EXPLICIT COMPUTATIONS OF INVARIANTS OF PLANE QUARTIC CURVES

ANDREAS-STEPHAN ELSENHANS

Abstract. We establish a complete set of invariants for ternary quartic forms. Further, we express four classical invariants in terms of these generators.

1. Introduction

Studing rings of invariants is classical (19th century) algebra. For example, it was known in those days that a ternery cubic has two invariants S and T of degrees 4 and 6, respectively. *Every invariant of the cubic can be expressed as a rational function of S and T* [10, p. 186]. Here, *rational* should be interpreted as being a polynomial.

The ring of invariants of ternary quartics is more complicated. First, one can show that the degree of each invariant is divisible by 3. In 1968, Shioda [11] conjectured that the ring of invariants of ternary quartic forms is generated by 13 invariants of degrees 3, 6, 9, 9, 12, 12, 15, 15, 18, 18, 21, 21, 27.

In 1987, Dixmier [4] proved that invariants of degree 3, 6, 9, 12, 15, 18, 27 form a complete system of primary invariants [3, Def. 2.4.6]. Further, he proved that at most 56 invariants suffice to generate the entire ring of invariants.

Using the Clebsch transfer principle and invariants of binary quartics, we describe an efficient algorithm to compute invariants of the degrees given above. As an application, we show that Shioda was right. Finally, we express four classical invariants in terms of the listed generators.

2. Invariants, covariants, and contravariants

The standard left action of $\mathrm{Gl}_n(\mathbb{C})$ on $\mathbb{C}^n$ induces a right action on $\mathbb{C}[X_1,\ldots,X_n]$ by $f^M := f(MX)$. Let $\mathbb{C}[X_1,\ldots,X_n]_d$ be the space of homogeneous polynomials of degree d. A polynomial mapping $I\colon \mathbb{C}[X_1,\ldots,X_n]_d \to \mathbb{C}$ is called an *invariant* if

$$I(f^M) = I(f)\det(M)^e$$

holds for some $e \in \mathbb{Z}$ and all $M \in \mathrm{Gl}_n(\mathbb{C}), f \in \mathbb{C}[X_1,\ldots,X_n]_d$. The *degree* of I is the degree of I as a polynomial in the coefficients of f.

A *covariant* c is a mapping $c\colon \mathbb{C}[X_1,\ldots,X_n]_{d_1} \to \mathbb{C}[X_1,\ldots,X_n]_{d_2}$ with

$$c(f^M) = c(f)^M \det(M)^e$$

for some $e \in \mathbb{Z}$ and all $M \in \mathrm{Gl}_n(\mathbb{C}), f \in \mathbb{C}[X_1,\ldots,X_n]_{d_1}$.

A *contravariant* C is a mapping $C\colon \mathbb{C}[X_1,\ldots,X_n]_{d_1} \to \mathbb{C}[X_1,\ldots,X_n]_{d_2}$ with

$$C(f^M) = C(f)^{(M^\top)^{-1}} \det(M)^e$$

for some $e \in \mathbb{Z}$ and all $M \in \mathrm{Gl}_n(\mathbb{C}), f \in \mathbb{C}[X_1,\ldots,X_n]_{d_1}$.

We call d_2 the *order* of the convariant (resp. contravariant). The *degree* of a covariant (resp. contravariant) is the degree of its coefficients viewed as polynomials in the coefficients of f.

Some examples are as follows:

- An invariant of a quadratic form in n variables is given by the determinant of its matrix. It is of degree n.
- More generally, the discriminant of a form of degree d in n variables is an invariant of degree $n(d-1)^{n-1}$.
- A covariant of a form is given by its Hessian $\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j}$. It has degree n and order $n(\deg(f)-2)$.

Remarks.

- Invariants are called invariants because they are invariant with respect to the action of $\mathrm{Sl}_n(\mathbb{C})$.
- The sets of all invariants, covariants, or contravariants are rings.
- The ring of covariants (resp. contravariants) is a module over the ring of invariants.
- By a theorem of Hilbert, the ring of all invariants is finitely generated.

