dual, and in fact $\mathrm{Aut}\,\Gamma \cong 3{\cdot}\mathrm{Alt}(7).2$, acting transitively on Γ. ($\mathrm{Aut}\,\Gamma$ has rank 8; the two antipodes of a vertex γ cannot be interchanged by an automorphism fixing γ.)'

ADDITION. On p. 210, at the end of Section 6.9, add: ', BIGGS & BOSHIER [815], LAZEBNIK & USTIMENKO [936]'. Six lines earlier, after the reference to WONG [793], add: 'An older reference is WALTHER & VOSS [1016], Teil II, Kap. 1'.

ADDITION. On p. 213, after 'are determined.' add: '(The arguments were geometric in nature. A treatment using parameters only was given by TERWILLIGER [1008].'

ADDITION. On p. 225, bottom, add: 'The antipodal distance-transitive covers of *K_k,k* have been classified by IVANOV, LIEBLER, PENTTILA & PRAEGER [912].'

ADDITION. On p. 229, at the end of §7.6, add: 'The 2-transitive two-graphs and their full groups of automorphisms have been determined by TAYLOR [1005]; there are no surprises. In particular, in cases (v) and (vi), we have $\mathrm{Aut}\,\Gamma \cong 2\times P\Gamma U(3,q^2)$ and $2\times^2 G_2(q).\mathrm{Aut}\,\boldsymbol{F}_q$, respectively, except for $q = 3$ where $\mathrm{Aut}\,\Gamma \cong 2\times Sp(6,2)$.'

CHAPTER 7

ADDITION. On p. 229 in Section 7.7, before 'Problem' insert: 'The case where *G* is almost simple of Lie type has been treated in CUYPERS [863]'.

ADDITION. On p. 230 in Section 7.7, before 'Problem' insert:

The case of the Lyons group has been treated in SOICHER [988].

In case (iii) significant progress has been made by VAN BON [823]:

Theorem. *Let G be as described in (iii) of Theorem 7.7.1. Suppose that* Γ *has diameter* > 2 *and valency* ≥ 3. *Write* $V\Gamma = \boldsymbol{F}_p^m$ *so that* G_0 *is the stabilizer of the zero vector and embeds in* $GL(m,p)$. *Take* $q \mid p^m$ *maximal such that the vector space* $\boldsymbol{F}_p^m$ *carries an* $\boldsymbol{F}_q$*-structure, denoted by V, preserved by* G_0 *(that is,* $G_0 \leq \Gamma L(V)$*). Suppose further that* Γ *is neither a bilinear forms graph nor a Hamming graph. Then either* $V = \boldsymbol{F}_q$, $q = p^m$ *and* G_0 *is conjugate to a subgroup of* $\Gamma L_1(q)$ *or the generalized Fitting subgroup*

$F^*(G_0/Z(G_0))$ is a simple group whose projective representation on V is absolutely irreducible and can be realized over no proper subfield of $GF(q)$.

This theorem heavily depends on ASCHBACHER [803].

Assuming the classification of finite simple groups, the determination of affine distance-transitive graphs whose point-stabilizer is almost simple (and nonabelian) is a matter of diligently ploughing through various possibilities. It is to be expected that this will be finished soon. (The case where G_0 leaves an orthogonal or unitary form invariant is treated in VAN BON [822].) Thus, the following remains:

Problem. Classify all distance-transitive graphs (or even groups) whose automorphism groups contain the additive group A of the 1-dimensional $\mathbb{F}_q$-space V and is contained in $A\Gamma L(V) \cong A\Gamma L(1,q)$. Those with diameter 2 have been classified: they are the Paley graphs, cf. LIEBECK [939].

ADDITION. On p. 231, at the end of §7.7, add: 'An extensive survey on the topic of this section is given by IVANOV [911].'

ADDITION. On p. 232, at the end of the section on distance-transitive digraphs, add: 'LEONARD & NOMURA [938] showed that every distance-regular digraph of short type, (that is, with $d = g-1$) has $d \le 7$.'

ADDITION. On p. 232, line –18, insert: '(see also CAMERON [853])'.

ADDITION. On p. 234, add at the end of the page: '(iii) Of course the graphs of Lie type, as discussed in Theorem 10.7.2 but for an infinite field, also provide examples of distance-transitive graphs of infinite valency'.

CHAPTER 8

ADDITION. On p. 247, line –5 it is said that the case $d = 4$ is possibly exceptional. That indeed it is exceptional can be seen from the Hadamard graphs (cf. 8.5.3 (vi)).

CORRECTION. On p. 253, line –5, replace '8.5.1' by '8.5.2'.

CHAPTER 9

ADDITION. On p. 258, line 6, add: 'For a survey of the known geometries that are extended grids, see MEIXNER & PASINI [949].'

CORRECTION. On p. 258, line –17, and on p. 263, line 3, replace 'Corollary' by 'Theorem'.

ADDITION. On p. 258, after line –6, add:

On the other hand, there do exist regular graphs of diameter d, with valency $k = d^2 + d + 1$ on $v = 2\binom{2d}{d}$ vertices satisfying $c_i = i^2$ $(1 \le i \le d)$ (Brouwer & Koolen [840], cf. §*Repres*.15). Thus, the collection of Johnson graphs $J(n,d)$ is not characterized by c_i alone.

CORRECTION. On p. 260, in Proposition 9.1.8, change the first part of the third sentence to 'Γ is bipartite and antipodal; if $m > 0$ it is the bipartite double of the Odd graph on X'.

ADDITION. On p. 261, at the end of Section 9.1, add to Remark (iv):

Muzichuk's result appeared as [958]. Uchida [1013] improved Muzichuk's result and showed that merging classes in $J(n,m)$ does not lead to an association scheme for $n \ge f(m)$, where $f(3) = 11$, $f(4) = 13$, $f(5) = 15$, $f(6) = 18$ and $f(m) = 3m-1$ for $m \ge 7$. Clearly, this is best possible for $m = 3, 4$.

ADDITIONS. On p. 261, at the end of Section 9.1, add a Remark (v) stating that $2.O_m$ is a geodetically closed induced subgraph of O_{2m} (with the same diameter $2m-1$). Add a Remark (vi): Meyerowitz [951] classified the completely regular designs of strength 0 in the Johnson scheme. Martin [945] classified the completely regular designs of strength 1 and minimum distance 2 in the Johnson scheme.

ADDITION. On p. 262, in the remark following Corollary 9.2.3, add:

Several other people characterized the d-cube by its parameters (i.e., by being bipartite of diameter d with $c_i = i$ for $1 \le i \le d$; note that a rectagraph is always regular) - cf. e.g. Foldes [873] and Mulder [954].

ADDITION. On p. 263, the question in the Problem can be answered negatively: Haemers [886] has constructed non-distance-regular graphs cospectral with $\binom{n}{3}$, $n \ge 6$. (Moreover, Hoffman's graph described above arises from 2^4 by 'Haemers switching'.)

ADDITION. On p. 265, add Remark (ia): 'The folded $(d+1)$-cube is $\Gamma_1 \cup \Gamma_d$ of the d-cube.'

CORRECTION. On p. 272, line 14-15, delete the sentence 'It is ... Grassmann graph $\left[{V \atop m}\right]$'.

ADDITION. On p. 273, the problem has been answered completely. Add at the end of Remark (iii):

Gardiner asked whether any distance-transitive graph with parameters $q.K_{q,q}$ must be derived from the Desarguesian projective plane, but Chuvaeva & Pasechnik [856] showed that also certain planes coordinated by the twisted fields of Albert gave rise to distance-transitive graphs, and subsequently Liebler [940] showed that conversely any such distance-transitive graph is derived from either the Desarguesian projective plane, or a plane coordinatized by a twisted field of Albert. (However, now Liebler says that his proof may be incomplete.)

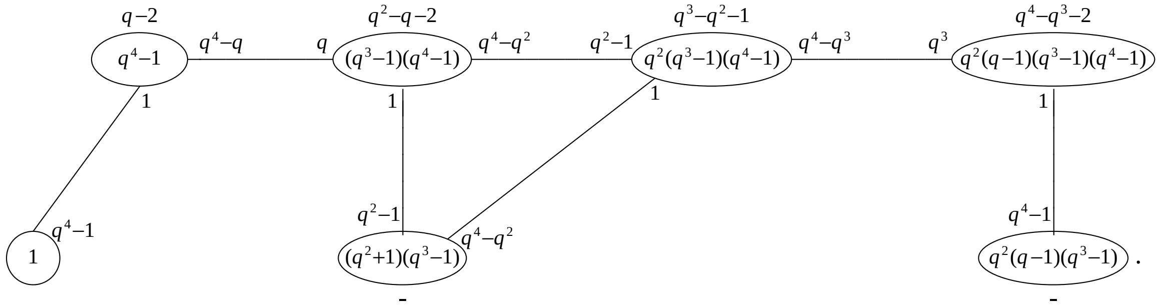
CORRECTION. On p. 279, in Theorem 9.4.10, insert after '(6)': 'but with $m+1$ instead of m'.

CORRECTION. On p. 284, lines 1-2: delete the statement about the group in Theorem 9.5.6. (The precise result was already given in Theorem 9.5.3.)

ADDITION. On p. 284, bottom, add: ‘(iv) A local characterization of the alternating forms graphs was given by MUNEMASA & SHPECTOROV [957] and by MUNEMASA, PASECHNIK & SHPECTOROV [956].’

ADDITION. On p. 285, in Theorem 9.5.7, the group of automorphisms given is (for $d \geq 2$) the full automorphism group of Γ (IVANOV & SHPECTOROV [917]). In the same paper the authors show that if Γ is a distance-regular graph with parameters (7) and singular lines of size r, and $d \geq 3$, then Γ is the Hermitean forms graph on $\boldsymbol{F}_q^n$. They remark that, according to a personal communication by P. Terwilliger, the assumption of singular lines of size r is superfluous. Finally, they remark that probably WAN [1017] was the first to compute the parameters of the association scheme of Hermitean forms.

CORRECTION & ADDITIONS. On p. 286 the discussion following Lemma 9.5.9 is somewhat imprecise in the case where q is even, as was pointed out by Pasechnik. The facts are as follows: Let Γ be the symmetric bilinear forms graph on $\boldsymbol{F}_q^n$ where q is even. Then Γ is distance-regular for $n \leq 3$ but not for $n > 3$. The distance function is given by $d_\Gamma(f,g) = \mathrm{rk}(f-g)$ when $f-g$ is not alternating, and $d_\Gamma(f,g) = \mathrm{rk}(f-g)+1$ otherwise. Each nonalternating form f of rank i has q^{i-1} neighbours of rank $i-1$, and $q^n - q^i$ neighbours of rank $i+1$, and a unique neighbour (of rank $i-1$ or i) that is alternating. Thus, the alternating forms form a perfect code in Γ. Each alternating form g of rank i has $q^i - 1$ neighbours of the same rank. For $n = 3$ we find a distance-regular graph with intersection array $\{q^3-1, q^3-q, q^3-q^2+1; 1, q, q^2-1\}$ (in the diagram on p. 286, merge the balloons for the forms of rank 3 and the alternating forms of rank 2). The distribution diagram for $n = 4$ is



For even q, there is a 1-1 correspondence ϕ between symmetric bilinear forms on $\boldsymbol{F}_q^n$ and alternating forms on $\boldsymbol{F}_q^{n+1}$ given by

$$(\phi f)((x,\xi),(y,\eta)) = f(x,y) + \sqrt{f(x,x)}\sqrt{f(y,y)} + \eta\sqrt{f(x,x)} + \xi\sqrt{f(y,y)}.$$

(This remark may be due to Kantor.) Under this correspondence, the symmetric bilinear forms of rank $2i-1$ or $2i$ correspond precisely to the alternating forms of rank $2i$. (The geometric version of this correspondence follows by regarding the symmetric bilinear forms as the vertices far away from a given vertex in $B_d(q)$, and the alternating forms as the vertices far away from a given vertex in $D_{d+1}(q)$ where the B_d-geometry lives on a nondegenerate hyperplane in the D_{d+1}-geometry (cf. the discussion on p. 278).) [However, it does not follow that the distance 1-or-2 graph of Γ is the alternating forms graph, since not all symmetric bilinear forms of rank 2 lie in $\Gamma_2(0)$.]

SPELLING CORRECTION. On p. 287, line −20, replace ‘The’ by ‘Then’.

CORRECTION. On p. 289, line 20, replace ‘§14’ by ‘§11.3H’.

ADDITION. On p. 293, Remarks, add references to BROUWER, HEMMETER & WOLDAR [839] and HEMMETER & WOLDAR [893]. Add the remark: ‘E.W. Lambeck remarks that when q is even the partition into q^n-cliques defined by the equivalence relation $\gamma \equiv \delta$ if and only if $\mathrm{rk}(\delta - \gamma) \le 1$ is completely regular (if $d(\gamma, C) = 1$ for such a clique C, then $|\gamma^\perp \cap C| = q^2$) and the corresponding quotient is the alternating forms graph on V.’

Add the remark: ‘MUNEMASA, PASECHNIK & SHPECTOROV [955] determine the full automorphism group of the quadratic forms graph in the characteristic 2 case. Interestingly, it is larger than expected. They also determine the convex subsets of this graph, thus completing Lambeck’s work.’

CHAPTER 10

ADDITION. On p. 294, at the end of the introduction to Chapter 10, add: ‘A recent monograph on Coxeter groups is HUMPHREYS [909]’.

ADDITION. On p. 297, Remark (iii), it is easy to verify that the result of Hemmeter mentioned remains true without the restriction ‘either $m \le 7$, or’ (for noncomplete half dual polar graphs).

ADDITION. On p. 298, before Lemma 10.1.8, add ‘see also SHI JIANYI [921]’.

CORRECTION. On p. 306, in the proof of Theorem 10.2.10, change twice ‘10.2.1 (i)’ into ‘10.2.2 (i)’ and in the one but last line of that proof, change ‘S sub m-1’ into ‘S sub m-2’.

CORRECTION. On p. 309, line 20: There are five, not four possibilities: also $I^m_{2,\{1,2\}}$, giving a $2m$-gon.

ADDITION. On p. 313, after the description of the root system of type E_7, add: ‘The 56 vectors of minimal norm $\frac{3}{2}$ in $E_7{}^*$ are $\frac{1}{\sqrt{2}}\{\pm e_i \pm e_j \pm e_k \mid ijk \text{ is a line}\}$’.

CORRECTION. On p. 313, line −1, the trace function should be defined by $t(\alpha + \beta\tau) = 2\alpha$ $(\alpha, \beta \in \mathbb{Q})$.

ADDITION. On p. 314, line 4, add: ‘Eliminating the τ’s this yields $\Phi = \frac{1}{\sqrt{2}}\{\pm 2e_i,\ \pm e_i \pm e_j \pm e_k \pm e_l \mid 1 \le i, j, k, l \le 8,\ ijkl \text{ a block of } S(3,4,8)\}$’.

CORRECTION. On p. 314, line 6, change ‘$v_1 \in \Phi_7$’ into ‘$v_1 \notin \Phi_7$’.

ADDITION. On p. 315 add at the end of Section 10.3.9: 'MÜHLHERR [952] describes quotients of Coxeter diagrams with respect to an admissable partition; one particular example of his results is the inclusion $W(H_4) \subseteq W(E_8)$.'

CORRECTION. On p. 321, Theorem 10.4.11, delete '$\neq H_4$'.

CORRECTION. On p. 328, lines –20, –18: replace 'A_n' by '$A_{n,1}$'.

