Sylvester versus Gundelfinger

Andries E. Brouwer[1] & Mihaela Popoviciu[2]

Apr 3, 2011

Abstract

Let V_n be the SL_2-module of binary forms of degree n and let $V = V_1 \oplus V_3 \oplus V_4$. We show that the minimum number of generators of the algebra $R = \mathbb{C}[V]^{\mathrm{SL}_2}$ of polynomial functions on V invariant under the action of SL_2 equals 63. This settles a 142-year old question.

1 Introduction

In 1868 Gordan proved that the algebra of invariants of binary forms of given degree is finitely generated. (For definitions, see the next section.) However, finding a set of generators for these algebras is even today an open problem in all but a few small cases.

This note focuses on the covariants of $V_3 \oplus V_4$, a case which illustrates the controversy between the German and English schools in the 19th century. The German school, following Clebsch and Gordan, was able to construct a system of generators for the algebra of invariants of binary forms, with no guarantee that the system was minimal. The English school, following Cayley and Sylvester, aimed to determine the number of independent generators. Sylvester used in his computations his "fundamental postulate", which turned out to hold only in small cases.

Gundelfinger, a student of Clebsch, wrote in 1869 a thesis ([7]) where he constructed generators for the covariants of $V_3 \oplus V_4$ 'in ordinary symbolic notation', after Clebsch had given him this system as computed by Gordan in his 'obscure' notation (cf. [4, pp. 270–272]). He found 20 generators for the invariants and 64 for the covariants.

Sylvester used his 'fundamental postulate' to show that there could be only 61 independent generators for the covariants of $V_3 \oplus V_4$, and wrote a series of papers [11–14] showing the superiority of the English methods over the German.

---

[1] Dept. of Math., Techn. Univ. Eindhoven, P.O. Box 513, 5600 MB Eindhoven, The Netherlands—`aeb@cwi.nl`

[2] Mathematisches Institut, Universität Basel, Rheinsprung 21, CH-4051 Basel, Switzerland—`mihaela.popoviciu@unibas.ch`

The second author is partially supported by the Swiss National Science Foundation.

In the first paper Sylvester uses his method (which he calls '*tamisage*') to derive the numbers of generators of given degrees in the coefficients of V_4 and those of V_3, and given order in the variables x, y. The following table is taken from [14]:

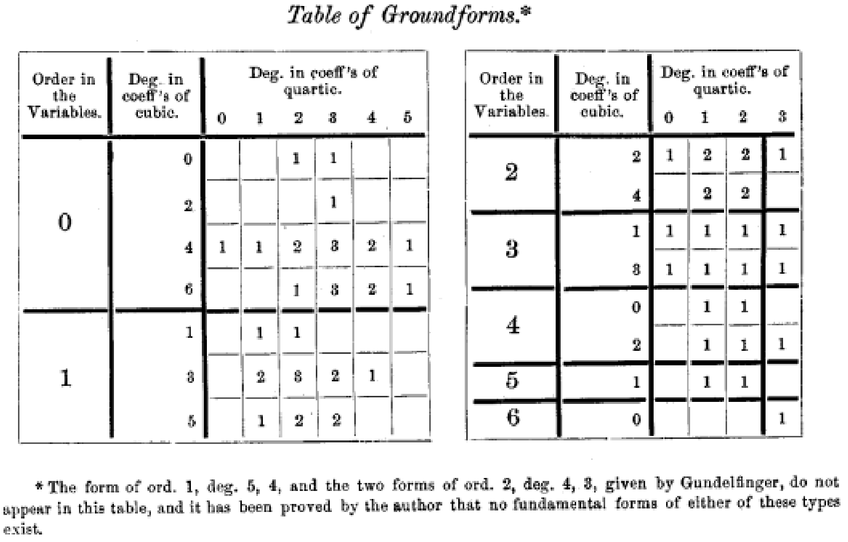


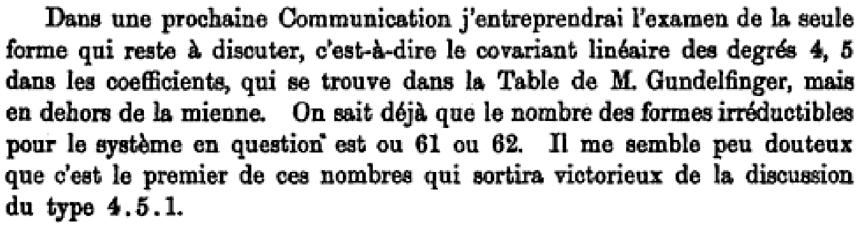
*Table of Groundforms.**

Order in the Variables.	Deg. in coeff's of cubic.	Deg. in coeff's of quartic.					
		0	1	2	3	4	5
0	0			1	1		
	2				1		
	4	1	1	2	3	2	1
	6			1	3	2	1
1	1		1	1			
	3		2	3	2	1	
	5		1	2	2		

