Counting symmetric nilpotent matrices

Andries E. Brouwer

Counting

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N is *nilpotent* when $N^e = 0$ for some $e \geq 0$.
e is called the *exponent* of N.

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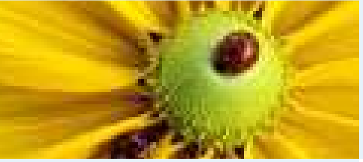
This is easy but nontrivial. Proofs by Hall (1955), Fine & Herstein (1958), Gerstenhaber (1961), Crabb (2006), Gow & Sheekey (2011), Blokhuis (2011).



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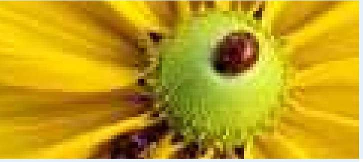
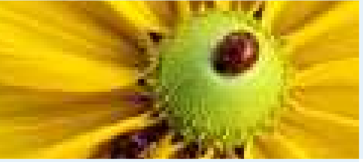
How many symmetric nilpotent matrices?



Small n

Count symmetric nilpotent matrices of order n

$n = 0$: 1 (exponent 0), namely $()$



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$n = 2$:

Look at $\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$.

All eigenvalues are 0, so trace is 0, so $c = -a$.

Determinant is 0, so $a^2 + b^2 = 0$.

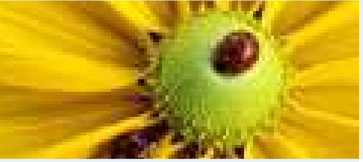
How many solutions?

q even: q

$q \equiv 1 \pmod 4$: $1 + 2(q-1) = 2q - 1$

$q \equiv 3 \pmod 4$: 1

Messy



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- $q \equiv 3 \pmod 4$: 1

$n = 3$: $1 + (q^2 - 1) + (q^3 - q) = q^3 + q^2 - q$

Exercise

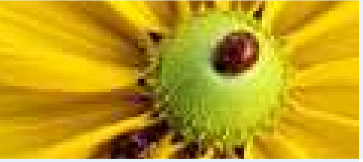
Sometimes we find a polynomial in q.

Self-adjoint matrices

A matrix N defines a linear map $N : V \to V$ and it makes sense to talk about N^e.
What does it mean that $N = N^\top$?

Self-adjoint matrices

Let $g : V \times V \to F$ be a nondegenerate symmetric bilinear form. N is called *self-adjoint* w.r.t. g when $g(x, Ny) = g(Nx, y)$ for all x, y.



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The *standard form* is the one given by the identity matrix: $g(x, y) = x^\top y = \sum x_i y_i$.

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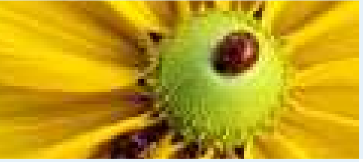
$N = N^\top$ iff N is self-adjoint for the standard form.

Symmetric bilinear forms

So, it seems we should be counting self-adjoint matrices w.r.t. a given nondegenerate symmetric bilinear form. How many nonequivalent forms are there?



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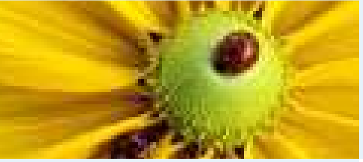
When n is odd, all forms are equivalent to the standard form.



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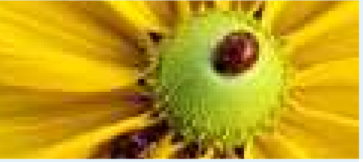

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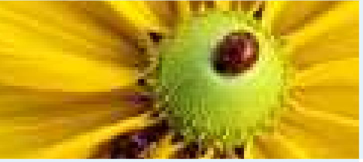
When n is even, there are two nonequivalent types. (Assuming $n > 0$.)



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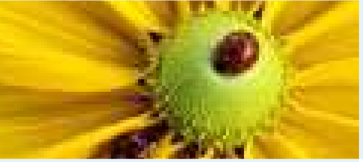
When n is even, there are two nonequivalent types.

q odd: the elliptic and hyperbolic forms.
q even: the standard and symplectic forms.



The standard form

When n and q are even, one has the standard and symplectic forms. How can one distinguish them?

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A form g is *symplectic* iff $g(x, x) = 0$ for all x.