CORRECTION. On p. 332 in the subsection Incidence graphs: Again there are five, not four cases, the fifth being $I^m_{2,\{1,2\}}$ ($m = 6, 8$), corresponding to the incidence graphs of generalized m-gons.

ADDITION. On p. 340, at the end of Section 10.7, add a subsection describing the construction of the known generalized hexagons and of the known generalized octagon of order (2,4). On p. 336, sentence below the Table, add: 'see below'.

Construction of the known generalized hexagons

Set $F = \mathbb{F}_q$. We define the *Cayley algebra O over F* as the set of matrices

$$x = \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}$$

where $\alpha, \beta \in F$ and $a, b \in F^3$. We provide $O(F)$ with the usual (entrywise) vector space structure over F and with the following multiplication:

$$xy = \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \begin{pmatrix} \alpha' & a' \\ b' & \beta' \end{pmatrix} = \begin{pmatrix} \alpha\alpha' - a\cdot b' & \alpha a' + \beta' a + b \times b' \\ \alpha' b + \beta b' + a \times a' & \beta\beta' - b\cdot a' \end{pmatrix}$$

where $a\cdot b$ and $a \times b$ are the usual standard inner and exterior product on F^3. It is clear that O is a non-associative algebra over F and with unity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The quadratic form $Q : O \to F$ given by

$$Q(x) = \alpha\beta + a\cdot b$$

is obviously non-degenerate and of Witt index 4. It turns O into a composition algebra, that is, it has the following property: $Q(xy) = Q(x)Q(y)$ for all $x, y \in O$.

The algebra O is uniquely determined up to isomorphism by its dimension 8, the Witt index 4 of the quadratic form, and the above identity, cf. SPRINGER [992]. We shall work with the projective space $\mathbf{P}(O)$ of O. If $x \in O$, $x \neq 0$ let $\langle x \rangle$ denote the point of $\mathbf{P}(O)$ determined by x, i.e., the subspace Fx of the vector space O.

Let H be the set of points $\langle x \rangle$ of $\mathbf{P}(O)$ such that $x^2 = 0$. Then H is the intersection of the quadric $Q(x) = 0$ with the hyperplane $\mathrm{tr}\,(x) = 0$. Let two points $\langle x \rangle, \langle y \rangle$ of H be adjacent whenever $xy = 0$. Then the graph induced on H is the collinearity graph of a generalized hexagon of order (q,q). Given distinct adjacent $\langle x \rangle, \langle y \rangle \in H$, the line on these two points is the maximal clique $\{\langle z \rangle \mid z \in \langle x, y \rangle, z \neq 0\}$ of H. Details of this construction are to be found in SCHELLEKENS [982]. The same paper contains a twisted version of this construction for the generalized hexagons of order (q^3, q), related to ${}^3D_4(q)$. Suppose $F = GF(q^3)$ and σ is a generator of the Galois group of F over $GF(q)$. Consider the algebra O_σ whose underlying vector space coincides with that of O, and whose multiplication $\circ$ is given by

$$x \circ y = \overline{x^{\sigma}}\,\overline{y^{\sigma^2}} = \overline{((y^{\sigma}x)^{\sigma})} \quad (x, y \in O_{\sigma}).$$

Here σ is an automorphism of O obtained by naturally extending the above Galois automorphism indicated by the same symbol, and written in the exponent, and $\overline{x}$ is the usual antiautomorphism of O, i.e.,

$$\overline{x} = \begin{pmatrix} \beta & -a \\ -b & \alpha \end{pmatrix}.$$

Now let H_{σ} be the set of all $\langle x \rangle \in P$ with $x \circ x = 0$, and call $\langle x \rangle$ and $\langle y \rangle$ adjacent if $x \circ y = 0$. Then adjacency is a symmetric relation and the resulting graph is the collinearity graph of a uniquely determined generalized hexagon of order (q^3, q). Given distinct adjacent $\langle x \rangle, \langle y \rangle \in H_{\sigma}$, the line on these two points is the maximal clique $\{\langle z \rangle | z \in \langle x, y \rangle, z \neq 0\}$ of H_{σ}.

The generalized octagon of order (2,4)

The Tits system of a Chevalley group gives rise to relatively natural presentations (i.e., descriptions in terms of generators and relations) of the group. In the special instance of the group $F = {}^2F_4(2)$ of order 35942400 such a presentation (cf. TITS [1010]) has the following shape. Here, by $[x,y]$ we denote the commutator $x^{-1}y^{-1}xy$.

F:

generators: $u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, r_1, r_8$;

relations: $u_1^4 = u_3^4 = u_5^4 = u_7^4 = v_1^4 = v_3^4 = v_5^4 = v_7^4 = 1$,

$u_2^2 = u_4^2 = u_6^2 = u_8^2 = v_2^2 = v_4^2 = v_6^2 = v_8^2 = 1$,

$[u_1, u_2] = [u_1, u_5] = [u_2, u_4] = [u_2, u_6] = 1$,

$u_2 = [u_1, u_3^{-1}],\ u_3^2 = [u_1, u_4^{-1}],\ u_4 u_6 = [u_2, u_8^{-1}]$,

$u_3^2 u_4 u_5^2 = [u_1, u_6^{-1}],\ u_2 u_3^3 u_5 = [u_1, u_7^{-1}]$,

$u_2 u_3^2 u_4 u_5^3 u_6 u_7 = [u_1, u_8^{-1}]$,

$r_1 = u_1 v_1^2 u_1^{-1},\ r_8 = u_8 v_8 u_8^{-1},\ (r_1 r_8)^8 = 1$,

$r_1 u_1 r_1 = v_1,\ r_1 u_2 r_1 = u_8,\ r_1 u_3 r_1 = u_7$,

$r_1 u_4 r_1 = u_6,\ r_1 u_5 r_1 = u_5,\ r_1 v_2 r_1 = v_8$,

$r_1 v_3 r_1 = v_7,\ r_1 v_4 r_1 = v_6,\ r_1 v_5 r_1 = v_5$,

$r_8 u_1 r_8 = u_7,\ r_8 u_2 r_8 = u_6,\ r_8 u_3 r_8 = u_5$,

$r_8 u_4 r_8 = u_4,\ r_8 u_8 r_8 = v_8,\ r_8 v_1 r_8 = v_7$,

$r_8 v_2 r_8 = v_6,\ r_8 v_3 r_8 = v_5,\ r_8 v_4 r_8 = v_4$.

Here the subgroups B and N of the Tits system of F used to construct this presentation are $B = \langle u_1, \cdots, u_8 \rangle$ and $N = \langle r_1, r_8 \rangle$. Thus the collinearity graph Γ of the generalized octagon of order (2,4) associated with F arises as the graph $\Gamma(F, \langle B, r_1 \rangle, r_2)$. Using that u_5^2 lies in the center of $\langle B, r_1 \rangle$ and that the latter group is a maximal subgroup of F (so that it coincides with the centralizer in F of u_5^2), we obtain the following more direct (but equivalent) description of the generalized octagon. Its points and lines are the involutions F-conjugate to u_{2i+1}^2 and u_{2i}, respectively. A point x is incident to a line y if the pair x,y is F-conjugate to u_1^2, u_2; this is equivalent to y lying in $C_F(x)'''$, the third commutator subgroup of the centralizer of x in F. The chain $u_1^2, u_2, u_3^2, \cdots, u_8, v_1^2, v_2, \cdots, v_8, u_1^2$ forms a 16-circuit in the incidence graph - an apartment in the generalized octagon.

The Rudvalis graph is a strongly regular graph with parameters $(v,k,\lambda,\mu) = (4060,1755,730,780)$ which is locally Γ. Its automorphism group is the simple Rudvalis group. For a construction, see CONWAY & WALES [857] and the *Atlas* [858].

CORRECTION. On p. 343, line –13, replace '*(ii)*' by '*(iii)*'.

ADDITION. On p. 344, add at the end of the page:

Remark. In the case of infinite fields more examples may be expected of distance-transitive graphs from algebraic groups than those suggested by the above theorem. For example, if G is an algebraic F-group of type E_7 for some field F possessing an anisotropic kernel of type D_4 (cf. TITS [1011]), then the maximal parabolic subgroups of G lead to a polar space of rank 3 (over the Cayley division algebra with center F) on which G acts highly transitively. Just as in the finite case, the collinearity graph of the polar space and that of the dual polar space are distance transitive.

CHAPTER 11

ADDITION. On p. 346 insert an improved definition of complete regularity in arbitrary graphs. Neumaier (cf. BROUWER [832]) showed that for distance-regular graphs this definition is equivalent to the one given in §11.1, and it is often much easier to work with.

Let $\Gamma = (X,E)$ be a connected graph, and C a subset of its vertex set X. Then C induces a partition $\Pi(C) = \{C_0, \cdots, C_t\}$ of X, where $C_i = \{x \in X \mid d(x,C) = i\}$, and $t = t(C)$ is the covering radius of C. We call C *completely regular* when this partition is regular in the sense of §A4 (p. 436), i.e., when there are numbers a_i, b_i, c_i $(0 \le i \le t)$ such that each $x \in C_i$ has a_i neighbours in C_i, b_i neighbours in C_{i+1}, and c_i neighbours in C_{i-1}. (Then necessarily Γ is regular of valency k, where $a_i + b_i + c_i = k$ for all i. Clearly $c_0 = b_t = 0$.)

ADDITION. On p. 348, line –11, insert 'Clearly, every s-regular code is s-uniform.'

CORRECTION. On p. 348, line –7, change 'e_{il}' into 'e_{li}'. On line –1 change '$|x^{\perp} \cap C_l|$' into '$|\Gamma(x) \cap C_l|$'.

ADDITION. On p. 353 after Proposition 11.1.9, add the following (essentially due to Bill Martin = wjmartiniii@violet.uwaterloo.ca, email communication 1990).

11.1.9A. Theorem. *Let Γ be a distance-regular graph with completely regular partition Π, and $\Delta = \Gamma/\Pi$ the corresponding quotient graph. Let $\pi : \Gamma \to \Delta$ be the quotient map.*

(i) *If C is a completely regular code in Δ, then $\pi^{-1}(C)$ is a completely regular code in Γ.*

(ii) *If Σ is a completely regular partition in Δ, then $\pi^{-1}\Sigma = \{\pi^{-1}C \mid C \in \Sigma\}$ is a completely regular partition in Γ, and $\Gamma/\pi^{-1}\Sigma \cong \Delta/\Sigma$.* □

ADDITION. On p. 354 at the end of Section 11.1C, add:

The following result shows that Terwilliger's powerful diameter bound 5.2.2 is applicable to almost all translation distance-regular graphs.

Lemma (VAN BON [823]). *Suppose Γ is a translation distance-regular graph but neither complete nor a conference graph. Then Γ contains a quadrangle.*

Proof. Let X be the elementary abelian group of exponent p (written additively) with which $V\Gamma$ can be identified. By Theorem 11.1.10 $sx \in \Gamma(0)$ whenever $x \in \Gamma(0)$ and $s \in GF(p)^*$. Assume now that Γ has no quadrangle. Since Γ is non-complete, there are $x, y \in \Gamma(0)$ with $d(x,y) = 2$. By induction on i $(0 \le i \le p-1)$ we show that $x+iy \in \Gamma(0)$. Indeed, this is clear for $i = 0$, and if $x \sim -iy$ then $x, x+y, y, -iy, x$ is a 4-cycle, and since Γ does not have induced quadrangles we have $x+y \sim -iy$, i.e., $x+(i+1)y \in \Gamma(0)$. In particular we find for $i = p-1$ that $x-y \in \Gamma(0)$, contradiction. □

[Note that there are many Cayley graphs without quadrangles, such as the graphs defined by the difference set $D = \{-a, -a+1, \cdots, -1, 1, \cdots, a-1, a\}$ in $\mathbb{Z}_m$ for $m \ge 4a+1$.]

ADDITION. On p. 355 add '*e*-error correcting' in line 10. After line −14, add:

For $e = 1$ one knows that $(n,q,e) \ne (7,6,1)$ (R.E. Block & M. Hall, jr., cf. VAN LINT [941], p. 87), and, e.g., $(n,q,e) \ne (19,6,1)$ [C. Roos]. [Indeed, Block & Hall showed that the existence of a perfect code with $(n,q,e) = (7,6,1)$ would lead to the existence of a pair of mutually orthogonal Latin squares of order 6 (and there is no such pair by TARRY [1004]). More generally, Roos observed (around 1978?) that any 1-perfect code C is an orthogonal array of strength $t := S/q - 1$, where $S := 1+(q-1)n$ is the volume of a ball of radius one, (because $(aQ)_j = 0$ for $j \ne 0, S/q$, so C is a t-design in the Hamming scheme). In particular it follows that q^t divides $|C| = q^n/S$, i.e., $S | q^{n-t}$, an improvement of the sphere-packing condition $S | q^n$.] HONG [908] shows by a uniform proof that for $e \ge 3$ and $q \ge 3$ there are no nontrivial perfect e-codes and no tight $2e$-designs in $H(n,q)$.

ADDITION. On p. 355, line −8, replace the sentence 'We don't know ... case (vi)' by: 'In case (vi), the partition $\{C+u \mid \mathrm{wt}\, u \le 1\}$ is completely regular. Indeed, more generally, when C is an e-error-correcting code, the partition $\{C+u \mid \mathrm{wt}\, u \le e\}$ is completely regular'.

ADDITION. On p. 356, line 6, add the references DE RESMINI [977], DE RESMINI & MIGLIORI [978], DE LANGE [935] and GULLIVER [884].

SPELLING CORRECTION. On p. 356, line −3, replace 'BCD' by 'BCH'.

ADDITION. On p. 357, lines 1-2, add the answer: 'In BROUWER [832, 834] it is shown that an even binary code is completely regular if and only if all truncations of this code are completely regular with the same parameters. A result slightly weaker than the 'if' part of this statement was given by SOLÉ [990]).'

ADDITION. On p. 357, after the second paragraph in §11.1E, add: '$S(5,8,24)$ is uniformly packed in $J(24,8)$ with $\delta = 4$, $r = 2$.'

ADDITION. On p. 358, line 2, add: 'For a completely regular partition of the quadratic forms graphs with even q, see p. 293.

CORRECTION. On p. 358, line 20, replace 'Shortening' by 'Truncating'.

SPELLING CORRECTION & ADDITIONS. On p. 360, §11.3B, line 3, change 'cf' into 'Cf'. Add the information that the Berlekamp-van Lint-Seidel graph has full automorphism group $3^5.2.M_{11}$. The problem has been solved by Koolen & Riebeek, who constructed a graph with intersection array $\{45,44,36,5;1,9,40,45\}$ and automorphism group $3^5{:}(2\times M_{10})$.

ADDITION. On p. 360, §11.3C, add: 'This graph is uniquely determined by its parameters and has full automorphism group $3^4.(2\times \mathrm{Sym}(6)).2$ (Brouwer & Haemers [837]).'

CORRECTION. On p. 361, line −17, change '11.2.2' into '11.3.2'.

ADDITION. On p. 365 (and on p. 213 in Table 6.10) add the information that the five graphs (A14)-(A18) are distance-transitive with automorphism groups $2^{11}.M_{22}$, $2^{10}.P\Sigma L(3,4)$, $2^{11}.P\Gamma L(3,4)$, $3^5.2.M_{10}$ and $3^6.2.M_{11}$, respectively.