Order in the Variables.	Deg. in coeff's of cubic.	Deg. in coeff's of quartic.			
		0	1	2	3
2	2	1	2	2	1
	4		2	2	
3	1	1	1	1	1
	3	1	1	1	1
4	0		1	1	
	2		1	1	1
5	1		1	1	
6	0				1

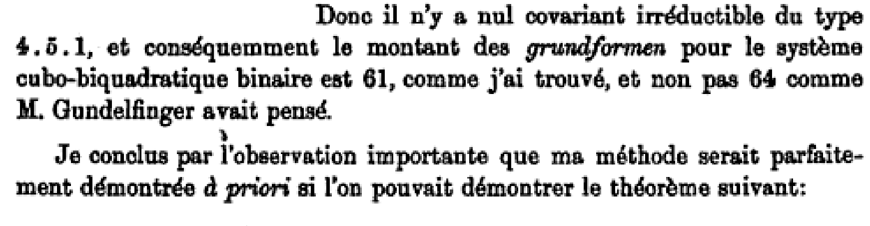
* The form of ord. 1, deg. 5, 4, and the two forms of ord. 2, deg. 4, 3, given by Gundelfinger, do not appear in this table, and it has been proved by the author that no fundamental forms of either of these types exist.

In the second paper he observes that it follows from the Poincaré series that there are 8 linearly independent covariants of multidegree (3,4,2). Next, he constructs 8 reducible such covariants (products of covariants of lower degree) and argues that these are linearly independent. However, the forms are dependent and only seven are independent. He finishes with the announcement



> **Dans une prochaine Communication j'entreprendrai l'examen de la seule forme qui reste à discuter, c'est-à-dire le covariant linéaire des degrés 4, 5 dans les coefficients, qui se trouve dans la Table de M. Gundelfinger, mais en dehors de la mienne. On sait déjà que le nombre des formes irréductibles pour le système en question* est ou 61 ou 62. Il me semble peu douteux que c'est le premier de ces nombres qui sortira victorieux de la discussion du type 4.5.1.**

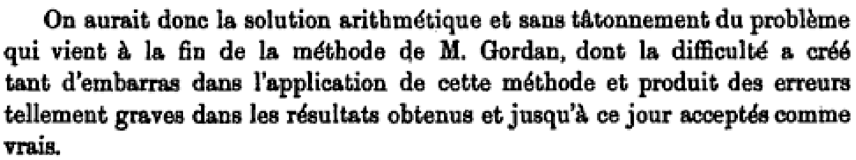
In the third paper he observes that it follows from the Poincaré series that there are 12 linearly independent covariants of multidegree (4,5,1). Next, he constructs 12 reducible such covariants and argues that these are linearly independent. However, the forms are dependent and only eleven are independent. He concludes



Donc il n'y a nul covariant irréductible du type 4.5.1, et conséquemment le montant des *grundformen* pour le système cubo-biquadratique binaire est 61, comme j'ai trouvé, et non pas 64 comme M. Gundelfinger avait pensé.

Je conclus par l'observation importante que ma méthode serait parfaitement démontrée *à priori* si l'on pouvait démontrer le théorème suivant:

(false theorem omitted)



On aurait donc la solution arithmétique et sans tâtonnement du problème qui vient à la fin de la méthode de M. Gordan, dont la difficulté a créé tant d'embarras dans l'application de cette méthode et produit des erreurs tellement graves dans les résultats obtenus et jusqu'à ce jour acceptés comme vrais.

Here we show that the actual minimal number of generators for the covariants of $V_3 \oplus V_4$ is 63. We will identify the covariants of $V_3 \oplus V_4$ with the invariants of $V_1 \oplus V_3 \oplus V_4$ (for details, see the next section), and will count the generators of the algebra of invariants of $V_1 \oplus V_3 \oplus V_4$.