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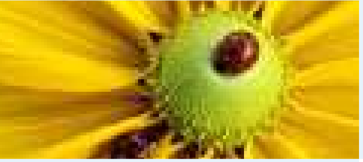
That is, iff G has zero diagonal.
(For $n = 0$ the standard form is symplectic.)



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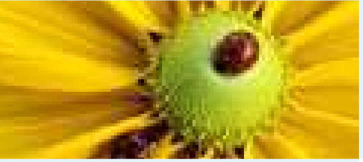
When n is even and q is odd, one has the elliptic and hyperbolic forms. The form is hyperbolic when $(-1)^{n/2} \det G$ is a square.



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The standard form is hyperbolic if $(-1)^{n/2}$ is a square, and elliptic otherwise.

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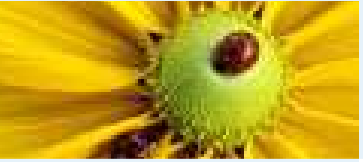
The standard form is hyperbolic if $(-1)^{n/2}$ is a square, and elliptic otherwise. (For $n = 0$ there is no elliptic form.)



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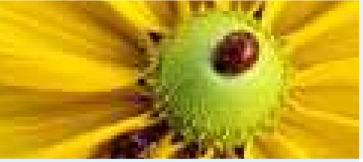
The standard form is hyperbolic if $(-1)^{n/2}$ is a square, and elliptic otherwise. So it is hyperbolic, unless $n \equiv 2 \pmod 4$ and $q \equiv 3 \pmod 4$.

$n = 2$ revisited

For $n = 2$ we now find for the number of nilpotent self-adjoint matrices:

- q even:
  - g standard: q
  - g symplectic: q^2



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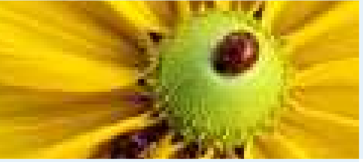
  Look at $N = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.
  Here $G = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and $GN = (GN)^\top$ yields $\left(\begin{smallmatrix} c & d \\ a & b \end{smallmatrix}\right) = \left(\begin{smallmatrix} c & a \\ d & b \end{smallmatrix}\right)$, so that $a = d$.
  The trace is 0. Determinant is 0, so $a^2 = bc$.
  Now b and c can be chosen freely.



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  $N = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ has $a = d$ and $a^2 = bc$.
  (More generally, for backdiagonal G, the self-adjoint N are those that are symmetric w.r.t. the back diagonal.)

$n = 2$ revisited

For $n = 2$ we now find for the number of nilpotent self-adjoint matrices:

q even:
- g standard: q
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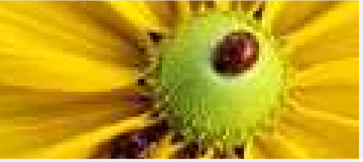
q odd:
- g elliptic: 1
- g hyperbolic: $2q - 1$

Note that q is the average of 1 and $2q - 1$.

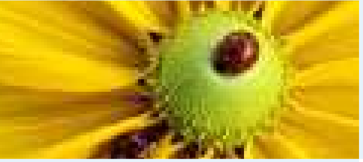


Symplectic space

Consider a vector space V of dimension $n = 2m$ provided with a nondegenerate symplectic form g.

Theorem (Steinberg (1968), Springer (1980).)
The Lie algebra $\mathfrak{sp}_{2m}$ *has* q^{2m^2} *nilpotent elements.*

A matrix A belongs to $Sp(2m)$ when it preserves the form, i.e., when $g(Ax, Ay) = g(x, y)$ for all x, y. Write $A = I + \epsilon X$, where $\epsilon^2 = 0$, to see that this means $g(x, Xy) + g(Xx, y) = 0$. For q even this says that X is self-adjoint.



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Counts by rank
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Corollary *If* q *is even, there are* q^{2m^2} *nilpotent matrices of order* $2m$ *that are self-adjoint for a given nondegenerate symplectic form* g*.*

This explains the q^2 that we got for $n = 2$.

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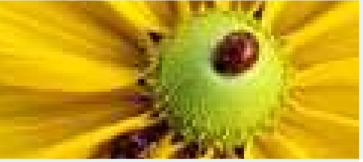
Exercise: give a direct geometric proof.