ADDITION. On p. 368, in the remark, replace 'it is unknown whether such graphs exist' by 'however, Ivanov & Shpectorov [916] showed that no such graphs exist'.

ADDITION. On p. 371, add at the end of Section 11.4F: In Brouwer [829] also other bipartite graphs with $c_2 = 2$, $c_3 = 5$ are discussed, and, in particular, the nonexistence of graphs with intersection array $\{10,9,8,5;1,2,5,10\}$ is shown.

CORRECTION. On p. 373, line 17, instead of the parameters of $\Lambda_2(\lambda)$, those of its complement were given. Replace '(162,105,72,60)' by '(162,56,10,24)'.

ADDITION. On p. 373 add new sections 11.4I and 11.4J.

I. The Soicher graph - an antipodal 3-cover of the second subconstituent of the McLaughlin graph

The second subconstituent Γ of the McLaughlin graph Λ on 275 vertices is strongly regular with parameters $(v,k,\lambda,\mu) = (162,56,10,24)$, and is the unique graph with these parameters (Cameron, Goethals & Seidel [855]). Let us give an explicit description of this graph:

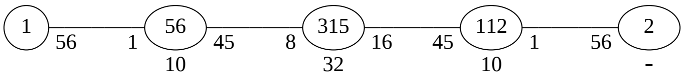
Take as vertices a symbol ∞, the 56 hyperovals in $PG(2,4)$ in a single $L_3(4)$-orbit, and the 105 flags of $PG(2,4)$. Let ∞ be adjacent to the hyperovals, let two hyperovals be adjacent when they are disjoint, let $(p,L) \sim O$ when $|O \cap L \setminus \{p\}| = 2$ and let $(p,L) \sim (q,M)$ when $p \neq q$ and $L \neq M$ and ($p \in M$ or $q \in L$).

Now we find on the hyperovals the strongly regular Gewirtz graph with parameters $(v,k,\lambda,\mu) = (56,10,0,2)$

and on the flags a strongly regular graph with parameters $(v,k,\lambda,\mu) = (105,32,4,12)$ as is well-known and easily checked; using this it quickly follows that the graph Γ just constructed is strongly regular with the stated parameters, and hence is isomorphic to the second subconstituent of the McLaughlin graph.

The automorphism groups (given in *Atlas* [858] notation) of these graphs are as follows. The graph Λ has automorphism group $McL.2$ with point stabilizer $U_4(3){:}2_3$. The graph Γ has automorphism group $U_4(3){:}(2^2)_{133}$ with point stabilizer $L_3(4){:}2^2$. Its subconstituents have automorphism group $L_3(4){:}2^2$ with point stabilizers $A_6{\cdot}2^2$ and $2^{2+4}.3.2^2$, respectively. All these graphs are distance-transitive, except the graph on 105 vertices, which has rank 4.

11.4.6. Theorem. *There is a unique distance-regular graph* Δ *with intersection array* $\{56,45,16,1;1,8,45,56\}$ *and distance distribution diagram*



$v = 486$

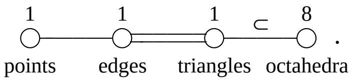
and no other antipodal covers of Γ *exist, that is, there do not exist distance-regular graphs with intersection array* $\{56,45,12,1;1,12,45,56\}$ *or* $\{56,45,18,1;1,6,45,56\}$ *or* $\{56,45,20,1;1,4,45,56\}$ *or* $\{56,45,21,1;1,3,45,56\}$. *The graph* Δ *is distance-transitive with automorphism group* $3.U_4(3){:}2^2$ *and point stabilizer* $L_3(4){:}2^2$.

Proof. In Γ the μ-graphs (the graphs induced on the common neighbours of two nonadjacent vertices) are isomorphic to $6K_{2,2}$ (the disjoint union of six quadrangles), and if $M = \gamma^\perp \cap \delta^\perp$ is such a μ-graph, then each neighbour of γ not in M has a unique neighbour in each of the six connected components of M.

The parameters for a distance-regular antipodal r-cover of Γ are feasible for $r \in \{2,3,4,6,8\}$, but clearly the μ-graphs of a cover are unions of connected components of the μ-graph of Γ, so $r \in \{4,8\}$ does not occur. We show that only $r = 3$ occurs, even without assuming distance-regularity.

Fix a base point ∞, and consider the homotopy group of closed paths in Γ starting at ∞, modulo all triangles in Γ. Using the mentioned property of μ-graphs (and the fact that the second subconstituent of a nontrivial strongly regular graph is connected) one finds that each closed path is homotopic (modulo triangles) to a quadrangle $\infty \sim \alpha_i \sim \beta \sim \gamma_j \sim \infty$ where β is a fixed vertex at distance two from ∞. This shows that the universal cover (among the covers that are locally isomorphisms) is finite, and a small computer program shows that in fact this universal cover is a 3-cover. □

It is amusing to remark that Γ and Δ have Buekenhout diagram



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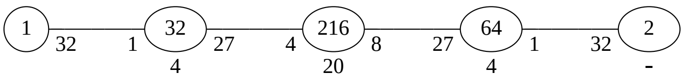
The graph Δ was discovered by Soicher [989]. He also observed the following.

11.4.7 Theorem (Soicher [989]). *The second subconstituent* E *of* Δ *is itself distance-regular with intersection array* $\{32,27,8,1;1,4,27,32\}$ *and distance-distribution diagram*



$v = 315,$

an antipodal 3-cover of the Goethals-Seidel graph (on the 105 flags of $PG(2,4)$*).* $\mathrm{Aut}(\mathrm{E}) \cong (3{\times}L_3(4)){:}2^2$, *acting rank 6, with point stabilizer* $\mathrm{Alt}(6){\cdot}2^2$. □

The Goethals-Seidel graph is strongly regular of rank 6, so E is not distance-transitive; in fact the subconstituent of size 216 splits into two suborbits, of sizes 24 and 192, respectively. The graph induced on the suborbit of size 24 is $3K_{4,4}$.

I. Another Soicher graph - z

CORRECTION & ADDITION. In §11.5 on p. 373, line −3 it is stated that the incidence graph Δ of the van Lint-Schrijver partial geometry is distance-transitive, but it is not. Replace this line by the following text.

We have $\mathrm{Aut}\,\Delta \cong 3^4.(\mathrm{Sym}(6)\times 2)$, with permutation rank 7 on Δ. The halved graph E of Δ (a strongly regular graph with parameters $(v,k,\lambda,\mu)=(81,30,9,12)$) has full group of automorphisms $3^4.\mathrm{Sym}(6)$ with permutation rank 4.

[The graph E is the graph on $\mathbf{1}^{\perp}/\mathbf{1}$ where $\gamma \sim \delta$ when $Q(\gamma-\delta)=2$, where $Q((x_1,\cdots,x_6))=\sum x_i^2$. This shows that construction FE4 of Calderbank & Kantor [849] is a special case of their construction FE1.]

$\Delta_3(\gamma)$ and $\Delta_4(\gamma)$ split into orbits of sizes 60, 15 and 20, 30 (respectively) for the stabilizer of γ. D. Pasechnik observed that if R_{3B} and R_{4B} are the relations corresponding to these suborbits of sizes 15 and 30, then $R_{3B} \cup R_{4B}$ defines on the vertex set of Δ a graph that is locally $GQ(4,2)$. This latter graph is an induced subgraph of the $U_6(2)$ polar graph.

CHAPTER 12

CORRECTION. On p. 380, line 13, middle, replace ‘ $p^3_{23}=q-3$’ by ‘ $p^2_{23}=q-3$’.

ADDITION. On p. 382, after the fourth line of §12.3, add:

Coxeter [861] writes ‘‘This ‘Coxeter graph’ was discovered independently by J.H. Conway and R.M. Foster’’.

Before Theorem 12.3.1, add: ‘See also Coxeter [861] for two drawings’.

ADDITION. After p. 384, add a new section 12.4A.

A. The Meixner graphs

It was pointed out to us by L. Soicher, that Meixner [950] (especially Proposition 4.3) implicitly contains the following. Let Δ be the graph on the nonisotropic points in $U(6,2^2)$, where two points are adjacent when joined by a tangent. The universal cover $\hat{\Delta}$ of Δ (among the covers that are locally isomorphisms) is a distance-transitive antipodal 4-cover with parameters $\{176,135,36,1;1,12,135,176\}$ and automorphism group $2^2.U(6,2).\mathrm{Sym}(3)$. The graph Δ also has a distance-transitive antipodal 2-cover with parameters $\{176,135,24,1;1,24,135,176\}$ and automorphism group $2.U(6,2).2$.

ADDITION. On p. 386, add after line 11: ‘The graphs with intersection array $\{8,6,1;1,3,8\}$ were determined again by Jurišić [927]. Add after line 14: ‘Generalized quadrangles of order (q^2,q) do not have spreads, and no $GQ(6,3)$ exists. In cases where it is possible to establish the existence of lines, this can be used to rule out such distance-regular graphs (cf. Godsil [878]). For example, using the remark following Proposition 1.2.1 it follows that there are no graphs with intersection array $\{18,12,1;1,2,18\}$ ($v=133$) or $\{27,18,1;1,2,27\}$ ($v=280$).’

ADDITION. On p. 386, add to Remark (iii) that it follows from LENSTRA [937] that Aut Γ is no larger than the group described.

ADDITION. On p. 386, last line, add: 'This graph is known as the Klein graph. It is locally a heptagon. See also CURTIS [862] and Coolsaet [859].' Add entries 'Klein graph, 386' and 'locally heptagon, 386' to the Subject index on pp. 491, 492.

ADDITION. On p. 387, at the end of §12.5, add:

In a completely analogous way CAMERON [854] constructs covers of complete graphs using Hermitean or quadratic forms. Let us look at the Hermitean case first. [Case (i) below was not given by Cameron.]

12.5.4. Proposition. *Let $q^2 = mr + 1$ for some prime power q, and let V be a vector space of dimension 3 over $F = \mathbb{F}_{q^2}$ provided with a nondegenerate Hermitean form H. Let K be the subgroup of index r of the multiplicative group F^*, and choose $b \in F^*$ such that $bK = \bar{b}K$ (i.e., such that $b^{q-1} \in K$). Define a graph Γ with vertex set $\{Kv \mid v \in V \setminus \{0\}\}$ where $Ku \sim Kv$ if and only if $H(u,v) \in bK$. Then Γ is an antipodal r-cover of the complete graph K_{q^3+1}. The graph Γ is distance-regular if and only if one of the following three conditions hold.*

(i) *r is odd and $r \mid (q-1)$,*

(ii) *q is even and $r \mid (q+1)$,*

(iii) *q is odd and $r \mid \frac{1}{2}(q+1)$.*

In case (i) we have $\lambda = \mu = (q^3-1)/r$; in cases (ii) and (iii) we find $\lambda = (q+1)m - q(q-1)$ and $\mu = (q+1)m$.

Proof. Everything is clear except for distance-regularity. (The vertices *Ku* contained in the same projective point *Fu* are antipodes: they have mutual distance at least 3, and in fact precisely three, as we shall see soon.) The group $GU(3, q^2)$ acts transitively on the vertices and edges, and 2-transitively on the projective points *Fu* ($u \in V \setminus \{0\}$). For distance-regularity it is necessary and sufficient that μ is constant. Thus, let *Ku* and *Kv* be two vertices not contained in the same projective point, and choose coordinates such that ... □

12.5.5. Proposition (DE CAEN, MATHON & MOORHOUSE [848]). *Let $q = 2^{2t-1}$ and $s = 2^e$, where $\gcd(e, 2t-1) = 1$. Construct a graph Γ whose vertices are the triples (a, i, α) with $a, \alpha \in \mathbb{F}_q$ and $i \in \mathbb{F}_2$, and in which $(a, i, \alpha) \sim (b, j, \beta)$ when $\alpha + \beta = a^s b + ab^s + (i+j)(a^{s+1} + b^{s+1})$. Then Γ is distance-regular with intersection array $\{2q-1, 2q-2, 1; 1, 2, 2q-1\}$. Taking the quotient w.r.t. $\{0\} \times \{0\} \times A$, where A is a subgroup of the additive group of $\mathbb{F}_q$, yields a distance-regular graph $\bar{\Gamma}$ with intersection array $\{2q-1, 2q-2^i, 1; 1, 2^i, 2q-1\}$ when $|A| = 2^{i-1}$ ($1 \le i \le 2t-1$).*

ADDITION. On p. 387, after the description of the Denniston arcs, add: '; for a characterization, cf. ABATANGELO & LARATO [801]'.

CORRECTION. On p. 389, Lemma 12.7.4, line 8, replace 'if and only if' by 'if' and insert the following text after the lemma.

Mathon [loc. cit., p. 136] stated that the above cubic polynomial in θ is reducible if and only if $b = c$, but as Bannai pointed out that is false. Munemasa gives the counterexample $a = 36$, $b = 42$, $c = 36$ corresponding to a scheme $\mathbb{Z}_{7^3}.\mathbb{Z}_{114}$ on $v = 7^3$ letters (the cyclotomic scheme with $q = 7^3$, $r = 3$, see below). Here the θ_i are 9, 2, −12.

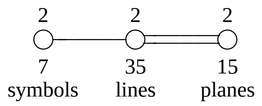
ADDITION. On p. 389, in the remark after Proposition 12.7.5, add a reference to LENSTRA [937].

ADDITION. On p. 390, at the end of Chapter 13, add a new Section 12.8. The Pasechnik graphs and related graphs

Let V be a vector space of dimension 3 over the field $\mathbb{F}_q$, provided with an outer product $\times$. Let Γ be the graph with vertex set $V \times V$ where $(u,u') \sim (v,v')$ if and only if $(u,u') \neq (v,v')$ and $u \times v + u' - v' = 0$. Then Γ is distance-regular with intersection array $\{q^3-1, q^3-q, q^3-q^2+1; 1, q, q^2-1\}$. Its extended bipartite double is distance-regular with intersection array $\{q^3, q^3-1, q^3-q, q^3-q^2+1; 1, q, q^2-1, q^3\}$ and was first constructed by D. Pasechnik [email, March 1991]. When q is a power of two, these graphs have the same parameters as certain Kasami graphs, but for $q > 2$ they are nonisomorphic.

CHAPTER 13

CORRECTION. On p. 392 the Buekenhout diagram in the middle was drawn incorrectly: the double stroke should have been right instead of left.



ADDITION. On p. 392 add to construction D: '(This construction is discussed in great detail by CALDERBANK & WALES [850].)'

CORRECTION. On p. 396, line –9, replace '(v,k,λ,μ)' by '(v,k,λ)'.

ADDITION. On p. 400, add at the end of Section 13.2C: 'The uniqueness of this graph has been shown by Koolen.'

ADDITION. On p. 400, bottom, add: 'Let Σ_n^ε be the graph on the hyperbolic points of $PG(n-1,3)$ provided with a nondegenerate symmetric bilinear form with discriminant ε, adjacent when perpendicular. PASECHNIK [970] shows that Σ_{n+1}^ε is the unique connected locally Σ_n^ε graph for $\varepsilon = +1$, $n \geq 6$ and for $\varepsilon = -1$, $n \geq 7$. The example in this section shows that there are at least two connected locally $SIGME_6^-$ graphs (and Pasechnik announces that there are precisely two).'

ADDITION. On p. 412, Remark (iii), add: 'For a characterization as locally polar graph, see WEISS & YOSHIARA [1018].'

