The graph with spectrum $14^1\ 2^{40}\ (-4)^{10}\ (-6)^9$

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Abstract

We show that there is a unique graph with spectrum as in the title.

1 The graph Δ

It was shown in [CGS] that there is a unique graph Z with spectrum $30^1\ 2^{90}$ $(-10)^{21}$ (with multiplicities written as exponents), namely the collinearity graph of the unique generalized quadrangle with parameters GQ(3, 9). It is strongly regular with parameters $(v, k, \lambda, \mu) = (112, 30, 2, 10)$. Its automorphism group is $U_4(3).D_8 \simeq PGO_6^{*-}(3)$ (of order $2^{10} \cdot 3^6 \cdot 5 \cdot 7$), where the * denotes that the form may be multiplied by a constant.

It was shown in [BH] that there is a unique graph Y with spectrum $20^1\ 2^{60}$ $(-7)^{20}$. It is strongly regular with parameters $(v, k, \lambda, \mu) = (81, 20, 1, 6)$, and is the second subconstituent of Z, the subgraph induced on the set of vertices at distance 2 from a fixed vertex a of Z. Its automorphism group is $3^4 : ((2 \times S_6).2)$ acting rank 3, the point stabilizer in $\mathrm{Aut}(Z)$. One construction of Y is found by taking $\mathbf{1}^\perp/\langle\mathbf{1}\rangle$ (where $\mathbf{1}$ denotes the all-1 vector) inside $\mathbf{F}_3^6$, where two cosets are adjacent when they differ by a weight-3 vector.

Let Δ be the second subconstituent of Y, the subgraph induced on the set of vertices at distance 2 from a fixed vertex b of Y. Then Δ has spectrum $14^1\ 2^{40}\ (-4)^{10}\ (-6)^9$ (apply Theorem 5.1 of [CGS]) and automorphism group $(2^2 \times S_6).2$, the stabilizer of the unordered pair $\{a, b\}$ in $\mathrm{Aut}(Z)$, twice as large as the point stabilizer of $\mathrm{Aut}(Y)$. The above description of Y leads to a description of Δ as the graph on the cosets in $\mathbf{F}_3^6$ with coordinates (up to permutation) either $000012 + \langle\mathbf{1}\rangle$ or $001122 + \langle\mathbf{1}\rangle$, where two cosets are adjacent when they differ by a weight 3 vector.

In this note we show that the graph Δ is determined by its spectrum.

2 Interlacing

An important tool is the following lemma on interlacing eigenvalues ([H], Theorem 2.1 (i),(ii); see also [BCN], Theorem 3.3.1).

Lemma 2.1 *Let Γ be a graph on n vertices with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$, and let $\{X_1, \ldots, X_m\}$ be a partition of the vertex set of Γ into nonempty parts. Let r_{ij} be the average number of neighbours in X_j of a vertex in X_i. Then the matrix $R = (r_{ij})$ has real eigenvalues $\mu_1 \geq \ldots \geq \mu_m$, which satisfy*
(i) (interlacing) $\lambda_i \geq \mu_i \geq \lambda_{n-m+i}$ for $i = 1, \ldots, m$;
(ii) if $\mu_i = \lambda_i$, or $\mu_i = \lambda_{n-m+i}$ for some $i \in \{1, \ldots, m\}$, then R has a μ_i-eigenvector $v = (v_1, \ldots, v_m)^\top$, such that the vector $w \in \mathbf{R}^n$ whose entries are equal to v_j for all vertices in X_j ($j = 1, \ldots, m$) is a μ_i-eigenvector of Γ.

For example if $m = 1$ it follows that the average valency $\overline{k}$ of Γ is at most equal to λ_1, and equality implies that the all-1 vector is a λ_1-eigenvector of Γ. Since $n\overline{k} = \sum \lambda_i^2$ it follows that Γ is regular of valency λ_1 if $n\lambda_1 = \sum \lambda_i^2$.

3 Graphs cospectral to Δ

Let Γ be a graph with the same spectrum $14^1\ 2^{40}\ (-4)^{10}\ (-6)^9$ as Δ.

We shall write $x \sim y$ ($x \not\sim y$) when x is a (non)neighbour of y in Γ, and denote the number of common neighbours of x and y by $\lambda(x, y)$ ($\mu(x, y)$).

(i) By Lemma 2.1 we know that Γ is regular of valency 14. Moreover Γ is connected, because the multiplicity of the eigenvalue 14 equals 1.