Steinberg

Steinberg (1968) shows for unipotent elements in algebraic groups, and Springer (1980) for nilpotent elements in the corresponding Lie algebras, that there are q^N of them, where $N = |\Phi|$ is the number of roots of the root system.

The proof uses the Steinberg character and modular representation theory.



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For A_{n-1}, that is, $GL(n)$, we have $|\Phi| = n(n-1)$, and we see again that there are $q^{n(n-1)}$ nilpotent matrices.

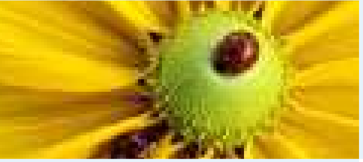
For C_m, that is, $Sp(2m)$, we have $|\Phi| = 2m^2$. If q is even, there are q^{2m^2} nilpotent back-symmetric matrices of order $2m$.

Skew-symmetric nilpotent matrices

N is skew-symmetric when N has zero diagonal and $N = -N^\top$.

For D_m, that is, $O^+(2m)$, we have $|\Phi| = 2m(m-1)$. There are $q^{2m(m-1)}$ skew-symmetric nilpotent matrices of order $2m$.

For B_n, that is, $O(2m+1)$, we have $|\Phi| = 2m^2$. There are q^{2m^2} skew-symmetric matrices of order $2m+1$.

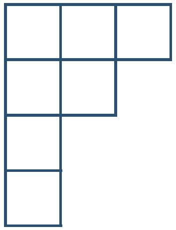


Young diagrams

The Jordan normal form N of a nilpotent matrix of order n has zeros on the main diagonal, and zeros and ones on the diagonal just above it. This leads to a block partition of the matrix, and to a partition of n.

Partitions are represented by Young diagrams Y.

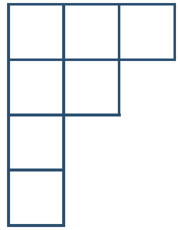
$$N = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \qquad Y =$$



.

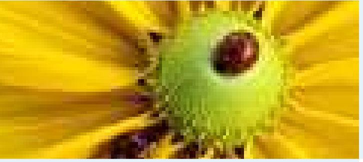
$$e_3 \mapsto e_2 \mapsto e_1 \mapsto 0, \; e_5 \mapsto e_4 \mapsto 0, \; e_6 \mapsto 0, \; e_7 \mapsto 0.$$

Young diagrams

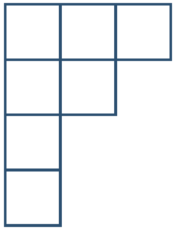
$Y =$  .

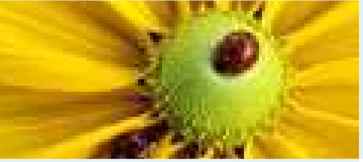
$$e_3 \mapsto e_2 \mapsto e_1 \mapsto 0,\ e_5 \mapsto e_4 \mapsto 0,\ e_6 \mapsto 0,\ e_7 \mapsto 0.$$

The map N determines a unique Y. The number of rows is $\dim\ker N$. The number of columns is the exponent of N. There is a square in row i column j if $\dim\ker N \cap \mathrm{im} N^{j-1} \geq i$.



Young diagrams (2)

Consider the Gram matrix $G = (g(u_i, u_j))_{ij}$ of ‘inner products’ of basis vectors belonging to the Young diagram $Y =$ , with the u_i identified with the squares of the diagram. If u_i has more squares to its right than u_j to its left, then $g(u_i, u_j) = g(N^a u_h, u_j) = g(u_h, N^a u_j) = 0$.



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If u_i has more squares to its right than u_j to its left, then $g(u_i, u_j) = 0$.

$$Y = \begin{array}{|c|c|c|} \hline 1 & 5 & 7 \\ \hline 2 & 6 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{2}{c}{} \\ \cline{1-1} 4 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \qquad \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

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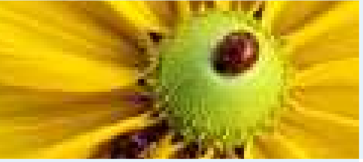
If u_i has more squares to its right than u_j to its left, then $g(u_i, u_j) = 0$.

$$Y = \begin{array}{|c|c|c|}\hline 1 & 5 & 7 \\ \hline 2 & 6 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{2}{c}{} \\ \cline{1-1} 4 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \qquad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & * & . \\ 0 & 0 & a & b & 0 & . & . \\ 0 & 0 & b & c & 0 & . & . \\ 0 & 0 & 0 & 0 & * & . & . \\ 0 & * & . & . & . & . & . \\ * & . & . & . & . & . & . \end{bmatrix}$$



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Get a transversal of nonsingular symmetric subblocks: for each group R of r rows of length s, get an $r \times r$ subblock with rows indexed by Y_{hi} and columns by $Y_{h,s+1-i}$ $(h \in R)$ for each i, $1 \leq i \leq s$. Different i give the same block.