1.1 Invariants and Covariants

Let V be a finite-dimensional vector space over a field k, with basis $e_1, \ldots, e_m$. Let x_i be the coordinate function defined by $x_i(\sum \xi_h e_h) = \xi_i$. The algebra $k[V]$ of polynomial functions on V is by definition the algebra generated by the x_i. (It does not depend on the choice of basis $e_1, \ldots, e_m$.) Let G be a group of linear transformations of V. It acts on $k[V]$ via the action $(g \cdot f)(v) = f(g^{-1}v)$. Invariant theory studies $k[V]^G$, the algebra of G-invariant polynomial functions on V, i.e., the $f \in k[V]$ such that $g \cdot f = f$ for all $g \in G$.

A covariant of order m and degree d of V is a G-equivariant homogeneous polynomial map $\phi : V \to V_m$ of degree d. In other words, $\phi(g \cdot v) = g \cdot \phi(v)$, for all $g \in G$, and $\phi(tv) = t^d \phi(v)$, for all $t \in k$. In particular, the covariants of V of order 0 are the invariants of V.

Below we shall take $k = \mathbb{C}$, $G = \mathrm{SL}_2(k)$, and $V = V_{n_1} \oplus \ldots \oplus V_{n_p}$, where V_n is the vector space (of dimension $n+1$) consisting of 0 and the binary forms of degree n, that is, of the homogeneous polynomials of degree n

$$v(x,y) = a_0 x^n + \binom{n}{1} a_1 x^{n-1} y + \ldots + \binom{n}{n-1} a_{n-1} x y^{n-1} + a_n y^n,$$

in two variables. This V_n is the n-th graded part of $k[W]$, where W is a 2-dimensional vector space over $\mathbb{C}$ with natural action of SL_2, hence has a natural action of SL_2. The set of covariants of V will be denoted $\mathcal{C}(V)$. We have $\mathcal{C}(V) \simeq k[V_1^* \oplus V]^{\mathrm{SL}_2}$, which allows us to identify the covariants of V with the invariants of $V_1 \oplus V$, where V_1 is the vector space of linear forms $ax+by$. The 1-1 correspondence is realised by the map $\phi(v,x,y) \mapsto \phi(v,b,-a)$, with $\phi \in \mathcal{C}(V)$ and $v \in V$.

Given $g \in V_m$ and $h \in V_n$, the expression

$$(g,h) \mapsto (g,h)_p := \frac{(m-p)!(n-p)!}{m!n!} \sum_{i=0}^{p} (-1)^i \binom{p}{i} \frac{\partial^p g}{\partial x^{p-i} \partial y^i} \frac{\partial^p h}{\partial x^i \partial y^{p-i}}$$

defines a linear and SL_2-equivariant map $V_m \otimes V_n \to V_{m+n-2p}$, which is classically called the *p-th transvectant* (Überschiebung).

2 The generators of the invariants of $V_1 \oplus V_3 \oplus V_4$

In this section we show that a minimal set of generators for the algebra of invariants of $V_1 \oplus V_3 \oplus V_4$, has size 63. That is, a single one of Gundelfinger's generators is superfluous.

Doing this type of work requires finding dependencies. Gundelfinger did not try to do this exhaustively, he only noted the obvious ones. Sylvester tried and made some mistakes, no doubt because he already knew what answer he wanted. For us this is relatively easy—a modern computer has no problems computing the rank of a 40000 by 600000 matrix (which is what is needed in the most straightforward approach).

We had a different problem: up to which degree should we compute covariants or invariants? Gundelfinger 'just' followed Gordan's algorithm, but as far as we know that has not been implemented yet.

The secret knowledge known today but not in the 19th century, is that the ring R of invariants of $V_1 \oplus V_3 \oplus V_4$ (or any such ring) is Cohen-Macaulay. It has a homogeneous system of parameters (hsop) $j_1, \ldots, j_r$, algebraically independent, and finitely many further generators $i_1, \ldots, i_s$, such that every invariant can be uniquely written as a product $i_m j_{m_1} \ldots j_{m_h}$. It follows that the Poincaré series $P(t) = \sum d_i t^i$, where d_i is the dimension of the degree i part of R, is of the form

$$P(t) = \frac{t^{a_1} + \cdots + t^{a_s}}{(1-t^{b_1}) \ldots (1-t^{b_r})}$$

where the a_h and b_h are the degrees of the i_h and j_h. Since $P(t)$ is known from the work of Cayley and Sylvester, in order to find $\max_h a_h, b_h$ it suffices to find the b_h.