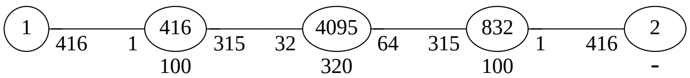
ADDITION. On p. 412, add Remark (iv): 'A. Jurisić, J.H. Koolen & P. Terwilliger proved that any graph with the parameters of the Patterson graph is locally strongly regular [pers.comm.]. Indeed, we have $(1+b^-)(1+b^+)+\lambda \leq (\lambda+1+b^-)(\lambda+1+b^+)/k$ with equality only if the local graph is strongly regular (with eigenvalues $-1-b^-$ and $-1-b^+$). [Proof: By Theorem 4.4.3 we have $\sum(\eta+1+b^-)(\eta+1+b^+) \leq 0$ where the sum is over the eigenvalues of the local graph distinct from the valency. On the other hand,

$\sum 1 = k$, $\sum \eta = 0$, $\sum \eta^2 = k\lambda$, where this time the sum is over all eigenvalues of the local graph.]

ADDITION.

13.8 A triple cover of the strongly regular Suzuki graph

SOICHER [989] discovered the existence of a distance-transitive graph with automorphism group 3.Suz:2 and point stabilizer $G_2(4)$:2 and diagram



$v = 5346$

In this case the r-covers of the underlying strongly regular graph (with parameters $(v,k,\lambda,\mu) = (1782,416,100,96)$) are feasible for $r \in \{2,3,4,6,8,12,16,24,32\}$. However, the μ-graphs in this strongly regular graph are edge-regular with $(v,k,\lambda) = (96,20,12)$, and one easily sees that their components must have size at least 28. Since the above-mentioned 3-cover exists, it follows that these μ-graphs all have three components, each of size 32. Thus, this graph has no other distance-regular covers. [And in fact, the μ-graphs are $3(2.\frac{1}{2}2^5)$: three copies of the 2-coclique extension of the halved 5-cube.]

CHAPTER 14

ADDITION. On p. 416, the parameter set for a bipartite distance-regular graph on 686 vertices of diameter 6 has been ruled out by J.H. Koolen (May 1991) by observing that it must contain a non-integral number of Pappus subgraphs.

CORRECTION. On p. 417-418, in the four places where it is stated for a graph Γ of diameter 5 that '$\Gamma_{1,2}$ is strongly regular', change this into '$\Gamma_{1,2}$ is distance-regular'. (This occurs for primitive graphs on 462 and 1024 vertices, and for bipartite antipodal graphs on 20 and 32 vertices.)

CORRECTION. On p. 418, delete lines 5-6. (Indeed, on p. 160 it was already mentioned that the parameter set $v = 2912$, intersection array $\{105,90,49,7,1;1,7,49,90,105\}$ was ruled out by Theorem 4.4.4.) For the same reason, the six parameter sets $\{36,25,8;1,4,20\}$ (on p. 428), $\{72,45,16;1,8,54\}$, $\{85,54,25;1,10,45\}$, $\{90,60,12;1,12,72\}$ (on p. 430), $\{112,77,16;1,16,88\}$ and $\{145,84,25;1,20,105\}$ (on p. 431) can be deleted.

ADDITION. On p. 419, the parameter set $\{5,4,3,3;1,1,1,2\}$ ($v = 176$) can be deleted; the nonexistence of a corresponding graph was shown in FON-DER-FLAASS [874].

ADDITION. On p. 422, the parameter set $\{45,40,11,1;1,1,40,45\}$ can be deleted (its singular lines are too large), and of the five parameter sets $\{56,45,24-\mu,1;1,\mu,45,56\}$ the four with $\mu \neq 8$ may be deleted, while there is a unique graph when $\mu = 8$ (the Soicher graph, see §11.4I above).

ADDITION. On p. 424, the question marks for the parameter sets $\{176,135,24,1;1,24,135,176\}$ and $\{176,135,36,1;1,12,135,176\}$ can be deleted; according to Soicher [pers. comm.] the collinearity graphs of the two geometries constructed in MEIXNER [950], Proposition 4.3, are distance-transitive antipodal 2- and 4-covers of the strongly regular graph on the nonisotropic points in $U(6,2^2)$ (nonadjacent when joined by a tangent).

ADDITION. On p. 425, the parameter set $\{5,4,3;1,1,2\}$ ($v = 56$) can be deleted; the nonexistence of a corresponding graph was shown in FON-DER-FLAASS [875].

ADDITION. Below we give tables for bipartite but not antipodal distance-regular graphs of diameter 4. They belong on p. 425.

CORRECTION. On p. 427, the 7th parameter set (starting $v = 525 = 1 + 20 + \cdots$) is known; an example is the unitary nonisotropics graph for $q = 5$ (cf. §12.4).

ADDITION. On p. 427, the 7th parameter set from below (starting $v = 729 = 1 + 26 + \cdots$) is known; an example is given by one of the graphs discussed in the (new) Section §12.8 (for $q = 3$). On p. 428, the 8th parameter set (starting $v = 1024 = 1 + 31 + \cdots$) is known; an example is given by a Kasami graph, cf. Theorem 11.2.1 (13), with $q = j = 2$.

ADDITION. On p. 431, last line, add: 'See also GARDINER [876]'.

APPENDIX.

ADDITION. On p. 441, add at the end of Section A.8: 'For the theory of polar spaces, see BUEKENHOUT & SHULT [846], BUEKENHOUT & SPRAGUE [847], PERCSY [973], JOHNSON [924] and BUEKENHOUT [845]'.

REFERENCES. The page numbers showing where a given reference is quoted were not always correctly given in case it concerned pages in the preface. (We do not intend to spell out the corrections; just to give an example: ref. [71] should have had an [x], but ref. [180] should not have had one.) In a few cases where diacritical marks were involved, the references were not sorted correctly. (For example, ref. [156] should have preceded ref. [153].) For [827, 828, 871, 896, 923] the wrong year was given. The five authors of [858] were not mentioned in alphabetical order. The volume number of [888] was given incorrectly (because it was printed incorrectly on the paper itself). In ref. [881], only a title was given. The preprint [891] has been published as [838]. The last page of [986] was given incorrectly. A few journal names were spelled or abbreviated incorrectly (see, e.g., ref. [444]); we do not give the corrections. The papers [817, 818, 820, 824-826, 830, 831, 835, 842, 872, 883, 887, 889, 890, 892, 894, 913, 914, 918, 933, 943, 961, 962, 965, 975, 980] have appeared now.

Additional references: JUNGNICKEL [926] is an update to [813]. COOPERSTEIN [860] (apparently unaware of earlier work) constructs the Gosset graph and determines its automorphism group. SUZUKI [996] discusses Fisher-type inequalities for designs in Hamming graphs $H(n,q)$ (with partial vectors - i.e., convex subgraphs - as blocks). SUZUKI [995] discusses Fisher-type inequalities for designs in projective spaces (with subspaces as blocks, not necessarily all of the same dimension), and reproves Chihara's result that there are no tight designs in the Grassmann graphs. SUZUKI [993] generalizes the construction by THOMAS [1009] of 2-designs in the Grassmann graph $\left[{n \atop 3}\right]_q$ from $q = 2$ to arbitrary powers of 2. For odd q, examples are constructed in SUZUKI [999].

SYMBOLS AND NOTATION. Move the line 'highly transitive scheme, 232' from p. 477 to p. 491.

INTERSECTION ARRAYS. On p. 480, the array '16 {5,4; 1,1}' should have been '16 {5,4; 1,2}'. In the entry in line −3 replace '28150' by '28160'.
Sometimes the page number given is one off, as for the seven graphs at the bottom of p. 222 (indexed under page 223) and the graph at the bottom of p. 223 (indexed under page 224). The five strongly regular graphs in the middle of page 397 were not mentioned in the index.

AUTHOR INDEX. Merge the two entries for B(h). Bagchi.

SUBJECT INDEX. On p. 490, add an entry 'doubled Johnson graph, obsolete synonym for doubled Odd graph'.
On p. 491, add an entry 'highly transitive, 232', and add '359' to the list of pages given for the icosahedron.
On p. 494, add an entry 'Wells graph, 146, 223, 266-267, 289, 410'.

In the following few pages we give the current version of Section 3.14 and the first part of 3.15 (pp. 113-120).

For the position of distance-regular graphs in the metric hierarchy, see KOOLEN [929], KOOLEN & SHPECTOROV [931], SHPECTOROV [985].

Root graphs

A *root representation* of a graph is an integral affine (2,4)-representation, i.e., a representation such that the squared distances $[\gamma,\delta] = (\bar\gamma-\bar\delta,\bar\gamma-\bar\delta)$ of the images of all $\gamma,\delta\in\Gamma$ are even integers and satisfy

$$[\gamma,\delta] = 2i \text{ if } d(\gamma,\delta) = i \le 2.$$

A connected graph having a root representation is called a *root graph*, and a *strict root graph* if the representation is strict, i.e.,

$$[\gamma,\delta] = 2i \le 4 \Leftrightarrow d(\gamma,\delta) = i \le 2.$$

The name reflects the fact that the lattice $\mathbb{L}(\Gamma)$ of a root representation of Γ, defined in Section 3.9, is a root lattice:

3.13.1. Proposition. *Let Γ be a root graph. Then $\mathbb{L}(\Gamma)$ is a root lattice. Moreover, if Γ is locally connected, then $\mathbb{L}(\Gamma)$ is irreducible.*

Proof. Since Γ is connected, $\mathbb{L}(\Gamma)$ is generated by the norm 2 vectors $\bar\gamma-\bar\delta$ ($\gamma,\delta\in\Gamma$, $\gamma\sim\delta$), and is therefore a root lattice. Now suppose that $\mathbb{L}(\Gamma)$ is the direct sum of two lattices $\mathbb{L}_1$ and $\mathbb{L}_2$. If we call an edge $\gamma\delta$ of *type i* if the norm 2 vector $\bar\gamma-\bar\delta$ is in $\mathbb{L}_i$ then each edge is either of type 1 or of type 2. If $\alpha\beta\gamma$ is a triangle, then $(\bar\alpha-\bar\beta,\bar\alpha-\bar\gamma) = 1$ so that edges in the same triangle have the same type. If Γ is locally connected, then this implies that all edges have the same type, whence $\mathbb{L}(\Gamma) = \mathbb{L}_1$ or $\mathbb{L}(\Gamma) = \mathbb{L}_2$, showing that $\mathbb{L}(\Gamma)$ is irreducible. □

Using the classification of root lattices we shall give a complete characterization of amply regular root graphs with $\mu > 1$ in the next section. This leads in the next chapter to an important classification theorem for distance-regular graphs (Theorem 4.4.11).

Remark. Related questions are considered by TERWILLIGER [1007], who considers structure theorems for (not necessarily regular) graphs having a spherical representation such that $[\gamma,\delta]$ is an even integer for all $\gamma,\delta\in\Gamma$ and $[\gamma,\delta] = 2$ if and only if $d(\gamma,\delta) = 1$.

Root graphs generalize graphs with smallest eigenvalue ≥ -2:

3.13.2. Theorem.

(i) *Every graph with smallest eigenvalue ≥ -2 is a root graph.*

(ii) *A regular root graph of diameter ≤ 2 has smallest eigenvalue ≥ -2.*

Proof. (i) A graph with smallest eigenvalue ≥ -2 has a spherical $(2,1,0)$-representation which is a root representation.
(ii) By Proposition 3.5.6 (ii), a regular root graph Γ of diameter ≤ 2 has a spherical $(p, p-1, p-2)$-representation for some $p > 0$; thus by Proposition 3.5.3 (iii), Γ has smallest eigenvalue ≥ -2. □

The graphs $K_{1,n}$ are root graphs of diameter 2 with smallest eigenvalue $-\sqrt{n}$, so regularity is required in part (ii) above.

Examples

The representation of $E_7(1)$ defined in §*Repres*.11 is a root representation, so that any induced subgraph of $E_7(1)$ is a root graph. (However, $E_8(1)$, as given in §*Repres*.11, contains vertices with squared distance 6 having a common neighbour, and hence defines no root representation). Further examples of root graphs are the *code graphs* $\Gamma(C)$, defined by the words of a binary code C of length n, words being adjacent if they differ in two entries (i.e., if their Hamming distance is 2); they have a natural representation by (0,1)-vectors in $\mathbb{R}^n$. We mention in particular the following amply regular code graphs.

(i) The *half-cube* $D_{n,n}(1)$ is the code graph of the binary code consisting of all even weight words of length n. $D_{n,n}(1)$ has 2^{n-1} vertices, valency $\binom{n}{2}$ and $\mu = 6$. The graph $D_{5,5}(1)$ is isomorphic to the Clebsch graph described in Section 3.11.

(ii) The *Johnson graph* $J(n,d)$ is the code graph of the binary code consisting of all words of length n and weight d. $J(n,d)$ has $\binom{n}{d}$ vertices, valency $d(n-d)$ and $\mu = 4$. In particular ($d = 1$), n-cliques are code graphs.

(iii) The *icosahedron* with 12 vertices, valency 5 and $\mu = 2$ can be viewed as the code graph of the binary code consisting of the words 000000, 110000, 001111, 111111, and those obtained by a cyclic permutation of the first five entries.

(iv) The *Shrikhande graph* with 16 vertices, valency 6 and $\mu = 2$ can be viewed as the code graph of the binary code consisting of the words 000000, 110000, 010111, 011011 and those obtained by a cyclic permutation of the six entries (cf. Example 3.11 (vi)).

(v) The *Petersen graph* with 10 vertices, valency 3 and $\mu = 1$ can be viewed as the code graph of the binary code consisting of the words 000000, 100100, 001111 and those obtained by a cyclic permutation of the six entries (cf. Example 3.11 (v)).

(vi) The *dodecahedron* with 20 vertices, valency 3 and $\mu = 1$ can be viewed as the code graph of the binary code consisting of the words 11000 00000, 11000 11000 and their complements and those obtained by simultaneous cyclic permutation of the first five and the last five entries.

(vii) The *g-gon* with g vertices, valency 2 and $\mu = 1$ (when $g > 4$) is the code graph of the binary code consisting of the cyclic shifts of $1^2 0^{g-2}$. (An isometric embedding in $D_{g,g}(1)$ is given by the cyclic shifts of $1^a 0^{g-a}$ for $a = [g/2]$.)

(viii) The direct product Γ of root graphs $\Gamma^1, \cdots, \Gamma^s$ is a root graph: simply represent $(\gamma_1, \cdots, \gamma_s)$ by the concatenation of the images of the γ_i. (Now clearly $\mathbb{L}(\Gamma)$ is the direct sum of $\mathbb{L}(\Gamma^1), \cdots, \mathbb{L}(\Gamma^s)$.) Similarly, the direct product of code graphs is a code graph. In particular, the *Hamming graphs* (direct products of cliques) and *Doob graphs* (the direct product of 4-cliques and Shrikhande graphs, cf. §*ClassGr*.2B) are code graphs.

(ix) Of course, any induced subgraph of a code graph is again a code graph. In particular, as subgraphs of n-cubes, the doubled Odd graphs are code graphs.

We first prove some easy results about the local structure of root graphs.