Young diagrams (2)

Another example:

$$Y = \begin{array}{|c|c|c|} \hline 1 & 4 & 7 \\ \hline 2 & 5 \\ \cline{1-2} 3 & 6 \\ \cline{1-2} \end{array} \qquad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 & a & b & . \\ 0 & 0 & 0 & 0 & b & c & . \\ 0 & 0 & 0 & d & . & . & . \\ 0 & a & b & . & . & . & . \\ 0 & b & c & . & . & . & . \\ d & . & . & . & . & . & . \end{bmatrix}$$

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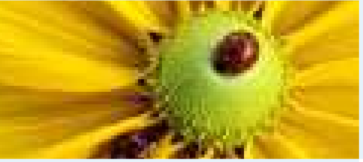

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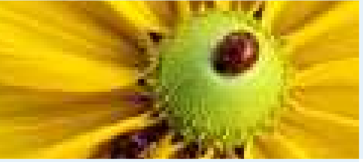
Diagrams Y describe conjugacy classes of unipotent matrices. (Or, orbits of nilpotent matrices under conjugation.)

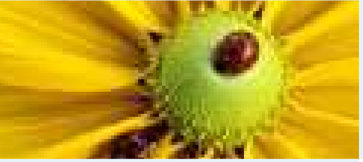


Young diagrams (2)

Another example:

$$Y = \begin{array}{|c|c|c|}\hline 1 & 4 & 7 \\ \hline 2 & 5 & \\ \cline{1-2} 3 & 6 & \\ \cline{1-2} \end{array} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 & a & b & . \\ 0 & 0 & 0 & 0 & b & c & . \\ 0 & 0 & 0 & d & . & . & . \\ 0 & a & b & . & . & . & . \\ 0 & b & c & . & . & . & . \\ d & . & . & . & . & . & . \end{bmatrix}$$

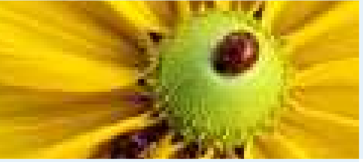
If the form is symplectic, then the Gram matrix has zero diagonal. This means that each odd part of the partition has even multiplicity.



Results

We want to count the number of self-adjoint nilpotent matrices in six cases: for odd q there are the elliptic, hyperbolic, and parabolic forms, for even q the symplectic and standard forms. Let us call these counts $e(2m)$, $h(2m)$, $p(2m+1)$, $z(2m)$, $s(2m)$, $s(2m+1)$.

Theorem *All of* $e(2m)$, $h(2m)$, $p(2m+1)$, $z(2m)$, $s(2m)$, $s(2m+1)$ *are polynomials in* q.



Results

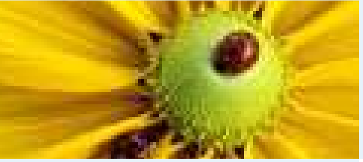
So far we learned one value: $z(2m) = q^{2m^2}$.

Theorem $p(2m+1) = s(2m+1)$.

Put $a(2m) = (h(2m) + e(2m))/2$
and $d(2m) = (h(2m) - e(2m))/2$.

Theorem $a(2m) = s(2m)$.

In both cases, the equality is one of polynomials.



Results

Theorem $p(2m+1) = q^{2m}a(2m) + q^{m}d(2m)$.

Theorem $p(2m+1) = (q^{2m}-1)a(2m) + z(2m)$.

Theorem $a(2m) = q^{2m-1}p(2m-1)$.

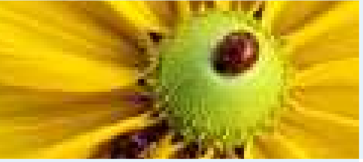
These settle all values recursively.