In the present case,

$$\begin{aligned} P(t) = 1 &+ t^2 + 2t^3 + 5t^4 + 10t^5 + 18t^6 + 31t^7 + 55t^8 + 92t^9 + \\ &144t^{10} + 223t^{11} + 341t^{12} + 499t^{13} + 725t^{14} + 1031t^{15} + \\ &1436t^{16} + 1978t^{17} + 2685t^{18} + 3592t^{19} + 4761t^{20} + 6235t^{21} + \\ &8078t^{22} + 10379t^{23} + 13226t^{24} + 16698t^{25} + 20937t^{26} + \\ &26069t^{27} + 32230t^{28} + 39614t^{29} + 48401t^{30} + \ldots = \\ = &\frac{a(t)}{(1-t^3)(1-t^4)^2(1-t^5)^2(1-t^6)^2(1-t^7)} \end{aligned}$$

where

$$\begin{aligned}a(t) = 1 &+ t^2 + t^3 + 3t^4 + 7t^5 + 12t^6 + 21t^7 + 32t^8 + 47t^9 + \\ &58t^{10} + 72t^{11} + 83t^{12} + 89t^{13} + 94t^{14} + 94t^{15} + 89t^{16} + \\ &83t^{17} + 72t^{18} + 58t^{19} + 47t^{20} + 32t^{21} + 21t^{22} + 12t^{23} + \\ &7t^{24} + 3t^{25} + t^{26} + t^{27} + t^{29}\end{aligned}$$

and it follows that computing invariants up to degree 29 suffices if we know that there is a hsop with degrees 3, 4, 4, 5, 5, 6, 6, 7.

2.1 Finding a hsop

Hilbert introduced in the 19th century the notion of *nullcone.* If V is an SL_2-module, then the nullcone $\mathcal{N}(V)$ of V is the set of elements of V on which all invariants of V of positive degree vanish. The elements of $\mathcal{N}(V)$ are called *nullforms.* One can show that a binary form $f \in V_n$ is a nullform if and only if f has a root of multiplicity $> \frac{n}{2}$ (this is a consequence of the Hilbert-Mumford criterion, see [2, §2.4.1]). Similarly, if we have p binary forms $f_1, \ldots, f_p$ of degrees $n_1, \ldots, n_p$, then $(f_1, \ldots, f_p) \in \mathcal{N}(V_{n_1} \oplus \ldots \oplus V_{n_p})$ if and only if $f_1, \ldots, f_p$ have a common root of multiplicity $> \frac{n_i}{2}$ in f_i, for all $i = 1, \ldots, p$. In our particular case, if $(l, c, q) \in V_1 \oplus V_3 \oplus V_4$, then $(l, c, q) \in \mathcal{N}(V_1 \oplus V_3 \oplus V_4)$ if and only if $l^2 | c$ and $l^3 | q$.

Let $\mathcal{V}(J)$ stand for the vanishing locus of J. The following result, due to Hilbert, gives a characterisation of homogeneous systems of parameters of $k[V_{n_1} \oplus \ldots \oplus V_{n_p}]^{\mathrm{SL}_2}$ as sets that define the nullcone of $\mathcal{N}(V_{n_1} \oplus \ldots \oplus V_{n_p})$:

Proposition 2.1. (Hilbert [9]) *Let* $V = V_{n_1} \oplus \ldots \oplus V_{n_p}$, *and* $R = k[V]^{\mathrm{SL}_2}$, *and* $m = n_1 + \ldots + n_p + p - 3 > 0$. *A set* $\{j_1, \ldots, j_m\}$ *of homogeneous elements of* R *is a system of parameters of* R *if and only if* $\mathcal{V}(j_1, \ldots, j_m) = \mathcal{N}(V)$.