3.13.3. Lemma. *Let Γ be a root graph and $\alpha, \beta, \gamma \in \Gamma$. For the associated root representation we have*

$$(\bar\alpha - \bar\gamma, \bar\beta - \bar\gamma) = 2 - d(\alpha,\beta) \geq 0 \text{ if } \alpha,\beta \in \Gamma(\gamma) \tag{6}$$

$$= 3 - d(\alpha,\beta) \geq 0 \text{ if } \alpha \in \Gamma(\gamma),\ \beta \in \Gamma_2(\gamma), \tag{7}$$

$$= 4 - d(\alpha,\beta) \geq 0 \text{ if } \alpha,\beta \in \Gamma_2(\gamma), \tag{8}$$

provided that $d(\alpha,\beta) \leq 2$. In particular, if $\bar\gamma = 0$, then points in $\Gamma(\gamma)$ are represented by norm 2 vectors, and *points in* $\Gamma_2(\gamma)$ *are represented by norm* 4 *vectors.*

Proof. By Lemma 3.5.1 we have

$$(\bar\alpha-\bar\gamma,\bar\beta-\bar\gamma)=\tfrac{1}{2}([\alpha,\gamma]+[\beta,\gamma]-[\alpha,\beta]-[\gamma,\gamma])$$
$$=d(\alpha,\gamma)+d(\beta,\gamma)-d(\alpha,\beta),$$

which implies (6), (7) and (8). By putting $\alpha=\beta$ we obtain the second statement. □

3.13.4. Proposition. *Let* Γ *be a root graph and* $\gamma\in\Gamma$. *Then:*

(i) *The local subgraph* $\Gamma(\gamma)$ *has smallest eigenvalue* ≥ -2.

(ii) *Two nonadjacent points* $\alpha,\beta\in\Gamma(\gamma)$ *have at most one common neighbour in* $\Gamma_2(\gamma)$, *and such a neighbour is represented by* $\bar\alpha+\bar\beta-\bar\gamma$.

(iii) *For* $\alpha,\beta\in\Gamma$ *at distance two, the graph* $\Gamma(\alpha)\cap\Gamma(\beta)$ *is complete multipartite with classes of size* 1 *or* 2.

Proof. (i) By Proposition 3.5.6 (i), $\Gamma(\gamma)$ has a (2,1,0)-representation. Thus, if A denotes the adjacency matrix of $\Gamma(\gamma)$, then the Gram matrix $G=2I+A$ of the image of $\Gamma(\gamma)$ is positive semidefinite. This implies (i).
(ii) Now let α, $\beta\in\Gamma(\gamma)$ be nonadjacent. If $\delta\in\Gamma_2(\gamma)$ is a common neighbour of α and β then Lemma 3.5.1 implies

$$(\bar\delta-\bar\alpha,\bar\beta-\bar\gamma)=\tfrac{1}{2}([\delta,\gamma]+[\alpha,\beta]-[\delta,\beta]-[\alpha,\gamma])=\tfrac{1}{2}(4+4-2-2)=2.$$

Thus the norm of $(\bar\delta-\bar\alpha)-(\bar\beta-\bar\gamma)$ is $[\delta,\alpha]-2(\bar\delta-\bar\alpha,\bar\beta-\bar\gamma)+[\beta,\gamma]=2-4+2=0$. Therefore $(\bar\delta-\bar\alpha)-(\bar\beta-\bar\gamma)=0$, so $\bar\delta=\bar\alpha+\bar\beta-\bar\gamma$, and, by Lemma 3.5.5, δ is unique. This implies (ii).
(iii) Finally, let $d(\alpha,\beta)=2$. If $\Gamma(\alpha)\cap\Gamma(\beta)$ is not as claimed in (iii), then it contains three distinct points γ, δ, δ' such that δ, $\delta'\notin\Gamma(\gamma)$. But this contradicts part (ii). □

3.13.5. Corollary. *Let* Γ *be a root graph,* $\gamma\in\Gamma$ *and* Q *a quadrangle in* $\Gamma(\gamma)$. *Then if* $\delta\in\Gamma(\gamma)\setminus Q$ *is adjacent to some but not all vertices of* Q, *the set* $\delta^\perp\cap Q$ *is an edge.*

Proof. $Q\cup\{\delta\}$ is a connected graph on 5 vertices with smallest eigenvalue less than –2, unless $\delta^\perp\cap Q$ is an edge. □

3.13.6. Corollary. *Let* Γ *be a root graph with* $\mu(\Gamma)=\mu$. *Then any two nonadjacent vertices* $\alpha,\beta\in\Gamma(\gamma)$ *have either* $\mu-1$ *or* $\mu-2$ *common neighbours in* $\Gamma(\gamma)$.

Proof. Among the μ common neighbours of α and β, one is γ, and at most one is in $\Gamma_2(\gamma)$. The remaining neighbours must be in $\Gamma(\gamma)$. □

3.13.7. Lemma. *Let* Γ *be a root graph with* $\mathbb{L}(\Gamma)\subseteq\mathbb{Z}^{n+1}$ *and* γ, δ *vertices at distance two with* $\bar\delta-\bar\gamma=2e_0$, *where* $e_0=(1,0,\cdots,0)$. *Then no vertex of* $\Gamma(\gamma)\cap\Gamma(\delta)$ *is adjacent to a vertex of* $\Gamma(\gamma)\setminus\Gamma(\delta)$.

Proof. Without loss of generality we may assume that $\bar\gamma=0$ and $\bar\delta=2e_0$. Lemma 3.14.3 shows that common neighbours of γ and δ are represented by suitable vectors $e_0\pm e_l$, and, by permuting the e_i and changing their signs if necessary, we may assume that $M:=\Gamma(\gamma)\cap\Gamma(\delta)$ is represented by the $\mu=2s+t$ vectors

$$e_0\pm e_l\ \ (l=1,\cdots,s),\quad e_0+e_l\ \ (l=s+1,\cdots,s+t).$$

Let $\alpha\in M$ so that (for some i) $\bar\alpha=e_0\pm e_i$, and suppose that γ has a neighbour $\beta\notin M$ adjacent with α. Then $d(\beta,\delta)=2$; but either $\bar\beta=e_0\pm e_j$, $(\bar\beta,\bar\delta)=2$, or $\bar\beta=\pm e_i\pm e_j$ $(0\neq j\neq i\neq 0)$, $(\bar\beta,\bar\delta)=0$, both contradicting Lemma 3.14.3. Hence no point of $\Gamma(\gamma)\setminus\Gamma(\delta)$ is adjacent to a point of M. □

3.13.8. Proposition. *Let Γ be a root graph with $\mathbb{L}(\Gamma) \subseteq \mathbb{Z}^{n+1}$ for some n, and (*) for no $\gamma, \delta \in \Gamma$ at mutual distance two do we have $\bar\delta - \bar\gamma = \pm 2e_i$ for some i, $0 \le i \le n$. Then Γ has a root representation in $\{0,1\}^{n+1}$ sending an arbitrary $\gamma \in \Gamma$ to 0.*

Proof. Given a root representation $\rho : \gamma \mapsto \bar\gamma = \sum \gamma_i e_i$ satisfying (*), and a vector $u \in \mathbb{Z}^{n+1}$, we find new root representations $\rho - u : \gamma \mapsto \bar\gamma - u$ and $|\rho| : \gamma \mapsto \sum |\gamma_i| e_i$ both satisfying (*). (Indeed, if $\{\delta_i - \gamma_i \in \{-1,0,1\}$ then $(\delta_i - \gamma_i)^2 = (|\delta_i| - |\gamma_i|)^2$; in particular this holds for $d(\gamma,\delta) \le 2$.) Let ρ be a root representation satisfying (*) in $\{0,1,...,m\}^{n+1}$, where m is chosen minimal. Then $m \le 1$, otherwise $|\rho - \mathbf{1}|$ would be a representation in $\{0,1,...,m-1\}$. Now $|\rho - \rho(\gamma)|$ is the required representation. □

3.13.9. Lemma. *Let Γ be a root graph with $\mathbb{L}(\Gamma) \subseteq \mathbb{Z}^{n+1}$ and $\mu(\alpha,\beta) \ge 2$ for any two vertices $\alpha, \beta \in \Gamma$ at distance two. Suppose $d(\gamma,\delta) \le 4$. (i) If $\bar\gamma = \bar\delta$, then $\gamma = \delta$. (ii) If $[\gamma,\delta] = 2$ then $\gamma \sim \delta$.*

Proof. If $d(\gamma,\delta) \le 2$, this is clear. We first prove part (ii). Assume $[\gamma,\delta] = 2$. If $d(\gamma,\delta) = 3$, then choose $\varepsilon \in \Gamma(\gamma) \cap \Gamma_2(\delta)$ and shift the representation such that $\bar\varepsilon = 0$. We may assume that $\bar\gamma = e_i + e_j$ and $\bar\delta - \bar\gamma = e_k + e_l$, with distinct i, j, k, l. The common neighbours of δ and ε are represented by certain $e_a + e_b$ with $\{a,b\} \subseteq \{i,j,k,l\}$, and since there are at least two of those, it follows that γ has distance at most one to (at least) one of them, so that $d(\gamma,\delta) \le 2$, contradiction. If $d(\gamma,\delta) = 4$, then choose $\varepsilon \in \Gamma_2(\gamma) \cap \Gamma_2(\delta)$ and shift the representation such that $\bar\varepsilon = 0$. We may assume that $\bar\gamma = e_i + e_j + e_k + e_l$ and $\bar\delta = e_i + e_j + e_k + e_m$. By the foregoing, no common neighbour of γ and ε is represented by $e_a + e_b$ with $\{a,b\} \subseteq \{i,j,k\}$, so the (at least two) vertices in $\gamma^\perp \cap \varepsilon^\perp$ are represented by $e_a + e_l$ for certain $a \in \{i,j,k\}$, and similarly those in $\delta^\perp \cap \varepsilon^\perp$ by $e_a + e_m$ for certain $a \in \{i,j,k\}$. But then some vertex in $\gamma^\perp \cap \varepsilon^\perp$ is adjacent to some vertex in $\delta^\perp \cap \varepsilon^\perp$, so that $d(\gamma,\delta) \le 3$, contradiction. This proves part (ii). But if $\bar\gamma = \bar\delta$, $\gamma, \delta \in \Gamma_2(\varepsilon)$, then by the above $\gamma^\perp \cap \varepsilon^\perp = \delta^\perp \cap \varepsilon^\perp$ and hence $d(\gamma,\delta) \le 2$, contradiction. This proves part (i). □

3.13.10. Proposition. *Let Γ be a root graph with $\mathbb{L}(\Gamma) \subseteq \mathbb{Z}^{n+1}$ and noncomplete μ-graphs. Then for any $\gamma \in \Gamma$ the root representation is uniquely determined by its restriction to $\gamma^\perp$. If moreover $\mu(\Gamma) = \mu$ is constant, then Γ itself is uniquely determined by the representation of $\gamma^\perp$.*

Proof. The first claim follows immediately from 3.14.4 (ii). If μ is constant, then the representation of $\gamma^\perp$ determines the structure and representation of $\Gamma_2(\gamma)$: for any two nonadjacent vertices $\alpha, \beta \in \Gamma(\gamma)$ that have only $\mu - 2$ common neighbours in $\Gamma(\gamma)$, we find a vertex δ represented by $\bar\alpha + \bar\beta - \bar\gamma$. But the restriction of the representation to $\Gamma_{\le 2}(\gamma)$ is injective, and determines the structure of this subgraph, by Lemma 3.14.9 above. Now vary the vertex γ. □

Classification of amply regular root graphs

Let us first define one more family of graphs, and then show that all amply regular root graphs with $\mu > 1$ are known.

Let $B(n,s)$ be the subgraph of the halved $(n+2)$-cube spanned by the $2\binom{n}{s-1} + 2\binom{n}{s}$ vectors of length $n+2$ and either weight $s+2$ starting $11\cdots$ or weight s starting $10\cdots$ or $01\cdots$ or $00\cdots$. Then $B(n,s)$ is a root graph (indeed, a code graph) with $\mu = 4$. It is regular if and only if $n = 2s-1$, and we put $B(s) := B(2s-1,s)$. Now $B(s)$ is regular of valency $s^2 + s + 1$, has diameter s, and satisfies $c_i = i^2$ $(1 \le i \le s)$. In particular, $B(1) \cong K_4$, and $B(2)$ is one of the subgraphs of the Clebsch graph mentioned in 3.11 (iii). For $s \ge 2$, Aut $B(s)$ is transitive of rank $3s$. For more details, see Brouwer & Koolen [840].

3.13.11. Theorem. *Let Γ be a non-complete regular root graph with $\mu(\Gamma) = \mu > 2$. Then Γ is locally connected and one of the following holds.*

(i) $\Gamma \cong K_{n\times 2}$, *where* $\mu = 2n - 2 \ge 8$.

(ii) $\mu = 6$ *and* Γ *is a half-cube.*

(iii) $\mu = 4$ *and either* Γ *is a Johnson graph or* $\Gamma \cong B(n)$ *for some* $n \geq 2$.

(iv) $\mathbb{L}(\Gamma) \cong E_6, E_7$, *or* E_8.

Proof. By Corollary 3.14.6, Γ is locally connected, and by Proposition 3.14.1, $\mathbb{L}(\Gamma)$ is irreducible. Now we assume that we are not in case (iv). Then by Theorem 3.10.4, $\mathbb{L}(\Gamma)$ is isomorphic to A_n or D_n; in particular $\mathbb{L}(\Gamma) \subseteq \mathbb{Z}^{n+1} := \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$, and we may identify vectors in $\mathbb{L}(\Gamma)$ with $(n+1)$-tuples of integers.

Suppose that Γ contains points γ, δ at distance 2 such that the norm 4 vector $\bar{\gamma} - \bar{\delta}$ has the form $\pm 2e_j$ for some j. Since Γ is locally connected, it follows from Lemma 3.14.7 that $\Gamma(\gamma) \cap \Gamma(\delta) = \Gamma(\gamma)$. Therefore Γ has valency $k = \mu$, and Γ is complete multipartite with classes of constant size s. But $\Gamma(\gamma)$ has classes of size 1 or 2 only; therefore $s \leq 2$, and $\Gamma \cong K_{m\times 2}$ for some $m \geq 3$. If $m > 4$, then (i) holds; if $m = 4$ then Γ is the halved 4-cube, and if $m = 3$ then $\Gamma \cong J(4,2)$. [If we do not assume regularity, the conclusion here is that either Γ is complete multipartite with classes of size 1 or 2, or $\Gamma \cong K_{3,1,1,1}$.]

We may now assume that Γ does not contain points γ, δ at distance 2 such that $\bar{\gamma} - \bar{\delta} = \pm 2e_j$ $(j = 0, \cdots, n)$. Fix $\gamma \in \Gamma$. By Proposition 3.14.8 we may assume that $\bar{\delta} = \sum \delta_i e_i$ with $\delta_i \in \{0,1\}$ for all i and $\delta \in \Gamma$, and that $\bar{\gamma} = 0$.

Now let Δ be the graph whose vertices are the $n+1$ indices $0, 1, \cdots, n$, and whose edges are the pairs ij such that some point of $\Gamma(\gamma)$ is represented by $e_i + e_j$. Since distinct points of $\Gamma(\gamma)$ are represented by distinct vectors (their difference must have norm 2 or 4) we may identify the point represented by $e_i + e_j$ with the edge ij of Δ.

Let us denote by $S(\delta)$ the set of indices i with $\delta_i \neq 0$. By Lemma 3.14.3, the points $\delta \in \Gamma_2(\gamma)$ are represented by norm 4 vectors $\bar{\delta} = \bar{\delta} - \bar{\gamma}$, and these must have the form $e_i + e_j + e_k + e_l$, i.e., $S(\delta)$ is a 4-set. The μ common neighbours of γ and δ are certain edges of the subgraph of Δ induced on $S(\delta)$. By Lemma 3.14.9, the intersection $\Gamma(\gamma) \cap \Gamma(\delta)$ coincides with the set of edges of Δ contained in $S(\delta)$, and $\delta \in \Gamma_2(\gamma)$ is uniquely determined by $S(\delta)$. To simplify the notation we shall use the abbreviation *abcd* for a 4-set $\{a,b,c,d\}$.