3. Invariants for binary quartics

A classical way to write down invariants is the symbolic form. Nowadays, this seems to be almost completely forgotten. We refer to [9] and [10] for a detailed explanation and [12, Chap. VIII, Sec. 2] for a more recent treatment. A brief summary is given in [6, App. B].

Write a general binary quartic in the form $f(x,y) := ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4$. In degrees 2 and 3, it has the invariants $S_2 := 96(12ae - 3bd + c^2)$ and $T_2 := 192(72ace - 27ad^2 - 27b^2e + 9bcd - 2c^3)$.

In symbolic notation, these invariants are given by $(1\,2)^4$ and $(1\,2)^2(1\,3)^2(2\,3)^2$. Here $(i\,j)$ is an abbreviation for the differential operator

$$\det\begin{pmatrix} \frac{\partial}{\partial x_i} & \frac{\partial}{\partial x_j} \\ \frac{\partial}{\partial y_i} & \frac{\partial}{\partial y_j} \end{pmatrix}.$$

Applying the differential operator $(12)^4$ to $f(x_1,y_1)f(x_2,y_2)$ gives the invariant S_2. Writing T_2 in this way, we get

$$(1\,2)^2(1\,3)^2(2\,3)^2 f(x_1,y_1)f(x_2,y_2)f(x_3,y_3)\,.$$

It is a classical result that S_2 and T_2 generate the ring of all invariants of binary quartics [3, Ex. 2.1.2].

4. The Clebsch transfer

The Clebsch transfer principle can be used to construct contravariants of ternary quartics from invariants of binary quartics. For this, we denote by $(\overline{u}\,i\,j)$ the differential operator

$$\det\begin{pmatrix} u & \frac{\partial}{\partial x_i} & \frac{\partial}{\partial x_j} \\ v & \frac{\partial}{\partial y_i} & \frac{\partial}{\partial y_j} \\ w & \frac{\partial}{\partial z_i} & \frac{\partial}{\partial z_j} \end{pmatrix}.$$

Then the contravariant $S_3(f) \in \mathbb{C}[u,v,w]$ of the ternary quartic $f(x,y,z)$ is given by $(\overline{u}\,1\,2)^4 f(x_1,y_1,z_1)f(x_2,y_2,z_2)$. It is of degree 2 and order 4.

Analogously, the contravariant $T_3(f)$ (degree 3, order 6) is given by

$$(\overline{u}\,1\,2)^2(\overline{u}\,1\,3)^2(\overline{u}\,2\,3)^2 f(x_1,y_1,z_1)f(x_2,y_2,z_2)f(x_3,y_3,z_3)\,.$$

Interpretation. An interpretation of these contravariants is as follows. The zero set $V(S_3(f))$ is a subset of $(\mathbf{P}^2)^\vee$. Recall that $(\mathbf{P}^2)^\vee$ is the space of lines in $\mathbf{P}^2$. As a line $\ell \subset \mathbf{P}^2$ can be identified with $\mathbf{P}^1$, it leads to a binary quartic $f|_\ell$. Now we get

$$\ell \in V(S_3(f)) \Longleftrightarrow S_2(f|_\ell) = 0\,.$$

I.e., the contravariant S_3 describes all lines that lead to a binary quartic with S_2 equal to zero. Analogously for T.

Further, the discriminant of a binary quartic is given by $S_2^3 - 6T_2^2$. Thus, $S_3^3 - 6T_3^2 = 0$ gives us all the lines that lead to singular binary quartics. For a smooth quartic, this is just the dual curve.