Counts via Fitting

Consider V, with nondegenerate symmetric bilinear form g. The number of self-adjoint M is $q^{n(n+1)/2}$.

The map $M : V \to V$ determines a unique *Fitting decomposition* $V = U \oplus W$ of V, where M is nilpotent on U and invertible on W.

If $u \in U$, $w \in W$, then $w = M^i w_i$ for a $w_i \in W$, and $g(u, w) = g(u, M^i w_i) = g(M^i u, w_i) = 0$ for large i. So $V = U \perp W$, and $U = W^\perp$, $W = U^\perp$, so that U and W are nondegenerate, and determine each other.



Counts via Fitting

Let $N(U)$ be the number of nilpotent self-adjoint maps on U (provided with the restriction of g to U), and let $S(W)$ be the number of invertible self-adjoint maps on W. We proved: $q^{n(n+1)/2} = \sum_U N(U)S(U^\perp)$, where the sum is over all nondegenerate subspaces U of V.

By induction one finds $N(V)$.

Counts by Y

One finds explicit formulas for the number of nilpotent maps of given type that have a given Young diagram Y by counting pairs (N, g).

E.g., for $n = 2m + 1$, $N_s(Y) = N(Y)g_s(Y)/g_s$

$N_s(Y)$: # symmetric nilpotent maps of shape Y

$N(Y)$: total # nilpotent maps of shape Y

g_s: total # nondegenerate symmetric bilinear forms (on V, where $\dim V = n$)

$g_s(Y)$: # such forms for which a given N of shape Y is self-adjoint.

Counts by Y

Each of $N(Y)$, g_s, $g_s(Y)$ is easy to compute. (For $g_s(Y)$ one uses the transversal of nonsingular blocks.)

This means that all counts are known as a sum $\sum_Y N_s(Y)$ over Young diagrams. Good for checking small values. Good for proving theorems.

Counts by rank

We have precise conjectures, but few proofs. However, there are recurrences, so all that is missing is algebraic manipulation.

The recurrences allow one to compute all counts for much larger n than is possible with the sums over Y.

Counts by rank

Let $p(2m+1,r)$, $h(2m,r)$, $e(2m,r)$ count selfadjoint nilpotent matrices of rank r (for odd q). Define $a(2m,r)$, $d(2m,r)$ as before.

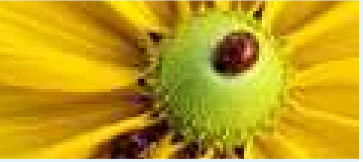
Conjectures

(i) $p(2m+1,2s+1) = (q^{2m-2s}-1)p(2m+1,2s)$.
(ii) $a(2m,2s+1) = (q^{2m-2s-1}-1)a(2m,2s)$.
(iii) $d(2m,2s) = (q^{2m-2s}-1)d(2m,2s-1)$.
(iv) $(q^{2m-r}-1)p(2m+1,r) = (q^{2m}-1)a(2m,r)$.
(v)

$$p(2m+1,2s) = q^{s(s+1)} \prod_{i=0}^{s-1} (q^{2m-2i}-1) \cdot \sum_{i=0}^{s} q^{(s-i)(2m-2s-1)} \begin{bmatrix} m-s-1+i \\ i \end{bmatrix}_{q^2}.$$

$$d(2m,2s+1) = (q-1)q^{m+s(s+1)-1} \prod_{i=1}^{s} (q^{2m-2i}-1) \cdot \sum_{i=0}^{s} q^{(s-i)(2m-2s-3)} \begin{bmatrix} m-s-1+i \\ i \end{bmatrix}_{q^2}.$$

There are similar conjectures for even q.



Counts by rank

Recursions:

Proposition

(i) $(q^{2m+1-r}-1)p(2m+1,r) =$
$(q^{2m}-1)p_0(2m+1,r) + q^{2m}(q-1)a(2m,r) + q^m(q-1)d(2m,r).$

(ii) $(q^{2m-r}-1)a(2m,r) =$
$(q^{2m-1}-1)a_0(2m,r) + q^{m-1}(q-1)d_0(2m,r) + q^{2m-1}(q-1)p(2m-1,r).$

(iii) $(q^{2m-r}-1)d(2m,r) =$
$(q^{2m-1}-1)d_0(2m,r) + q^{m-1}(q-1)a_0(2m,r) - q^{m-1}(q-1)p(2m-1,r).$

And for f any of p, h, e, a, d:

(iv)
$f_0(n,r) = q^r f(n-2,r) + (q-1)q^{r-1} f(n-2,r-1) + (q^{n-r}-1)q^{r-1} f(n-2,r-2).$

Here $f(n,r) = f_0(n,r) = 0$ for $r < 0$ or $r > n$ or $r = n > 0$. As start of the recursion only $h(0,0) = 1$ is needed.