Let our binary forms $l \in V_1$, $c \in V_3$, $q \in V_4$ be

$$\begin{aligned}l &= c_0 x + c_1 y, \\ c &= a_0 x^3 + 3a_1 x^2 y + 3a_2 x y^2 + a_3 y^3, \\ q &= b_0 x^4 + 4b_1 x^3 y + 6b_2 x^2 y^2 + 4b_3 x y^3 + b_4 y^4,\end{aligned}$$

and consider the following invariants:

$$\begin{aligned}k_2 &= (q, q)_4, & k_3 &= ((q, q)_2, q)_4, \\ k_{4,1} &= ((c, c)_2, (c, c)_2)_2, & k_{4,2} &= (lc, lc)_4, \\ k_{4,3} &= (c, l^3)_3, & k_{5,1} &= ((q, (q, q)_2)_1, c^2)_6, \\ k_{5,2} &= ((q, c^2)_2, c^2)_6, & k_{5,3} &= (q, l^4)_4, \\ k_{6,1} &= ([(c, c)_2]^2, (q, q)_2)_4, & k_{6,2} &= ((lc, lc)_2, lc)_4, \\ k_{6,3} &= ((q, q)_2, l^4)_4, & k_7 &= (c^4, q^3)_{12}.\end{aligned}$$

We prove the following

Proposition 2.2. *With the notations above, the invariants*

$$\begin{array}{llll} j_1 = k_3, & j_2 = k_{4,1} + k_2^2, & j_3 = k_{4,2} + k_{4,3} - k_2^2, & j_4 = k_{5,1} + k_{5,2}, \\ j_5 = k_{5,3}, & j_6 = k_{6,1} + k_{6,2}, & j_7 = k_{6,3}, & j_8 = k_7, \end{array}$$

(of degrees 3, 4, 4, 5, 5, 6, 6, 7, respectively) form a system of parameters of $k[V_1 \oplus V_3 \oplus V_4]^{\mathrm{SL}_2}$.

Proof. We show that $\mathcal{V}(j_1, \ldots, j_8) = \mathcal{N}(V_1 \oplus V_3 \oplus V_4)$. Consider three cases.

Case 1: $q = 0$.
In this case, the vanishing of $j_1, \ldots, j_8$ reduces to $k_{4,1} = k_{4,2} + k_{4,3} = k_{6,2} = 0$, which implies that $(l, c) \in \mathcal{N}(V_1 \oplus V_3)$. Indeed, if $k_{4,1} = 0$, then c is a nullform, and, without loss of generality, we may suppose that $x^2|c$, i.e. $a_2 = a_3 = 0$. But then $k_{6,2} \sim a_1^3 c_1^3$ (we use '$\sim$' for equality up to a nonzero constant). If $c_1 = 0$, then $(l, c) \in \mathcal{N}(V_1 \oplus V_3)$. If $c_1 \neq 0$, then $a_1 = 0$ and $k_{4,2} + k_{4,3} \sim a_0 c_1^3$. Hence $a_0 = 0$, so that $c = 0$ and $(l, c) \in \mathcal{N}(V_1 \oplus V_3)$.

Case 2: $l = 0$.
In this case, the vanishing of $j_1, \ldots, j_8$ reduces to $k_2 = k_3 = k_{4,1} = k_{5,1} + k_{5,2} = k_{6,1} = k_7 = 0$, which implies that $(c, q) \in \mathcal{N}(V_3 \oplus V_4)$. Indeed, the vanishing of $k_2, k_3, k_{4,1}$ implies that c and q are nullforms. If c or q vanish identically, then the statement is clear. Otherwise, if the double zero of c and the triple zero of q do not coincide, we may suppose, without loss of generality, that $x^2|c$ and $y^3|q$, i.e. $a_2 = a_3 = b_0 = b_1 = b_2 = 0$. Then $k_{6,1} \sim a_1^4 b_3^2$. If $a_1 = 0$, then $k_{5,1} + k_{5,2} \sim a_0^2 b_3^3$, and $k_7 \sim a_0^4 b_4^3$, which contradicts the assumption $c, q \neq 0$. If $b_3 = 0$, then $k_{5,1} + k_{5,2} \sim a_1^4 b_4$ and $k_7 \sim a_0^4 b_4^3$, which again contradicts the assumption $c, q \neq 0$.