Algorithmic aspects. One can compute a contravariant directly from its definition. In the case of a ternary quartic, this performs well if one writes the differential operator in its Horner representation. This means, we write $(\overline{u}\,1\,2)$ as

$$(w\frac{\partial}{\partial y_2} - v\frac{\partial}{\partial z_2})\frac{\partial}{\partial x_1} + (u\frac{\partial}{\partial z_2} - w\frac{\partial}{\partial x_2})\frac{\partial}{\partial y_1} + (v\frac{\partial}{\partial x_2} - u\frac{\partial}{\partial y_2})\frac{\partial}{\partial z_1}\,.$$

Further, we multiply out as late as possible. Thus, $T_3(f)$ becomes

$$(\overline{u}\,1\,2)^2(\overline{u}\,1\,3)^2 f(x_1,y_1,z_1)(\overline{u}\,2\,3)^2 f(x_2,y_2,z_2)f(x_3,y_3,z_3)\,.$$

However, there is a second method to compute these contravariants. It starts with a numerical version of the Clebsch transfer. For this, we denote by φ_n the isomorphism

$$\begin{aligned} \varphi_n\colon \Lambda^{n-1}K^n &\to (K^n)^\vee, \\ v_1 \wedge \ldots \wedge v_{n-1} &\mapsto (x \mapsto \det(x, v_1, \ldots, v_{n-1}))\,. \end{aligned}$$

Then we get

$$T_3(f)(u,v,w) = T_2(f(a_1x + b_1y, a_2x + b_2y, a_3x + b_3y))\,.$$

Here, the right hand side is the invariant T_2 of the binary quartic, given by restriction of f to the line spanned by a and b. This line is given via φ_3^{-1} by $a \wedge b = \varphi_3^{-1}(u,v,w)$.

Using this, we can numerically evaluate the contravariant. Doing this at sufficiently many points in $(\mathbf{P}^3)^\vee$, one can reconstruct the contravariant by interpolation.

In our implementation for ternary quartics, the interpolation method is slightly faster. A large part of the running time is used for the computation of φ_3^{-1}. When we switch to forms in more variables, the advantage of the interpolation method becomes clearer.

One can easily speed up the interpolation method by storing the used values of φ_3^{-1} in a table. Further, the reconstruction of the contravariant as a polynomial from its numerical values is done by solving a linear system of equations. As the points in $(\mathbf{P}^3)^\vee$ we work with are a priori known, the coefficient matrix of the linear system is fixed. Thus, one could precompute the inverse of this matrix to speed up the interpolation step.

5. Invariants of ternary quartics

A system of invariants for ternary quartics is given in [4], together with unpublished work of T. Ohno [8]. Here, we will describe another system of invariants, which can easily be evaluated. But before we can do this, we need the action of contravariants on covariants.

Covariant and contravariant operation. The polynomial ring $\mathbb{C}[X_1, \ldots, X_n]$ and the ring of differential operators $\mathbb{C}[\frac{\partial}{\partial U_1}, \ldots, \frac{\partial}{\partial U_n}]$ are isomorphic. We fix the isomorphism D given by $X_i \mapsto \frac{\partial}{\partial U_i}$. Following [6, App. B], we denote $D(f_1)f_2$ by $f_1 \vdash f_2$, for $f_1 \in \mathbb{C}[X_1, \ldots, X_n]$ and $f_2 \in \mathbb{C}[U_1, \ldots, U_n]$.

Lemma. Let C be a contravariant and c be a covariant. In case the order of c is larger than the order of C, we get a new covariant $f \mapsto C(f) \vdash c(f)$. In case the orders coincide, this results in an invariant. In case the order of c is smaller than the order of C, we get the contravariant $f \mapsto c(f) \vdash C(f)$.

A list of invariants. Producing invariants for a form $f \in \mathbb{C}[X_1, X_2, X_3]_4$ is now somehow a random process. One multiplies known covariants and applies contravariants to covariants and vice versa. However, one can try to keep the order small such that they will be represented by a small number of terms. Further, it might happen that some expressions degenerate to zero. We ended up with the following:

$C_{2,0,4} := \frac{1}{96} S_3(f)$, $C_{3,0,6} := \frac{1}{192} T_3(f)$, $c_{3,6} := \det \left(\frac{\partial^2 f(x_1,x_2,x_3)}{\partial x_i \partial x_j} \right)$
$C_{4,0,2} := f \vdash C_{3,0,6}$, $c_{4,4} := C_{2,0,4} \vdash f^2$, $c_{5,2} := C_{4,0,2} \vdash f$
$c_{7,4} := C_{2,0,4} \vdash (c_{4,4} \cdot f)$, $C_{7,0,2} := c_{5,2} \vdash C_{2,0,4}$, $C_{8,0,4} := c_{5,2} \vdash C_{3,0,6}$
$c_{11,2} := C_{8,0,4} \vdash c_{3,6}$, $C_{13,0,2} := c_{5,2} \vdash C_{8,0,4}$, $c_{14,2} := C_{7,0,2} \vdash c_{7,4}$
$c_{17,2} := C_{13,0,2} \vdash c_{4,4}$, $c_{20,2} := C_{13,0,2} \vdash c_{7,4}$

$I_3 := C_{2,0,4} \vdash f$, $I_6 := C_{2,0,4} \vdash c_{4,4}$, $I_{9a} := C_{4,0,2} \vdash c_{5,2}$
$I_{9b} := C_{2,0,4} \vdash c_{7,4}$, $I_{12a} := C_{8,0,4} \vdash c_{4,4}$, $I_{12b} := C_{7,0,2} \vdash c_{5,2}$
$I_{15a} := C_{8,0,4} \vdash c_{7,4}$, $I_{15b} := C_{4,0,2} \vdash c_{11,2}$, $I_{18a} := C_{7,0,2} \vdash c_{11,2}$
$I_{18b} := C_{13,0,2} \vdash c_{5,2}$, $I_{21a} := C_{7,0,2} \vdash c_{14,2}$, $I_{21b} := C_{4,0,2} \vdash c_{17,2}$
$I_{27} := C_{7,0,2} \vdash c_{20,2}$

6. Completeness of the system of invariants

Following [4], a complete system of primary invariants is given by invariants of degree 3, 6, 9, 12, 15, 18, 27. Further, Shioda [11] computed the Poincare series of the ring of invariants as

$$\sum_{i=0}^{\infty} \dim(V_i) t^i = \frac{P(t)}{(1 - T^{27}) \cdot \prod_{i=1}^{6} (1 - T^{3i})}$$

with

$$\begin{aligned} P(t) \quad = \quad & 1 + T^9 + T^{12} + T^{15} + 2T^{18} + 3T^{21} + 2T^{24} + 3T^{27} \\ & + 4T^{30} + 3T^{33} + 4T^{36} + 4T^{39} + 3T^{42} + 4T^{45} + 3T^{48} \\ & + 2T^{51} + 3T^{54} + 2T^{57} + T^{60} + T^{63} + T^{66} + T^{75} . \end{aligned}$$

Here, V_i denotes the vector space of degree i invariants of ternary quartics. We get $\dim(V_i) = 1, 1, 2, 4, 7, 11, 19, 29, 44, 67, 98, 139, 199, 275, 375, 509, 678, 890,$

1165, 1501, 1916, 2431, 3054, 3802, 4713, 5791, 7068, 8587, 10364, 12434, 14861, 17660, 20886, 24611 for $i = 0, 3, 6, 9, \ldots, 99$.

Further, using [3, Formula 3.5.1] one can read off the Poincare series that the invariants of degree at most 75 generate the ring of invariants. The degree bound [3, Corollary 4.7.7] for a generating system of invariants evaluates to 83 in this case.

Experiment. Using MAGMA [2], one can compute lower bounds for the dimension of the space of degree d invariants generated by the listed invariants. This is done as follows:

i) Choose a random list of quartics being longer than the expected dimension.

ii) Compute the invariants listed above for each quartic.

iii) Compute a spanning system (as a vector space) of all invariants of degree d generated by the known ones. For example, choose all degree d monomials in a rank 13 polynomial ring with degree weights 3, 6, 9, 9, 12, 12, 15, 15, 18, 18, 21, 21, and 27.

iv) Compute a matrix M of values of invariants containing one row for each quartic and one column for each vector space generator.

v) Reduce this matrix M modulo p (e.g. $p = 101$).

vi) Compute the rank of the matrix.

Remark. As we chose only a finite number of quartics and reduced the matrix M modulo p, it is not clear that we get the exact dimension of the space of invariants spanned. However, we still get a lower bound.