Let us call a set $S \subseteq \{0,1, \cdots, n\}$ *special* if there is a point $\delta \in \Gamma$ with

$$S(\delta) = S, \quad d(\delta,\gamma) = \tfrac{1}{2}|S|.$$

Clearly, the empty set is special. Application of Lemma 3.14.3 to pairs of nonadjacent points in $\Gamma_{\leq 2}(\gamma)$ yields the following facts (here distinct Latin letters denote distinct indices taken from $\{0, \cdots, n\}$).

S1. If $\delta \in \Gamma_2(\gamma)$ then $S(\delta)$ is special.

S2. If *abcd* is special then precisely μ of the sets *ab*, *ac*, *ad*, *bc*, *bd*, *cd* are special.

S3. If *ab* and *cd* are special then precisely μ of the sets $\emptyset$, *ac*, *ad*, *bc*, *bd*, *abcd* are special.

S4. If *abcd* and *de* are special then none or μ of the sets *ad*, *bd*, *cd*, *abde*, *acde*, *bcde* are special.

(Indeed, if *abcd* and *de* represent vertices at distance two, then they have μ common neighbours, and these must be among the six sets listed. On the other hand, if one of these six sets is special, we must show that it represents a common neighbour of *abcd* and *de*. But this follows immediately from Lemma 3.14.9.)

By definition, the edges of Δ are just the special 2-sets. Since $\mu \geq 3$, S3 implies that Δ has no subgraph consisting of two disjoint edges. Therefore Δ consists of a connected graph Δ_0 and a (possibly empty) set of isolated vertices. S3 implies

S5. Any two disjoint edges of Δ have $\mu - 1$ or $\mu - 2$ transversals (i.e. edges meeting both given edges).

Moreover, since Γ is regular and not complete, $\Gamma(\gamma)$ is not complete; therefore we have

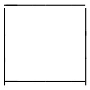
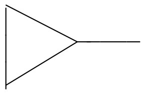
S6. Δ_0 is not isomorphic to K_3 or $K_{s,1}$.

We now consider each value of μ separately. Clearly, S3 implies that $\mu \le 6$.

CASE 1: $\mu = 6$. By S1 and S2, a point of $\Gamma_2(\gamma)$ is a complete subgraph K_4 of Δ_0. Let K be a complete subgraph K_m of Δ_0 with maximal $m \ge 4$. If $\Delta_0 \ne K$ then there is a vertex $b \in \Delta_0$ at distance 1 from K, and a vertex $a \in K$ adjacent with b. By maximality of K, there is a vertex $c \in K$ with $c \not\sim b$. Let $d \in K \setminus \{a,c\}$. Then the edges ab, cd have at most 3 transversals, contradicting S5. Hence $\Delta_0 = K$ is complete and $\Gamma(\gamma) \cong T(m)$. By Proposition 3.14.10 this determines the representation uniquely, and we find that Γ is the halved m-cube.

CASE 2: $\mu = 5$. By S1 and S2, a point of $\Gamma_2(\gamma)$ is a subgraph $K_{2,1,1}$ of Δ_0. Let K be a subgraph of Δ_0 isomorphic to $K_{m,1,\ldots,1}$ with a maximal number $m{+}s$ of vertices, $m \ge 2$, $s \ge 2$. Using S5 we find as before that $\Delta_0 = K_{m,1,\ldots,1}$. Let L be the m-coclique of Δ_0. If $s > 2$, then we find that, for $a, b, e \in K \setminus L$ and $c, d \in L$, the set $abcd$ is special. Now S4 gives a contradiction. Hence $\Delta_0 = K_{m,1,1}$, and we find by Proposition 3.14.10 that $\Gamma \cong K_1 \oplus T(m+2)$, but this graph is not regular.

CASE 3: $\mu = 4$. By S5, each pair of disjoint edges has 2 or 3 transversals. By S1 and S2, a point of $\Gamma_2(\gamma)$ is a special 4-set on which Δ_0 induces

 or  .

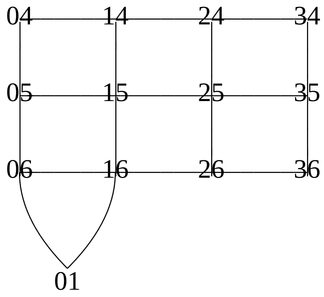
and by S3, all subgraphs of Δ_0 of these shapes are special 4-sets.
These subgraphs of Δ_0 are of the form $K_{2,2}$ and $K_{1,3} + \text{edge}$. Let $E = A dotcu B$ be a largest subset of Δ_0 such that $a \sim b$ for all $a \in A$ and $b \in B$, where A and B are nonempty.
(i) *E contains all vertices of* Δ_0. Indeed, if not then let $c \in \Delta_0$, $c \notin E$. We may assume $d(c,E) = 1$, say $c \sim a \in A$. By maximality of E there is a $b \in B$ such that $b \not\sim c$. Put $A_0 := A \cap \Delta(c)$ and $A_1 := A \setminus \Delta(c)$, then for any $a \in A_0$ and $d \in A_1$ we have $a \sim d$ since ac and bd have at least two transversals. But then $E \cup \{c\} = A_0 dotcu (A_1 \cup B \cup \{c\})$ is a larger subset of Δ with the required property, contradiction.

(ii) Δ_0 *does not contain a complete graph K_4 or a wheel $K_{1,2,2}$.* Indeed, two disjoint edges in a K_4 have 4 transversals, which is impossible. If Δ_0 contains an induced subgraph $K_{2,2,1}$, say $a \sim b \sim c \sim d \sim a$, $e \sim a, b, c, d$, then $abcd$ is special, and by S4 we find a contradiction.

(iii) Δ_0 *is either complete bipartite or complete bipartite with one additional edge.* Indeed, if A contains two disjoint edges, then (since these must have at least two transversals) A contains a triangle or a quadrangle, and since $B \ne \emptyset$ this is forbidden by (ii). Thus, all edges in A pass through one vertex d. If both A and B contain an edge then Δ_0 contains a K_4. So we may assume that B is a coclique. If $|B| \ge 2$ and A contains two intersecting edges, then Δ_0 contains a $K_{1,2,2}$, contradiction. So we may assume that $B = \{b\}$. If A has no isolated point, then we can move d from A to B and are done. So we may assume that A has at least two edges cd and de and an isolated point a. Now S4 yields a contradiction.

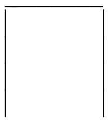
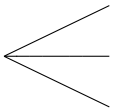
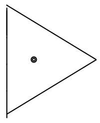
(iv) *If* Δ_0 *is complete bipartite, say $K_{s,t}$, then* $\Gamma \cong J(s+t,t)$. Indeed, this follows from Proposition 3.14.10.

Now assume that Δ_0 is a complete bipartite graph $K_{s,t}$ with one additional edge in the t-coclique, so that $\Gamma(\gamma)$ is a grid $s \times t$ with one additional vertex adjacent to a $s \times 2$ subgrid. Let us temporarily call this latter graph $s_+ \times t$ - in the figure below the graph $3_+ \times 4$ is drawn; a label ij denotes the vector $u = e_i + e_j$.



(v) *If* $\Gamma(\gamma) \cong s_+ \times t$, *then* $t = s+1$, *and* $\Gamma \cong B(s)$. Indeed, with Proposition 3.14.10 we find that $\Gamma \cong B(s+t-2, s)$, and then regularity of Γ implies $t = s+1$.

CASE 4: $\mu = 3$. By S5, each pair of disjoint edges has 1 or 2 transversals. By S1 and S2, a point of $\Gamma_2(\gamma)$ is a special 4-set on which Δ_0 induces

 or  or  .

Assume first that Δ_0 contains an induced path P of length 3 with vertices $a \sim b \sim c \sim d$. By S3, *abcd* is special. Let e be at distance 1 from P. If $e \sim d$, then S4 implies that precisely two of the sets *abde*, *acde*, *bcde* are special. However, this is impossible: If $e \sim a$, then $e \sim b$ implies that *abde*, *bcde* are not special. So $e \not\sim b$, and by symmetry $e \not\sim c$. But then *abde*, *acde* and *bcde* are all special, contradiction. And if $e \not\sim a$ then $e \sim b$ since *ab* and *de* must have a transversal, and $e \not\sim c$ since *bc* and *de* cannot have 3 transversals. But then *acde* and *bcde* are not special, contradiction. Hence $e \not\sim d$, and by symmetry, $e \not\sim a$.
If $e \sim b$ and $e \sim c$, then S4 with (b,d) interchanged gives a contradiction. Therefore e is adjacent to just one of b and c (and if $e \sim b$ then *ebcd* is special). Hence the neighbours $\neq c$ of b form a coclique A, and the neighbours $\neq b$ of c form a coclique D. Now it is easy to see that $\Delta_0 = A \cup \{b,c\} \cup D$ (the coclique extension of P where a and d are blown up to A and D, respectively). Since γ has valency $|A| + |D| + 1$ and *bc* has $\geq |A| + |D| + 2$ neighbours γ, *ab* $(a \in A)$, *cd* $(d \in D)$ and P, this contradicts the regularity of Γ. Therefore Δ_0 does not contain an induced path of length 3.
[If we do not assume regularity, this case leads to $\Gamma \cong K_1 \oplus K_{|A|+1, |D|+1}$.]

Now if $\Gamma_2(\gamma)$ contains a vertex *abcd* with $b \sim c \sim d \sim b$ and e is adjacent with d, then *abcd* and *de* have only two common neighbours, contradiction. Hence $e \not\sim d$ and by symmetry, $e \not\sim b, c$. Therefore *bcd* is an isolated triangle, $\Delta_0 = \{b,c,d\}$, contradicting S6. This shows that $\Gamma_2(\gamma)$ contains only special 4-sets forming a 3-claw. [If we do not assume regularity, this case leads to $\Gamma \cong L(s,t)$, where $L(s,t)$ is the graph represented by e_i $(1 \leq i \leq 3s+t)$, $e_{3j-2} + e_{3j-1} + e_{3j}$ $(1 \leq j \leq s)$.]

Let *abcd* be such a special 4-set, $a \sim b, c, d$. If $e \sim d$ then $e \sim b, c$ since Δ_0 has no path of length 3; but then S4 is violated. Hence b, c, d are vertices of Δ_0 of valency 1, and since Δ_0 is connected and has no path of length 3, every vertex $\neq a$ of Δ_0 must be adjacent to a. By S6, Δ_0 contains at least one further edge *ef*. If there is another edge *eg* (or *fg*), then *ag*, *ef* have 3 transversals, contradicting S5, and if there is another edge *gh* then *ef*, *gh* have no transversal, again contradicting S5. Therefore there is no further edge. Now *ef* must have a neighbour in $\Gamma_2(\gamma)$, but there is no special 4-set containing *ef*, contradiction. Therefore $\mu = 3$ is impossible. [If we do not assume regularity, then we find $\Gamma \cong L(s,t)$ again.]

This completes the proof of the theorem. □

Remark. KOOLEN [929] generalized this theorem by omitting 'regular' from the assumptions. In the proof above we mentioned which further graphs arise in this more general situation.

Tables of parameter sets for bipartite (but not antipodal) distance-regular graphs with $v \le 4096$ that pass all known existence tests.

Diameter 4 - Bipartite (but not antipodal) graphs

These graphs are precisely the incidence graphs of partial λ-geometries with nexus $e < k-1$ (i.e., excluding the symmetric nets), cf. Theorem 1.7.1. Apart from the families of generalized octagons of order $(1,t)$ (i.e., incidence graphs of generalized quadrangles of order (t,t)), and (pseudo) $D_m(q)$ graphs, an infinite family of feasible Q-polynomial parameter sets with $k = 3\binom{m}{3} + \binom{m-1}{2}$, $c_2 = \binom{m}{2} - 1$, $c_3 = 3\binom{m}{3}$, $q^3_{33} = p^4_{44} = 0$ is visible. Such a graph is known only for $m = 4$ (and for $m = 3$ we get the 4-cube, which is antipodal).

! $v = 30 = 1+3+6+12+8$ $3^1\ 2^9\ 0^{10}\ {-2}^9\ {-3}^1$ $\{3,2,2,2;1,1,1,3\}$
Generalized octagon of order (1,2)
! $v = 80 = 1+4+12+36+27$ $4^1\ 2.449^{24}\ 0^{30}\ {-2.449}^{24}\ {-4}^1$ $\{4,3,3,3;1,1,1,4\}$
Generalized octagon of order (1,3)
! $v = 170 = 1+5+20+80+64$ $5^1\ 2.828^{50}\ 0^{68}\ {-2.828}^{50}\ {-5}^1$ $\{5,4,4,4;1,1,1,5\}$
Generalized octagon of order (1,4)
$v = 162 = 1+6+30+75+50$ $6^1\ 3^{50}\ 0^{60}\ {-3}^{50}\ {-6}^1$ $\{6,5,5,4;1,1,2,6\}$
van Lint-Schrijver partial geometry, cf. §*CodeGr*.5
$v = 312 = 1+6+30+150+125$ $6^1\ 3.162^{90}\ 0^{130}\ {-3.162}^{90}\ {-6}^1$ $\{6,5,5,5;1,1,1,6\}$
Generalized octagon of order (1,5)
? $v = 140 = 1+7+42+63+27$ $7^1\ 3^{49}\ 0^{40}\ {-3}^{49}\ {-7}^1$ $\{7,6,6,3;1,1,4,7\}$
? $v = 518 = 1+7+42+252+216$ $7^1\ 3.464^{147}\ 0^{222}\ {-3.464}^{147}\ {-7}^1$ $\{7,6,6,6;1,1,1,7\}$
Generalized octagon of order (1,6)
! $v = 128 = 1+8+28+56+35$ $8^1\ 4^{28}\ 0^{70}\ {-4}^{28}\ {-8}^1$ $\{8,7,6,5;1,2,3,8\}$
Folded 8-cube; $q^3_{33} = 0$; Q-polynomial
$v = 800 = 1+8+56+392+343$ $8^1\ 3.742^{224}\ 0^{350}\ {-3.742}^{224}\ {-8}^1$ $\{8,7,7,7;1,1,1,8\}$
Generalized octagon of order (1,7)
$v = 1170 = 1+9+72+576+512$ $9^1\ 4^{324}\ 0^{520}\ {-4}^{324}\ {-9}^1$ $\{9,8,8,8;1,1,1,9\}$
Generalized octagon of order (1,8)
$v = 1640 = 1+10+90+810+729$ $10^1\ 4.243^{450}\ 0^{738}\ {-4.243}^{450}\ {-10}^1$ $\{10,9,9,9;1,1,1,10\}$
Generalized octagon of order (1,9)
? $v = 2222 = 1+11+110+1100+1000$ $11^1\ 4.472^{605}\ 0^{1010}\ {-4.472}^{605}\ {-11}^1$ $\{11,10,10,10;1,1,1,11\}$
Generalized octagon of order (1,10)
? $v = 244 = 1+12+66+110+55$ $12^1\ 4.690^{60}\ 0^{122}\ {-4.690}^{60}\ {-12}^1$ $\{12,11,10,6;1,2,6,12\}$
! $v = 288 = 1+12+66+132+77$ $12^1\ 4.899^{66}\ 0^{154}\ {-4.899}^{66}\ {-12}^1$ $\{12,11,10,7;1,2,5,12\}$
Leonard graph, cf. §*CodeGr*.4F
$v = 2928 = 1+12+132+1452+1331$ $12^1\ 4.690^{792}\ 0^{1342}\ {-4.690}^{792}\ {-12}^1$ $\{12,11,11,11;1,1,1,12\}$
Generalized octagon of order (1,11)
$v = 442 = 1+13+156+208+64$ $13^1\ 4^{169}\ 0^{102}\ {-4}^{169}\ {-13}^1$ $\{13,12,12,4;1,1,9,13\}$
Denniston arc, cf. §*FinGeom*.6
? $v = 3770 = 1+13+156+1872+1728$ $13^1\ 4.899^{1014}\ 0^{1740}\ {-4.899}^{1014}\ {-13}^1$ $\{13,12,12,12;1,1,1,13\}$
Generalized octagon of order (1,12)
? $v = 340 = 1+14+91+156+78$ $14^1\ 5.099^{84}\ 0^{170}\ {-5.099}^{84}\ {-14}^1$ $\{14,13,12,7;1,2,7,14\}$
? $v = 392 = 1+14+91+182+104$ $14^1\ 5.292^{91}\ 0^{208}\ {-5.292}^{91}\ {-14}^1$ $\{14,13,12,8;1,2,6,14\}$
! $v = 100 = 1+15+42+35+7$ $15^1\ 5^{21}\ 0^{56}\ {-5}^{21}\ {-15}^1$ $\{15,14,10,3;1,5,12,15\}$
Cocliques in Hoffman-Singleton graph, cf. Theorem 13.1.1 (iv); $q^3_{33} = 0$; $p^4_{44} = 0$; Q-polynomial
? $v = 170 = 1+15+70+70+14$ $15^1\ 4.583^{50}\ 0^{68}\ {-4.583}^{50}\ {-15}^1$ $\{15,14,12,3;1,3,12,15\}$