If Γ has adjacency matrix A, then $(A - 2I)(A + 4I)(A + 6I) = 72J$ so that $(A^3)_{xx} = 8$, and it follows that each vertex is in 4 triangles.

(ii) For a vertex x, let T_x be a set of 8 neighbours of x such that $\{x\} \cup T_x$ contains the four triangles on x. Let S_x be the set of the remaining 6 neighbours of x, and let N_x be the set of 45 nonneighbours of x. The matrix of average row sums of A, partitioned according to $\{\{x\}, T_x, S_x, N_x\}$ is

$$\begin{pmatrix} 0 & 8 & 6 & 0 \\ 1 & 1 & 0 & 12 \\ 1 & 0 & 0 & 13 \\ 0 & \frac{96}{45} & \frac{78}{45} & \frac{456}{45} \end{pmatrix}$$

with eigenvalues 14, 2, 0.40, −5.27. The 2-eigenspace is $\langle (15, 3, 1, -1)^\top \rangle$. By Lemma 2.1 it follows that the vector that is constant 15, 3, 1, −1 on $\{x\}$, T_x, S_x, N_x, respectively, is 2-eigenvector of A. Therefore each vertex in T_x has precisely one neighbour in T_x, that is, two triangles on x have only x in common. It also follows that if z is a non-neighbour of x with a neighbours in T_x and b neighbours in S_x, then $2a + b = 6$ while $a + b = \mu(x, z)$, so that $a = 6 - \mu(x, z)$.

(iii) The rank 10 matrix $B = 4J - (A - 2I)(A + 6I)$ is positive semi-definite and hence can be written $B = N^\top N$ for a 10×60 matrix N.

Let $\bar{x}$ be column x of N. Then $x \mapsto \bar{x}$ is a representation of Γ in Euclidean 10-space, with

$$(\bar{x}, \bar{y}) = \begin{cases} 2 & \text{if } x = y \\ -\lambda(x, y) & \text{if } x \sim y \\ 4 - \mu(x, y) & \text{if } x \not\sim y \end{cases} .$$

It follows that for nonadjacent vertices x, y one has $2 \leq \mu(x, y) \leq 6$.

If $\{x, y, z\}$ is a triangle, then $\bar{x} + \bar{y} + \bar{z} = 0$ (since this sum has squared norm 0).

The matrix B satisfies $JB = 0$ and $AB = -4B$ and $B^2 = 12B$ so that the rows of B are integral vectors with sum 0 and squared norm 24.

Row x of B has a 2 at the x-position, and a -1 at the 8 positions $z \in T_x$ (with $\lambda(x,z) = 1$). If $\bar{x} = \bar{y}$, so that rows x and y of B are identical, then $\mu(x,y) = 2$ and we see two 2's and at least fourteen -1's in each row, and since there can be at most two more nonzero entries, the row sum is nonzero, contradiction. It follows that the representation is injective.

If $(\bar{x}, \bar{y}) = -2$, then $\bar{y} = -\bar{x}$. Given x, this happens for at most one y. It follows that a row of B has entries either $2^1\ 1^8\ 0^{42}\ (-1)^8\ (-2)^1$ or $2^1\ 1^9\ 0^{39}\ (-1)^{11}$ (with multiplicities written as exponents).

(iv) Let us call a triangle a *line*. If $\mu(x,y) = 3$ then the three common neighbours of x and y are joined to both x and y by a line. Now there are 24 lines not on x meeting T_x, and each y with $\mu(x,y) = 3$ determines three such lines, so if there are 9 such points y then some line is seen twice. We find a line $\{y, y', z\}$ with $x \sim z$. Now $0 = (\bar{x}, \bar{y}) + (\bar{x}, \bar{y'}) + (\bar{x}, \bar{z}) = 1 + 1 + (-1) = 1$, contradiction. It follows that no row of B has pattern $2^1\ 1^9\ 0^{39}\ (-1)^{11}$.

(v) A set of roots (vectors of squared norm 2) with integral inner products spans a root lattice ([BCN], §3.10), so $\Lambda = \langle \bar{x} \mid x \in V\Gamma \rangle$ is a 10-dimensional root lattice, orthogonal direct sum of summands of the form A_n ($n \geq 1$), D_n ($n \geq 4$), E_6, E_7, or E_8.

(vi) The roots of the orthogonal direct sum of root lattices are the roots of the summands, so that an orthogonal direct sum decomposition of Λ gives a partition of $V\Gamma$ such that $(\bar{y}, \bar{z}) = 0$ if y, z are vertices from different parts. It follows that the three vertices of a triangle belong to the same part.