Result. As the lower bounds found coincide with the upper bound given by the coefficients of the Poincare series, the listed invariants in fact generate all invariants up to degree 83. At this place, the degree bound above shows that Shioda was right. This was first discovered by T. Ohno [8].

7. Classical invariants

Knowing explicit generators of the ring of invariants, we can ask for a representation of classical invariants in terms of these generators. A classical invariant is an invariant that has a geometric interpretation. Most of them were described in the 19th century. It is a somehow surprising fact that the degrees of all these invariants were known, but explicit formulas were given only rarely.

Four classical invariants. Here, we will focus on the following classical invariants:

i) The *catalecticant invariant* is an invariant of degree 6. It vanishes iff the quartic can be expressed as a sum of five 4th powers of linear forms. Such quartics are called *Clebsch quartics.*

ii) The *discriminant* is an invariant of degree 27. It vanishes iff the quartic is singular.

iii) The *Lüroth invariant* is an invariant of degree 54. It vanishes iff the quartic can be written in the form $l_1 l_2 l_3 l_4 + a l_1 l_2 l_3 l_5 + b l_1 l_2 l_4 l_5 + c l_1 l_3 l_4 l_5 + d l_2 l_3 l_4 l_5$. Here, the l_i are linear forms. Such quartics are called *Lüroth quartics.*

iv) The *Salmon invariant* is an invariant of degree 60. It vanishes iff the quartic has a flex bitangent.

The Lüroth invariant was recently constructed explicitly using several hours of CPU time [1]. The general approach for such an explicit construction is similar to our first experiment. The first step is to analyze the vector space of all degree d invariants. We describe the construction by two algorithms.

Algorithm. Given a generating set of the ring of invariants and a degree d, this algorithm computes a basis of the vector space of all degree d invariants.

i) Generate all monomials of weighted degree d in the given generators.

ii) Generate a sufficiently large list of randomly chosen quartics.

iii) Compute the invariants of these quartics.

iv) Build up the matrix M with one row per quartic and one column per monomial as above.

v) Compute the row echelon form of M modulo 101.

vi) Check that the row-space has the dimension predicted by the Poincare series. Otherwise, choose more quartics and redo the computation.

vii) Select all the monomials listed in step i) that correspond to the leading coefficients of the row echelon form of M. Return these monomials as a vector space basis of all degree d invariants.

Using this as a subalgorithm we can construct the invariant explicitly as follows.

Algorithm. Given the degree d of an invariant and its geometric interpretation, this algorithm computes an expression of the invariant in terms of given generators.

i) Compute a vector space basis B of all degree d invariants with the algorithm above.

ii) Compute a list of special quartics with $\mathbb{Q}$-coefficients. The invariant, we search for, should vanish for all these quartics. In our cases, this means to write down a list of random Clebsch quartics, or quartics being singular in $[1 : 0 : 0]$, or Lüroth quartics, or quartics meeting the line $z = 0$ only in $[1 : 0 : 0]$.

iii) Evaluate the generators of the invariant ring for these quartics.

iv) Compute the matrix M consisting of one row for each invariant in B and one column for each special quartic.

v) Compute the kernel of M. In the case it is not 1-dimensional, redo the computation with more quartics.

vi) Compute a generator v of the kernel.

vii) Return $I := \sum_{i=1}^{\#B} v_i B_i$ as a representation of the special invariant.

Results. The algorithm gives us the expression $3I_6 - 74I_3^2$ for the catalecticant invariant. All the other formulas are not suitable for printing as they involve many terms and large coefficients. They are available on the author's web page `http://www.staff.uni-bayreuth.de/~bt270951/`.

As the Salmon invariant is the one of the largest degree, its reconstruction is the most complex one. The timing results are as follows.

i) The computation of all invariants lasted 70 seconds.

ii) The computation of the row echelon form of B (linear algebra over $\mathbb{Z}/101\mathbb{Z}$) took 10 seconds.

iii) The computation of the kernel vector v (linear algebra over $\mathbb{Q}$) was done in 58 seconds.