$v = 270 = 1+15+70+120+64$	$15^1\ 6^{50}\ 0^{168}\ -6^{50}\ -15^1$	$\{15,14,12,8;1,3,7,15\}$
dual polar graph $D_4(2)$; $q^3_{33} = 0$; classical ($b = 2$)		
? $v = 1500 = 1+15+210+735+539$	$15^1\ 5^{441}\ 0^{616}\ -5^{441}\ -15^1$	$\{15,14,14,11;1,1,4,15\}$
? $v = 452 = 1+16+120+210+105$	$16^1\ 5.477^{112}\ 0^{226}\ -5.477^{112}\ -16^1$	$\{16,15,14,8;1,2,8,16\}$
? $v = 512 = 1+16+120+240+135$	$16^1\ 5.657^{120}\ 0^{270}\ -5.657^{120}\ -16^1$	$\{16,15,14,9;1,2,7,16\}$
? $v = 1232 = 1+16+240+600+375$	$16^1\ 5^{384}\ 0^{462}\ -5^{384}\ -16^1$	$\{16,15,15,10;1,1,6,16\}$
? $v = 580 = 1+18+153+272+136$	$18^1\ 5.831^{144}\ 0^{290}\ -5.831^{144}\ -18^1$	$\{18,17,16,9;1,2,9,18\}$
? $v = 648 = 1+18+153+306+170$	$18^1\ 6^{153}\ 0^{340}\ -6^{153}\ -18^1$	$\{18,17,16,10;1,2,8,18\}$
? $v = 192 = 1+20+76+76+19$	$20^1\ 6.325^{38}\ 0^{114}\ -6.325^{38}\ -20^1$	$\{20,19,15,5;1,5,15,20\}$
? $v = 724 = 1+20+190+342+171$	$20^1\ 6.164^{180}\ 0^{362}\ -6.164^{180}\ -20^1$	$\{20,19,18,10;1,2,10,20\}$
? $v = 800 = 1+20+190+380+209$	$20^1\ 6.325^{190}\ 0^{418}\ -6.325^{190}\ -20^1$	$\{20,19,18,11;1,2,9,20\}$
? $v = 1408 = 1+20+190+684+513$	$20^1\ 6.928^{285}\ 0^{836}\ -6.928^{285}\ -20^1$	$\{20,19,18,15;1,2,5,20\}$
? $v = 1750 = 1+20+190+855+684$	$20^1\ 7.071^{342}\ 0^{1064}\ -7.071^{342}\ -20^1$	$\{20,19,18,16;1,2,4,20\}$
? $v = 378 = 1+21+140+168+48$	$21^1\ 6^{98}\ 0^{180}\ -6^{98}\ -21^1$	$\{21,20,18,6;1,3,15,21\}$
? $v = 462 = 1+21+140+210+90$	$21^1\ 6.708^{98}\ 0^{264}\ -6.708^{98}\ -21^1$	$\{21,20,18,9;1,3,12,21\}$
? $v = 762 = 1+21+140+360+240$	$21^1\ 7.746^{126}\ 0^{508}\ -7.746^{126}\ -21^1$	$\{21,20,18,14;1,3,7,21\}$
? $v = 1092 = 1+21+420+525+125$	$21^1\ 5^{441}\ 0^{208}\ -5^{441}\ -21^1$	$\{21,20,20,5;1,1,16,21\}$
? $v = 2842 = 1+21+420+1400+1000$	$21^1\ 5.916^{840}\ 0^{1160}\ -5.916^{840}\ -21^1$	$\{21,20,20,15;1,1,6,21\}$
? $v = 3402 = 1+21+420+1680+1280$	$21^1\ 6^{980}\ 0^{1440}\ -6^{980}\ -21^1$	$\{21,20,20,16;1,1,5,21\}$
? $v = 884 = 1+22+231+420+210$	$22^1\ 6.481^{220}\ 0^{442}\ -6.481^{220}\ -22^1$	$\{22,21,20,11;1,2,11,22\}$
? $v = 968 = 1+22+231+462+252$	$22^1\ 6.633^{231}\ 0^{504}\ -6.633^{231}\ -22^1$	$\{22,21,20,12;1,2,10,22\}$
? $v = 2816 = 1+22+462+1386+945$	$22^1\ 6^{847}\ 0^{1120}\ -6^{847}\ -22^1$	$\{22,21,21,15;1,1,7,22\}$
? $v = 324 = 1+24+138+138+23$	$24^1\ 6^{92}\ 0^{138}\ -6^{92}\ -24^1$	$\{24,23,20,4;1,4,20,24\}$
? $v = 1014 = 1+24+184+483+322$	$24^1\ 8.307^{168}\ 0^{676}\ -8.307^{168}\ -24^1$	$\{24,23,21,16;1,3,8,24\}$
? $v = 1152 = 1+24+184+552+391$	$24^1\ 8.485^{184}\ 0^{782}\ -8.485^{184}\ -24^1$	$\{24,23,21,17;1,3,7,24\}$
? $v = 1060 = 1+24+276+506+253$	$24^1\ 6.782^{264}\ 0^{530}\ -6.782^{264}\ -24^1$	$\{24,23,22,12;1,2,12,24\}$
? $v = 1152 = 1+24+276+552+299$	$24^1\ 6.928^{276}\ 0^{598}\ -6.928^{276}\ -24^1$	$\{24,23,22,13;1,2,11,24\}$
! $v = 4096 = 1+24+276+2024+1771$	$24^1\ 8^{759}\ 0^{2576}\ -8^{759}\ -24^1$	$\{24,23,22,21;1,2,3,24\}$
Cosets of extended binary Golay code, cf. §*CodeGr*.3D		
? $v = 3250 = 1+25+600+1600+1024$	$25^1\ 6.325^{1000}\ 0^{1248}\ -6.325^{1000}\ -25^1$	$\{25,24,24,16;1,1,9,25\}$
? $v = 1252 = 1+26+325+600+300$	$26^1\ 7.071^{312}\ 0^{626}\ -7.071^{312}\ -26^1$	$\{26,25,24,13;1,2,13,26\}$
? $v = 1352 = 1+26+325+650+350$	$26^1\ 7.211^{325}\ 0^{700}\ -7.211^{325}\ -26^1$	$\{26,25,24,14;1,2,12,26\}$
? $v = 1302 = 1+27+234+624+416$	$27^1\ 8.832^{216}\ 0^{868}\ -8.832^{216}\ -27^1$	$\{27,26,24,18;1,3,9,27\}$
$v = 1458 = 1+27+234+702+494$	$27^1\ 9^{234}\ 0^{988}\ -9^{234}\ -27^1$	$\{27,26,24,19;1,3,8,27\}$
Pasechnik graph		
? $v = 1460 = 1+28+378+702+351$	$28^1\ 7.348^{364}\ 0^{730}\ -7.348^{364}\ -28^1$	$\{28,27,26,14;1,2,14,28\}$
? $v = 1568 = 1+28+378+756+405$	$28^1\ 7.483^{378}\ 0^{810}\ -7.483^{378}\ -28^1$	$\{28,27,26,15;1,2,13,28\}$
? $v = 1626 = 1+30+290+783+522$	$30^1\ 9.327^{270}\ 0^{1084}\ -9.327^{270}\ -30^1$	$\{30,29,27,20;1,3,10,30\}$
? $v = 1800 = 1+30+290+870+609$	$30^1\ 9.487^{290}\ 0^{1218}\ -9.487^{290}\ -30^1$	$\{30,29,27,21;1,3,9,30\}$
? $v = 1684 = 1+30+435+812+406$	$30^1\ 7.616^{420}\ 0^{842}\ -7.616^{420}\ -30^1$	$\{30,29,28,15;1,2,15,30\}$
? $v = 1800 = 1+30+435+870+464$	$30^1\ 7.746^{435}\ 0^{928}\ -7.746^{435}\ -30^1$	$\{30,29,28,16;1,2,14,30\}$
? $v = 2294 = 1+31+930+1116+216$	$31^1\ 6^{961}\ 0^{370}\ -6^{961}\ -31^1$	$\{31,30,30,6;1,1,25,31\}$
? $v = 1924 = 1+32+496+930+465$	$32^1\ 7.874^{480}\ 0^{962}\ -7.874^{480}\ -32^1$	$\{32,31,30,16;1,2,16,32\}$
$v = 2048 = 1+32+496+992+527$	$32^1\ 8^{496}\ 0^{1054}\ -8^{496}\ -32^1$	$\{32,31,30,17;1,2,15,32\}$
Coset graph of extended Kasami code, cf. Theorem 11.2.1		
? $v = 1986 = 1+33+352+960+640$	$33^1\ 9.798^{330}\ 0^{1324}\ -9.798^{330}\ -33^1$	$\{33,32,30,22;1,3,11,33\}$
? $v = 2178 = 1+33+352+1056+736$	$33^1\ 9.950^{352}\ 0^{1472}\ -9.950^{352}\ -33^1$	$\{33,32,30,23;1,3,10,33\}$
? $v = 2180 = 1+34+561+1056+528$	$34^1\ 8.124^{544}\ 0^{1090}\ -8.124^{544}\ -34^1$	$\{34,33,32,17;1,2,17,34\}$