Consider the graph T with vertex set $V\Gamma$ where two vertices x, y are adjacent when $(\bar{x}, \bar{y}) = -1$, i.e., when xy is an edge in a triangle of Γ. Given x, consider the five subsets $S_i = \{u \in V\Gamma \mid (\bar{x}, \bar{u}) = i\}$ for $i = 2, 1, 0, -1, -2$. We have $|S_2| = |S_{-2}| = 1$, $|S_{-1}| = |S_1| = 8$, $|S_0| = 42$. The graph T is regular of valency 8. In T, any vertex $y \in S_{-1}$ has 1 neighbour x, 1 neighbour in S_{-1}, 3 neighbours in S_1, and hence 3 neighbours in S_0. A vertex $z \in S_0$ has 0 or 2 Γ-neighbours in S_{-1}, so at most 2 T-neighbours. We see that the connected component of T containing x has at least $1 + 8 + 8 + 1 + (8 \cdot 3)/2 = 30$ vertices.

It follows that either the root lattice Λ is indecomposable, i.e., is A_{10} or D_{10}, or has precisely two summands. Since A_n has $n(n+1)$ roots, and D_n has $2n(n-1)$ roots, the possibilities in the latter case are $A_5 + A_5$, $A_5 + D_5$, $D_5 + D_5$.

(vii) Suppose Λ has a direct summand D_5. The root system D_5 has 40 roots, and 30 occur as images of vertices in the corresponding connected component C of T. Let Φ be the graph on the 40 roots of D_4, adjacent when they have inner product -1, and consider C a subset of the vertex set of Φ. Let D be the set of 10 roots not in C. The graph Φ is regular of valency 12. The valency inside C is 8, so each vertex in C has 4 neighbours in D. This gives 120 edges meeting D, so there are no internal edges in D and no two roots of D have inner product -1. Both Φ and C are closed under $u \mapsto -u$, so also D is, and no two roots of D have inner product 1. Consequently, D has only inner products $2, 0, -2$ and consists of five mutually orthogonal pairs of opposite roots. But D_5 does not contain 5 mutually orthogonal roots. Contradiction.

(viii) Consider the graph Π with as vertices the 30 pairs $\pm\bar{x}$, adjacent when they have nonzero inner product. Then Π has valency 8 and $\lambda = 4$. Using a Weetman argument (cf. [W]) we see that a connected component of Π has fewer than 30 vertices. It will follow that $\Lambda \simeq A_5 + A_5$.

As follows. For geodesics $x_0 \sim x_1 \sim x_2 \sim \ldots$ we find lower bounds n_i for the number of common neighbours of x_i and x_{i+1} at distance i from x_0. We can take $n_1 = 2$ since two nonadjacent vertices in a 4-regular graph on 8 vertices must have at least 2 common neighbours. We can take $n_2 = 3$ since the set of common neighbours of x_2 and x_0 has valency at least $n_1 = 2$, and hence size at least 3 (and an 8-vertex graph of degree 4 cannot have a cut set of size 2). Now the local graph at x_3 has at least 4 vertices at distance 2 from x_0, and hence cannot have any at distance 4 from x_0 and a connected component of Π has diameter at most 3 and size at most $1+8+(8\cdot3)/3+(8\cdot2)/4 = 21$, as desired.

(ix) Thus far, we identified the 60 vertices of Γ with the 60 roots of $A_5 + A_5$, and can recognize the triangles of Γ. It remains to find the edges of Γ that are not in a triangle.

Let C and D be the two sets of vertices belonging to the two systems A_5. Given $x \in C$, the 12 vertices $y \in C$ with $(\bar{x}, \bar{y}) = 0$ have common T-neighbours with x, so are nonadjacent to x in Γ. That determines the induced subgraph on C and on D, and we have to find the edges between C and D.

Suppose $x \in C$. If $\bar{y} = -\bar{x}$, then $\mu(x, y) = 6$, and the 6 common neighbours of x and y live in D, and form all neighbours of x in D. If u is a common neighbour of x and y, and $\bar{v} = -\bar{u}$, then also v is a common neighbour of x and y. This means that for the edges across we can identify pairs of opposite roots, and have a geometry with 15 points and 15 lines, where each point is on 3 lines and each line has 3 points. The points can be identified with the pairs from a 6-set. Then subgraph on the set of points is $T(6)$. The lines consist of three mutually disjoint pairs. This is the unique generalized quadrangle of order 2.

This proves that Γ is uniquely determined by its spectrum, and hence must be isomorphic to Δ.

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