Pessimal gossiping

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Consider n ladies, each with a different gossip item. They communicate by telephone, and whenever two ladies talk, each tells the other all she knows. The question about the minimum number of calls needed to spread all gossip was answered around 1971, by several independent groups. For $n \geq 4$ the answer is $2n - 4$.

In this note we look at the question about the longest possible series of calls, when in each call at least one participant learns something new. The answer turns out to be $n(n-1)/2$.

That $n(n-1)/2$ is a lower bound, is shown by the following scenario: Let the participants be $A_1, \ldots, A_n$. All calls involve A_1. For $i = 2, \ldots, n$ she calls $A_i, A_{i-1}, \ldots, A_2$, for a total of $1+2+\cdots+(n-1) = n(n-1)/2$ calls. Of course there are many other scenarios.

We show that $n(n-1)/2$ is an upper bound. Since each of the n participants must learn $n-1$ items, and at least one item is learned on each call, there are at most $n(n-1)$ calls. If during a call $1+e$ items of gossip are transmitted, we say that there were e extra items. We will show that in all calls together at least $n(n-1)/2$ extra items occur, by pointing at (at least) one extra item for each unordered pair A, B of participants. Then the total number of calls is at most $n(n-1) - n(n-1)/2 = n(n-1)/2$.

We shall assign a call to each ordered pair (A, B) of participants in such a way that a call with e extra items is assigned to (A, B) for at most e unordered pairs A, B. If A, B start out knowing a, b, respectively, the call assigned to (A, B) will be one during which a and b are exchanged, or be the one where B first learns a. If a and b are exchanged during the call where B first learns a, this will be the assigned call. If a and b are not exchanged during the call assigned to (A, B), then a different call is assigned to (B, A), and we find a sharper bound.

Number the calls in time order. There is a canonical call where a and b are exchanged: At the end B knows both a and b. Trace calls backward from B to uniquely find a strictly increasing series of calls numbered $i_1, ..., i_h$ and a maximal series of ladies X_0, X_1, ..., $X_h = B$, such that call i_j is between ladies X_{j-1} and X_j, where X_j knows both a and b at the end of the call, but did not know both at the start of the call. Now at call i_1 the ladies X_0 and X_1 exchanged a and b. The call assigned to the pair (A, B) will be either i_1 or i_h. The series of calls $i_1, ..., i_h$ will be called the canonical path of a to B.

We shall compare paths $i_1, ..., i_h$ in reverse lexicographic order (revlex), that is lexicographic order on the reversed paths $i_h, ..., i_1$. For two paths where one is a tail of the other, the smaller path is the one that starts with the smallest element, i.e., the one that is longest.

Let R be the set of items that B learned during call number i. Let a_0 be some element of R (to be fixed later) that revlex minimizes the canonical path to B. Regard all items $a \in R \setminus \{a_0\}$ as extra on call i. This assigns an extra item to the pair A, B whenever the canonical path of a to B is not revlex minimal among the paths with the same final element.

Suppose $i_1, ..., i_h = i$ is the revlex minimal one among the canonical paths to B ending in i, where this is the canonical path of a to B, and let i_1 be a call between P and Q, where P is the participant learning b. If $p \in R$, then $i_1, ..., i_h$ is the canonical path of p to B. Now pick $a_0 = p$, and assign the pair (P, B) to call i_1 as promised. If $p \notin R$ then pick $a_0 \in R$ arbitrarily, and assign the pair (A_0, B) to call i_1. (In this case the canonical path for (P, B) is revlex smaller than $i_1, ..., i_h$, and does not use the call i_1. The pair (P, B) will be assigned somewhere on that path.)

Let Q, P learn the sets of items S, T of sizes s, t (respectively) during call i_1. Then $s + t$ items were transmitted, so $s + t - 1$ extra items, and $s, t \geq 1$. We are allowed to assign $s + t - 1$ pairs to call i_1. If that call exchanges p and q, then a possible assignment is (P, B) for all $b \in T$ and (A, Q) for all $a \in S$. If we assign (A, B) then we do not assign (P, B), so that the total remains ok. If call i_1 does not exchange p and q, for example because $p \notin S$, then for every $b \in T$ we pick a unique $a \in S$, and at most t pairs are assigned to call i_1, again ok. □

References

[1] A. E. Brouwer, `http://www.win.tue.nl/~aeb/math/gossip.html`.

[2] Bart Frenk, *Tropical varieties, maps and gossip*, Ph.D. thesis, Eindhoven Univ. of Techn., 2013-03-13.