Case 3: q, $l \neq 0$.
In this case, $j_5 = 0$ implies that q and l have a common root (up to a constant, j_5 is the resultant of q and l). Without loss of generality, we can suppose that the common factor of q and l is x, i.e., $c_1 = b_4 = 0$ and $c_0 \neq 0$. Then $j_7 \sim b_3^2 c_0^4$, which implies $b_3 = 0$. Then $j_1 \sim b_2^3$, which implies $b_2 = 0$. Then a_3 becomes a factor of j_8. If $a_3 = 0$, then $j_3 \sim a_2^2 c_0^2$, which implies $a_2 = 0$, and then $(l, c, q) \in \mathcal{N}(V_1 \oplus V_3 \oplus V_4)$. If $a_3 \neq 0$, we may take $a_3 = c_0 = 1$. Now

$$j_3 \sim 3a_2^2 - 3a_1 - 2,$$

and it follows that $a_1 = a_2^2 - \frac{2}{3}$. Then

$$j_6 \sim 27a_2^3 - 54a_2 - 27a_0 - 256b_1^2,$$

and it follows that $a_0 = a_2^3 - 2a_2 - \frac{256}{27} b_1^2$. Then

$$j_4 \sim 36b_0 - 144a_2 b_1 - 949b_1^3,$$

and it follows that $b_0 = 4a_2b_1 + \frac{949}{36}b_1^3$. Then

$$\begin{aligned} j_2 &\sim 27 - 2048b_1^4, \\ j_8 &\sim b_1^5(33205248 - 4273351745b_1^4). \end{aligned}$$

But $j_2 = j_8 = 0$ has no solution. This settles Case 3.

By Proposition 2.1, it follows that these eight invariants form a hsop of the ring of invariants of $V_1 \oplus V_3 \oplus V_4$. $\square$

2.2 The degrees of the generators

The Poincaré series of the ring of invariants of $V_1 \oplus V_3 \oplus V_4$ tells us which is the maximal degree in which we have to look for generators, namely 29. For each $i \leq 29$ we do the following: multiply invariants of smaller degrees to see what part of the vector space of invariants of degree i is known. The Poincaré series tells us how big the dimension of this vector space is, and if the known invariants do not yet span this vector space, one constructs in some way further invariants, until they do span. In the following table i denotes the degree of the generators, and d_i the number of generators of degree i needed:

i	2	3	4	5	6	7	8	9	10	11
d_i	1	2	4	8	10	13	11	10	3	1

For $12 \leq i \leq 29$ no further generators are needed, and it follows that the minimal number of generators is 63.

References

[1] A. Cayley, *A Second Memoir upon Quantics*, Phil. Trans. Royal Soc. London **146** (1856) 101–126.

[2] H. Derksen & G. Kemper *Computational Invariant Theory*, Springer, Berlin, 2002.

[3] P. Gordan, *Beweis, dass jede Covariante und Invariante einer binären Form eine ganze Funktion mit numerischen Coeffizienten einer endlichen Anzahl solcher Formen ist*, Journ. f. Math. **69** (1868) 323–354.

[4] P. Gordan, *Die simultanen Systeme binärer Formen*, Math. Ann. **2** (1869) 227–280.

[5] P. Gordan, *Invariantentheorie*, Bd. II (Binäre Formen), Leipzig, 1887.

[6] J. H. Grace & A. Young, *The Algebra of Invariants*, Cambridge, 1903.

[7] S. Gundelfinger, *Zur Theorie des simultanen Systems einer cubischen und einer biquadratischen binären Form*, Habilitationsschrift, J. B. Metzler, Stuttgart, 1869.

[8] J. Hammond, *Note on an exceptional case in which the fundamental postulate of prof. Sylvester's theory of tamisage fails*, Proc. London Math. Soc. t. **14** (1882) 85–88.

[9] D. Hilbert, *Über die vollen Invariantensysteme*, Math. Ann. **42** (1893) 313–373.

[10] J. J. Sylvester, *Sur le vrai nombre des covariants élémentaires d'un système de deux formes biquadratiques binaires*, Comptes Rendus **84** (1877) 1285–1289.

[11] J. J. Sylvester, *Sur les covariants fondamentaux d'un système cubo-biquadratique binaire*, Comptes Rendus **87** (1878) 242–244, 287–289.

[12] J. J. Sylvester, *Sur le vrai nombre des formes irréductibles du système cubo-biquadratique*, Comptes Rendus **87** (1878) 445–448.

[13] J. J. Sylvester, *Détermination du nombre exact des covariants irréductibles du système cubo-biquadratique binaire*, Comptes Rendus **87** (1878) 477–481.

[14] J. J. Sylvester, assisted by F. Franklin, *Tables of generating functions and groundforms for simultaneous binary quantics of the first four orders, taken two and two together*, Amer. J. Math. **2** (1879) 223–251.