The computations for the other invariants were considerably faster.

Remark. Knowing the geometric interpretation of the catalecticant invariant, one can improve the experiments. We describe this in detail for the check of the completeness of the system of invariants, as this experiment is the largest one in terms of CPU and memory usage.

The first step is to replace the invariant I_6 by the catalecticant. Further, one chooses a certain proportion of the quartics as Clebsch quartics.

This results in a block-structure for the matrix M. The upper left block consists of invariants of Clebsch quartics not involving the catalecticant invariant. The upper right block is zero. The lower left block consists of invariants of general quartics not involving the catalecticant. The lower right block contains all invariants of general quartics that contain the catalecticant as a factor.

As the upper right block of M is zero, a lower bound for the rank is given by the sum of the ranks of the upper left and the lower right blocks. This leads to a check of the completeness of the ring of invariants up to degree 81 using 675 seconds of CPU time and 486 MB of memory. The largest inspected matrix contained approx. $52.5 \cdot 10^6$ entries in $\mathbb{Z}/101\mathbb{Z}$. The computations were done with MAGMA 2.19 on one core of a 3 GHz Intel Core 2 X9650 processor.

8. Comparision with other systems of invariants

The package `Echidna` [7] of David Kohel can compute the Dixmier-Ohno invariants of plane quartic curves. Similarly to the method above, these invariants can easily be related to our system. However, the expressions become lengthly. We use small letters to denote the Dixmier-Ohne invariants. The expressions for the first six are

$$\begin{aligned}
i_3 &= \frac{1}{6} I_3 \qquad i_6 = -\frac{37}{36} I_3^2 + \frac{1}{24} I_6 \qquad i_9 = \frac{1}{3456} I_{9a} \\
j_9 &= \frac{1295}{1944} I_3^3 - \frac{325}{5184} I_3 I_6 + \frac{1}{31104} I_{9a} + \frac{5}{3456} I_{9b} \\
i_{12} &= \frac{1295}{5184} I_3^4 - \frac{325}{13824} I_3^2 I_6 - \frac{11}{539136} I_3 I_{9a} + \frac{5}{9216} I_3 I_{9b} - \frac{1}{359424} I_{12a} \\
j_{12} &= + \frac{1}{22464} I_{12b} - \frac{205}{404352} I_3 I_{9a} + \frac{5}{269568} I_{12a} + \frac{11}{269568} I_{12b} \,.
\end{aligned}$$

The formulas for the other invariants are available online on the author's web page `http://www.staff.uni-bayreuth.de/~bt270951`.

References

[1] Basson, R., Lercier, R., Ritzenthaler, C., and Sijsling, J.: *An explicit expression of the Lüroth invariant*, arXiv:1211.1327v1 (2012)

[2] Bosma, W., Cannon, J., and Playoust, C.: *The Magma algebra system. I. The user language.* J. Symbolic Comput. **24** (1997), 235 – 265.

[3] Derksen, H. and Kemper, G.: *Computational Invariant Theory.* Springer (2002)

[4] Dixmier, J.: *On the projective invariants of quartic plane curves.* Adv. in Math. **64** (1987), no. 3, 279 – 304

[5] Dolgachev, I.: *Classical Algebraic Geometry: a modern view.* Cambride Univ. Press (2012)

[6] Hunt, B.: *The geometry of some special arithmetic quotients.* Springer (1996)

[7] Kohel, D.: *Algorithms for Elliptic Curves and Higher Dimensional Analogues* `http://echidna.maths.usyd.edu.au/ kohel/alg/index.html`

[8] Ohno, T.: *The graded ring of invariants of ternary quartics I*, unpublished
[9] Salmon, G.: *Lessons introductory to the modern higher algebra.* Hodges, Figgis, and Co. (1885)
[10] Salmon, G.: *A treatise on the higher plane curves.* Hodges-Foster (1873)
[11] Shioda, T.: *On the graded ring of invariants of binary octavics.* Amer. J. Math. **89** (1967) 1022 – 1046.
[12] Weyl, H.: *The classical groups.* Princeton Univ. Press (1973)