? $v = 2312 = 1+34+561+1122+594$	$34^1\ 8.246^{561}\ 0^{1188}\ -8.246^{561}\ -34^1$	$\{34,33,32,18;1,2,16,34\}$
? $v = 750 = 1+35+238+340+136$	$35^1\ 10^{119}\ 0^{510}\ -10^{119}\ -35^1$	$\{35,34,30,14;1,5,21,35\}$
? $v = 784 = 1+35+238+357+153$	$35^1\ 10.247^{119}\ 0^{544}\ -10.247^{119}\ -35^1$	$\{35,34,30,15;1,5,20,35\}$
? $v = 324 = 1+36+140+126+21$	$36^1\ 9^{56}\ 0^{210}\ -9^{56}\ -36^1$	$\{36,35,27,6;1,9,30,36\}$
$q^3_{33} = 0;\ p^4_{44} = 0$; Q-polynomial		
? $v = 744 = 1+36+315+336+56$	$36^1\ 7.483^{216}\ 0^{310}\ -7.483^{216}\ -36^1$	$\{36,35,32,6;1,4,30,36\}$
? $v = 912 = 1+36+315+420+140$	$36^1\ 8.944^{189}\ 0^{532}\ -8.944^{189}\ -36^1$	$\{36,35,32,12;1,4,24,36\}$
? $v = 1080 = 1+36+315+504+224$	$36^1\ 9.798^{189}\ 0^{700}\ -9.798^{189}\ -36^1$	$\{36,35,32,16;1,4,20,36\}$
? $v = 1512 = 1+36+315+720+440$	$36^1\ 10.954^{216}\ 0^{1078}\ -10.954^{216}\ -36^1$	$\{36,35,32,22;1,4,14,36\}$
? $v = 912 = 1+36+420+420+35$	$36^1\ 6.481^{360}\ 0^{190}\ -6.481^{360}\ -36^1$	$\{36,35,33,3;1,3,33,36\}$
? $v = 1458 = 1+36+420+693+308$	$36^1\ 9^{308}\ 0^{840}\ -9^{308}\ -36^1$	$\{36,35,33,16;1,3,20,36\}$
? $v = 2052 = 1+36+420+990+605$	$36^1\ 9.950^{360}\ 0^{1330}\ -9.950^{360}\ -36^1$	$\{36,35,33,22;1,3,14,36\}$
? $v = 2382 = 1+36+420+1155+770$	$36^1\ 10.247^{396}\ 0^{1588}\ -10.247^{396}\ -36^1$	$\{36,35,33,24;1,3,12,36\}$
? $v = 2592 = 1+36+420+1260+875$	$36^1\ 10.392^{420}\ 0^{1750}\ -10.392^{420}\ -36^1$	$\{36,35,33,25;1,3,11,36\}$
? $v = 4032 = 1+36+420+1980+1595$	$36^1\ 10.954^{594}\ 0^{2842}\ -10.954^{594}\ -36^1$	$\{36,35,33,29;1,3,7,36\}$
? $v = 2452 = 1+36+630+1190+595$	$36^1\ 8.367^{612}\ 0^{1226}\ -8.367^{612}\ -36^1$	$\{36,35,34,18;1,2,18,36\}$
? $v = 2592 = 1+36+630+1260+665$	$36^1\ 8.485^{630}\ 0^{1330}\ -8.485^{630}\ -36^1$	$\{36,35,34,19;1,2,17,36\}$
? $v = 2740 = 1+38+703+1332+666$	$38^1\ 8.602^{684}\ 0^{1370}\ -8.602^{684}\ -38^1$	$\{38,37,36,19;1,2,19,38\}$
? $v = 2888 = 1+38+703+1406+740$	$38^1\ 8.718^{703}\ 0^{1480}\ -8.718^{703}\ -38^1$	$\{38,37,36,20;1,2,18,38\}$
? $v = 2814 = 1+39+494+1368+912$	$39^1\ 10.677^{468}\ 0^{1876}\ -10.677^{468}\ -39^1$	$\{39,38,36,26;1,3,13,39\}$
? $v = 3042 = 1+39+494+1482+1026$	$39^1\ 10.817^{494}\ 0^{2052}\ -10.817^{494}\ -39^1$	$\{39,38,36,27;1,3,12,39\}$
? $v = 704 = 1+40+312+312+39$	$40^1\ 7.746^{208}\ 0^{286}\ -7.746^{208}\ -40^1$	$\{40,39,35,5;1,5,35,40\}$
? $v = 920 = 1+40+312+420+147$	$40^1\ 10.247^{160}\ 0^{598}\ -10.247^{160}\ -40^1$	$\{40,39,35,14;1,5,26,40\}$
? $v = 1120 = 1+40+312+520+247$	$40^1\ 11.402^{160}\ 0^{798}\ -11.402^{160}\ -40^1$	$\{40,39,35,19;1,5,21,40\}$
? $v = 860 = 1+40+390+390+39$	$40^1\ 7.211^{300}\ 0^{258}\ -7.211^{300}\ -40^1$	$\{40,39,36,4;1,4,36,40\}$
? $v = 1250 = 1+40+390+585+234$	$40^1\ 10^{234}\ 0^{780}\ -10^{234}\ -40^1$	$\{40,39,36,16;1,4,24,40\}$
$v = 2240 = 1+40+390+1080+729$	$40^1\ 12^{300}\ 0^{1638}\ -12^{300}\ -40^1$	$\{40,39,36,27;1,4,13,40\}$
dual polar graph $D_4(3)$; $q^3_{33} = 0$; classical ($b = 3$)		
? $v = 3044 = 1+40+780+1482+741$	$40^1\ 8.832^{760}\ 0^{1522}\ -8.832^{760}\ -40^1$	$\{40,39,38,20;1,2,20,40\}$
? $v = 3200 = 1+40+780+1560+819$	$40^1\ 8.944^{780}\ 0^{1638}\ -8.944^{780}\ -40^1$	$\{40,39,38,21;1,2,19,40\}$
? $v = 576 = 1+42+246+246+41$	$42^1\ 9.165^{123}\ 0^{328}\ -9.165^{123}\ -42^1$	$\{42,41,35,7;1,7,35,42\}$
? $v = 3282 = 1+42+574+1599+1066$	$42^1\ 11.091^{546}\ 0^{2188}\ -11.091^{546}\ -42^1$	$\{42,41,39,28;1,3,14,42\}$
? $v = 3528 = 1+42+574+1722+1189$	$42^1\ 11.225^{574}\ 0^{2378}\ -11.225^{574}\ -42^1$	$\{42,41,39,29;1,3,13,42\}$
? $v = 3364 = 1+42+861+1640+820$	$42^1\ 9.055^{840}\ 0^{1682}\ -9.055^{840}\ -42^1$	$\{42,41,40,21;1,2,21,42\}$
? $v = 3528 = 1+42+861+1722+902$	$42^1\ 9.165^{861}\ 0^{1804}\ -9.165^{861}\ -42^1$	$\{42,41,40,22;1,2,20,42\}$
? $v = 3700 = 1+44+946+1806+903$	$44^1\ 9.274^{924}\ 0^{1850}\ -9.274^{924}\ -44^1$	$\{44,43,42,22;1,2,22,44\}$
? $v = 3872 = 1+44+946+1892+989$	$44^1\ 9.381^{946}\ 0^{1978}\ -9.381^{946}\ -44^1$	$\{44,43,42,23;1,2,21,44\}$
? $v = 486 = 1+45+220+198+22$	$45^1\ 9^{110}\ 0^{264}\ -9^{110}\ -45^1$	$\{45,44,36,5;1,9,40,45\}$
? $v = 1050 = 1+45+396+480+128$	$45^1\ 10^{216}\ 0^{616}\ -10^{216}\ -45^1$	$\{45,44,40,12;1,5,33,45\}$
? $v = 1674 = 1+45+660+792+176$	$45^1\ 8.485^{495}\ 0^{682}\ -8.485^{495}\ -45^1$	$\{45,44,42,10;1,3,35,45\}$
? $v = 2400 = 1+45+660+1155+539$	$45^1\ 10.247^{495}\ 0^{1408}\ -10.247^{495}\ -45^1$	$\{45,44,42,21;1,3,24,45\}$
? $v = 3786 = 1+45+660+1848+1232$	$45^1\ 11.489^{630}\ 0^{2524}\ -11.489^{630}\ -45^1$	$\{45,44,42,30;1,3,15,45\}$
? $v = 4050 = 1+45+660+1980+1364$	$45^1\ 11.619^{660}\ 0^{2728}\ -11.619^{660}\ -45^1$	$\{45,44,42,31;1,3,14,45\}$
? $v = 4052 = 1+46+1035+1980+990$	$46^1\ 9.487^{1012}\ 0^{2026}\ -9.487^{1012}\ -46^1$	$\{46,45,44,23;1,2,23,46\}$
? $v = 1080 = 1+50+490+490+49$	$50^1\ 8.367^{350}\ 0^{378}\ -8.367^{350}\ -50^1$	$\{50,49,45,5;1,5,45,50\}$
? $v = 1430 = 1+55+594+660+120$	$55^1\ 10^{363}\ 0^{702}\ -10^{363}\ -55^1$	$\{55,54,50,10;1,5,45,55\}$
? $v = 992 = 1+56+385+440+110$	$56^1\ 12.649^{154}\ 0^{682}\ -12.649^{154}\ -56^1$	$\{56,55,48,14;1,8,42,56\}$
? $v = 1036 = 1+56+385+462+132$	$56^1\ 13.266^{147}\ 0^{740}\ -13.266^{147}\ -56^1$	$\{56,55,48,16;1,8,40,56\}$

? $v = 1344 = 1 + 56 + 440 + 616 + 231$	$56^1\ 14^{176}\ 0^{990}\ {-14}^{176}\ {-56}^1$	$\{56, 55, 49, 21; 1, 7, 35, 56\}$
$q^3_{33} = 0$; Q-polynomial		
? $v = 1536 = 1 + 60 + 590 + 708 + 177$	$60^1\ 12^{295}\ 0^{944}\ {-12}^{295}\ {-60}^1$	$\{60, 59, 54, 15; 1, 6, 45, 60\}$
? $v = 1536 = 1 + 60 + 708 + 708 + 59$	$60^1\ 8.944^{531}\ 0^{472}\ {-8.944}^{531}\ {-60}^1$	$\{60, 59, 55, 5; 1, 5, 55, 60\}$
? $v = 2916 = 1 + 63 + 1302 + 1395 + 155$	$63^1\ 9^{1085}\ 0^{744}\ {-9}^{1085}\ {-63}^1$	$\{63, 62, 60, 7; 1, 3, 56, 63\}$
? $v = 1848 = 1 + 66 + 715 + 858 + 208$	$66^1\ 12.490^{363}\ 0^{1120}\ {-12.490}^{363}\ {-66}^1$	$\{66, 65, 60, 16; 1, 6, 50, 66\}$
? $v = 784 = 1 + 70 + 345 + 322 + 46$	$70^1\ 14^{115}\ 0^{552}\ {-14}^{115}\ {-70}^1$	$\{70, 69, 56, 10; 1, 14, 60, 70\}$
$q^3_{33} = 0$; $p^4_{44} = 0$; Q-polynomial		
? $v = 1750 = 1 + 70 + 690 + 805 + 184$	$70^1\ 13.229^{322}\ 0^{1104}\ {-13.229}^{322}\ {-70}^1$	$\{70, 69, 63, 16; 1, 7, 54, 70\}$
? $v = 1280 = 1 + 72 + 568 + 568 + 71$	$72^1\ 12^{284}\ 0^{710}\ {-12}^{284}\ {-72}^1$	$\{72, 71, 63, 9; 1, 9, 63, 72\}$
? $v = 1014 = 1 + 78 + 462 + 429 + 44$	$78^1\ 13^{198}\ 0^{616}\ {-13}^{198}\ {-78}^1$	$\{78, 77, 65, 8; 1, 13, 70, 78\}$
? $v = 1080 = 1 + 78 + 462 + 462 + 77$	$78^1\ 15.297^{154}\ 0^{770}\ {-15.297}^{154}\ {-78}^1$	$\{78, 77, 65, 13; 1, 13, 65, 78\}$
? $v = 2160 = 1 + 84 + 996 + 996 + 83$	$84^1\ 11.225^{664}\ 0^{830}\ {-11.225}^{664}\ {-84}^1$	$\{84, 83, 77, 7; 1, 7, 77, 84\}$
? $v = 3026 = 1 + 85 + 1428 + 1428 + 84$	$85^1\ 10.247^{1156}\ 0^{712}\ {-10.247}^{1156}\ {-85}^1$	$\{85, 84, 80, 5; 1, 5, 80, 85\}$
? $v = 1452 = 1 + 88 + 696 + 638 + 29$	$88^1\ 11^{464}\ 0^{522}\ {-11}^{464}\ {-88}^1$	$\{88, 87, 77, 4; 1, 11, 84, 88\}$
? $v = 1800 = 1 + 88 + 696 + 812 + 203$	$88^1\ 17.861^{224}\ 0^{1350}\ {-17.861}^{224}\ {-88}^1$	$\{88, 87, 77, 22; 1, 11, 66, 88\}$
? $v = 2816 = 1 + 88 + 957 + 1320 + 450$	$88^1\ 17.889^{363}\ 0^{2088}\ {-17.889}^{363}\ {-88}^1$	$\{88, 87, 80, 30; 1, 8, 58, 88\}$
? $v = 1352 = 1 + 91 + 630 + 585 + 45$	$91^1\ 13^{315}\ 0^{720}\ {-13}^{315}\ {-91}^1$	$\{91, 90, 78, 7; 1, 13, 84, 91\}$
? $v = 3302 = 1 + 91 + 1170 + 1560 + 480$	$91^1\ 16.733^{507}\ 0^{2286}\ {-16.733}^{507}\ {-91}^1$	$\{91, 90, 84, 28; 1, 7, 63, 91\}$
? $v = 1712 = 1 + 96 + 760 + 760 + 95$	$96^1\ 15.100^{320}\ 0^{1070}\ {-15.100}^{320}\ {-96}^1$	$\{96, 95, 84, 12; 1, 12, 84, 96\}$
? $v = 2472 = 1 + 96 + 1140 + 1140 + 95$	$96^1\ 12.329^{720}\ 0^{1030}\ {-12.329}^{720}\ {-96}^1$	$\{96, 95, 88, 8; 1, 8, 88, 96\}$
? $v = 1650 = 1 + 105 + 728 + 720 + 96$	$105^1\ 17.321^{252}\ 0^{1144}\ {-17.321}^{252}\ {-105}^1$	$\{105, 104, 90, 14; 1, 15, 91, 105\}$
? $v = 3570 = 1 + 105 + 1560 + 1680 + 224$	$105^1\ 14^{900}\ 0^{1768}\ {-14}^{900}\ {-105}^1$	$\{105, 104, 98, 14; 1, 7, 91, 105\}$
? $v = 3850 = 1 + 105 + 1560 + 1820 + 364$	$105^1\ 15.652^{780}\ 0^{2288}\ {-15.652}^{780}\ {-105}^1$	$\{105, 104, 98, 21; 1, 7, 84, 105\}$
? $v = 4032 = 1 + 105 + 1560 + 1911 + 455$	$105^1\ 16.523^{735}\ 0^{2560}\ {-16.523}^{735}\ {-105}^1$	$\{105, 104, 98, 25; 1, 7, 80, 105\}$
? $v = 2400 = 1 + 110 + 1090 + 1090 + 109$	$110^1\ 14.832^{545}\ 0^{1308}\ {-14.832}^{545}\ {-110}^1$	$\{110, 109, 99, 11; 1, 11, 99, 110\}$
? $v = 3332 = 1 + 112 + 1554 + 1554 + 111$	$112^1\ 12.961^{1036}\ 0^{1258}\ {-12.961}^{1036}\ {-112}^1$	$\{112, 111, 104, 8; 1, 8, 104, 112\}$
? $v = 2496 = 1 + 117 + 1044 + 1131 + 203$	$117^1\ 19.416^{351}\ 0^{1792}\ {-19.416}^{351}\ {-117}^1$	$\{117, 116, 104, 21; 1, 13, 96, 117\}$
? $v = 1600 = 1 + 120 + 714 + 680 + 85$	$120^1\ 20^{204}\ 0^{1190}\ {-20}^{204}\ {-120}^1$	$\{120, 119, 100, 15; 1, 20, 105, 120\}$
$q^3_{33} = 0$; $p^4_{44} = 0$; Q-polynomial		
? $v = 2688 = 1 + 120 + 1190 + 1224 + 153$	$120^1\ 16.971^{510}\ 0^{1666}\ {-16.971}^{510}\ {-120}^1$	$\{120, 119, 108, 15; 1, 12, 105, 120\}$
? $v = 3096 = 1 + 120 + 1428 + 1428 + 119$	$120^1\ 14.491^{816}\ 0^{1462}\ {-14.491}^{816}\ {-120}^1$	$\{120, 119, 110, 10; 1, 10, 110, 120\}$
? $v = 2352 = 1 + 126 + 1125 + 1050 + 50$	$126^1\ 14^{675}\ 0^{1000}\ {-14}^{675}\ {-126}^1$	$\{126, 125, 112, 6; 1, 14, 120, 126\}$
? $v = 2312 = 1 + 136 + 1080 + 1020 + 75$	$136^1\ 17^{480}\ 0^{1350}\ {-17}^{480}\ {-136}^1$	$\{136, 135, 119, 10; 1, 17, 126, 136\}$
? $v = 2432 = 1 + 136 + 1080 + 1080 + 135$	$136^1\ 20.199^{360}\ 0^{1710}\ {-20.199}^{360}\ {-136}^1$	$\{136, 135, 119, 17; 1, 17, 119, 136\}$
? $v = 2890 = 1 + 153 + 1368 + 1292 + 76$	$153^1\ 17^{684}\ 0^{1520}\ {-17}^{684}\ {-153}^1$	$\{153, 152, 136, 9; 1, 17, 144, 153\}$
? $v = 4032 = 1 + 156 + 1860 + 1860 + 155$	$156^1\ 17.664^{930}\ 0^{2170}\ {-17.664}^{930}\ {-156}^1$	$\{156, 155, 143, 13; 1, 13, 143, 156\}$
? $v = 3500 = 1 + 160 + 1590 + 1590 + 159$	$160^1\ 20^{636}\ 0^{2226}\ {-20}^{636}\ {-160}^1$	$\{160, 159, 144, 16; 1, 16, 144, 160\}$

? $v = 3720 = 1 + 170 + 1690 + 1690 + 169$ $\{170, 169, 153, 17; 1, 17, 153, 170\}$
$170^1\ 21.024^{650}\ 0^{2418}\ -21.024^{650}\ -170^1$
? $v = 3402 = 1 + 171 + 1530 + 1530 + 170$ $\{171, 170, 152, 19; 1, 19, 152, 171\}$
$171^1\ 22.650^{510}\ 0^{2380}\ -22.650^{510}\ -171^1$
? $v = 3174 = 1 + 184 + 1464 + 1403 + 122$ $184^1\ 23^{488}\ 0^{2196}\ -23^{488}\ -184^1$ $\{184, 183, 161, 16; 1, 23, 168, 184\}$
? $v = 2916 = 1 + 189 + 1316 + 1269 + 141$ $189^1\ 27^{329}\ 0^{2256}\ -27^{329}\ -189^1$ $\{189, 188, 162, 21; 1, 27, 168, 189\}$
$q^3_{33} = 0;\ p^4_{44} = 0;$ Q-polynomial
? $v = 3584 = 1 + 200 + 1592 + 1592 + 199$ $\{200, 199, 175, 25; 1, 25, 175, 200\}$
$200^1\ 28.284^{398}\ 0^{2786}\ -28.284^{398}\ -200^1$

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