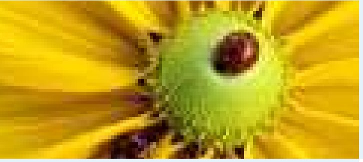
Counts by exponent

Let now $N_s(n, e)$ be the number of N with exponent e.
There is information on the case with large e.

Proposition *For odd n we have* $N_s(n + 2, n + 2) = q^n(q^{n+1} - 1)N_s(n, n)$.

This is $N_s(Y)$, $Y = \square\square\square\square\square\square\square$.

Proposition *For n odd, $n > 2i$, the ratio $N_s(n, n - i)/N_s(n, n)$ is independent of n.*



Proofs

Theorem *All counts are polynomials in q.*

Proof The sums over Y are rational functions of q that are integral for all q. $\square$

Theorem $p(2m+1) = s(2m+1)$.

Proof Write both counts as sums over Y. The parity of q never plays a role. $\square$

Proofs

Theorem $a(2m) = s(2m)$.

Proof Write as sums over Y and show termwise equality. Reduce to $g_h(Y) - g_e(Y) = q^m g_z(Y)$. Look at the block structure of a form g. Off-diagonal blocks contribute $\pm$ a square to $\det G$ and do not influence whether the form will be hyperbolic, elliptic, or symplectic. Use multiplicativity of both $g_h - g_e$ and $q^{n/2} g_z$ for taking orthogonal direct sums. $\square$

Proofs

Proposition *Let g be a standard symmetric bilinear form on V. Then*

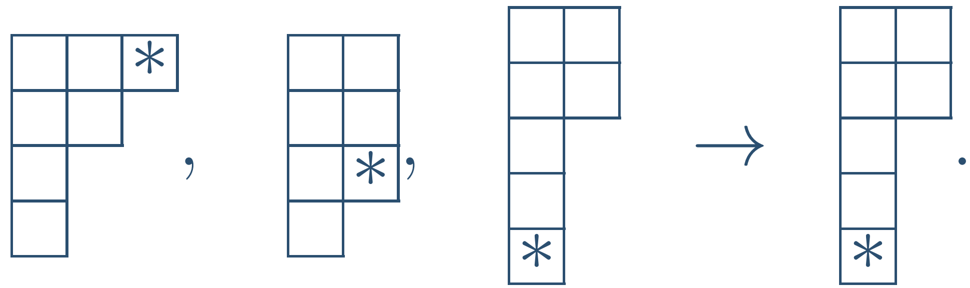
$$\#\{N \mid N \text{ self-adjoint, nilpotent}\} = \#\{(N, x) \mid N \text{ idem},\ Nx = 0,\ g(x, x) \neq 0\}$$

when $n = 2m + 1$, q even or odd, and when $n = 2m$, q even.

Proof (for $n = 2m + 1$): Write as sums over Y, and show that the terms can be grouped so as to get equality.

Proofs

The grouping is given by the map



that moves the bottom square from the rightmost odd column to form a new row of length one at the bottom. $\square$

Proof (for $n = 2m$, q even): Use the Fitting decomposition. $\square$

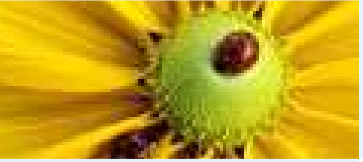
Proofs

Theorem $p(2m+1) = q^{2m}a(2m) + q^m d(2m)$.

Theorem $s(2m+1) = (q^{2m}-1)s(2m) + z(2m)$.

Theorem $s(2m) = q^{2m-1}s(2m-1)$.

Proof The first says that $p(2m+1) = \frac{1}{2}q^m(q^m+1)h(2m) + \frac{1}{2}q^m(q^m-1)e(2m)$.
All follow from the proposition above. $\square$



The end

That was all.