Strongly regular graphs from hyperovals

aeb

11 Feb 2016

Abstract

We give a construction of a family of graphs found by Huang, Huang and Lin.

In Huang-Huang-Lin [2] the authors constructed various families of designs and graphs. We give a simplified description of one family.

1 Construction

Let q be a power of 2, and let V be a 3-dimensional vector space over $\mathbb{F}_q$. Let $H = \{P_1, \ldots, P_{q+2}\}$ be a hyperoval (that is, a $(q+2)$-arc) in the projective plane PV, and let x_i be a representative vector of P_i, so that $x_i \in V$ with $P_i = \langle x_i \rangle$, $(i = 1, \ldots, q+2)$. Let C be the linear q-ary $[q+2, 3, q]$-code obtained by evaluating all linear forms $c \in V^*$ at the x_i, so that $c_i = c(x_i)$. There are precisely q^{3-i} code words with i specified coordinate positions $(i = 0, 1, 2, 3)$. Any two code words have Hamming distance q or $q+2$ (since any projective line meets H in 2 or 0 points).

Construct a graph Γ with as vertex set C, where two code words are adjacent when they have Hamming distance $q+2$.

Proposition 1.1 *The graph Γ is strongly regular with parameters* $(v, k, \lambda, \mu) = (q^3, \frac{1}{2}q(q-1)^2, \frac{1}{4}q(q-2)(q-3), \frac{1}{4}q(q-1)(q-2))$. *Its spectrum is $k^1\ r^f\ s^g$ (with exponents denoting multiplicities), where $r = \frac{1}{2}q$, $s = -\frac{1}{2}q(q-1)$ and $f = (q-1)(q^2-1)$, $g = (q-1)(q+2)$.*

Proof. This is well-known, a special case of the similar construction (due to Delsarte [1]) for a set X in projective space such that the size of the intersection $|X \cap Z|$ for hyperplanes Z takes only two values. In our case the set X is the dual hyperoval consisting of the $q(q-1)/2$ exterior lines of the hyperoval H. □

The graph Γ is invariant under the transitive translation group. Let Δ be its local graph: the graph on all code words of weight $q+2$, adjacent when they have Hamming distance $q+2$.

Proposition 1.2 *The graph Δ is strongly regular with parameters* $(v, k, \lambda, \mu) = (\frac{1}{2}q(q-1)^2, \frac{1}{4}q(q-2)(q-3), \frac{1}{8}q(q^2-9q+22), \frac{1}{8}q(q-3)(q-4))$. *Its spectrum is $k^1\ r^f\ s^g$, where $r = \frac{1}{2}q$, $s = -\frac{1}{4}q(q-3)$ and $f = \frac{1}{2}(q+1)(q-2)(q-3)$, $g = q^2-4$.*

Proof. Direct counting, via inclusion-exclusion. □

These graphs are the complements of those that follow from [2], Cor. 4.1. The special case $q = 8$ was mentioned explicitly in [2], Cor. 4.2.

References

[1] Ph. Delsarte, *Weights of linear codes and strongly regular normed spaces*, Discr. Math. **3** (1972) 47–64.

[2] Tayuan Huang, Lingling Huang & Miaow-Ing Lin, *On a class of strongly regular designs and quasi-semisymmetric designs*, Recent developments in algebra and related areas, Chongying Dong et al. eds., Proc. Internat. Conference on Algebra and Related Areas, Tsinghua University, Beijing, China, August 18–20, 2007. Advanced Lectures in Mathematics (ALM) **8** (2009) 129–153.