Locally 4-by-4 Grid Graphs

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Abstract

We investigate locally grid graphs. The main results are (i) a characterization of the Johnson graphs (and certain quotients of these) as locally grid graphs such that two points at distance 2 have precisely four common neighbors, and (ii) a complete determination of all graphs that are locally a 4×4 grid (it turns out that there are four such graphs, with 35, 40, 40, and 70 vertices).

Introduction

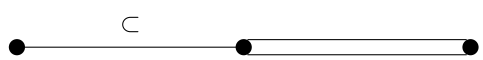
Let us denote by $\binom{n}{m}$ the graph with as vertices the m-subsets of an n-set, where two m-sets are adjacent when they have an $(m-1)$-set in common. (These graphs are commonly known as Johnson graphs; in the cases $m = 2$ and $m = 3$, also as triangular and tetrahedral graphs.)

Let us denote by $p \times q$ the graph with vertex set $P \times Q$ where $|P| = p$, $|Q| = q$, and (x_1, y_1) is adjacent to (x_2, y_2) if and only if $x_1 = x_2$, or $y_1 = y_2$ (but not both). The graphs $p \times q$ are called *grids*. A graph Γ is called *locally* G if for each vertex x of Γ the graph induced by Γ on the set of neighbors of x is isomorphic to the graph G.

With this terminology the graph $\binom{n}{m}$ is locally $m \times (n-m)$ and one might want to characterize it as such; without additional hypotheses this seems impossible (J. I. Hall has indicated large classes of locally grid graphs), but adding the hypothesis that the graph induced on the set of common neighbors of two points at distance 2 is a 4-cycle (which is true in $\binom{n}{m}$) we do indeed obtain a characterization of the Johnson graphs and certain quotients .

Let us denote by $\mu(x, y)$ (where x and y are two vertices at distance 2 in a graph Γ) the graph induced on the set of common neighbors of x and y; we shall call subgraphs of Γ of this form *μ-graphs*. Characterizing locally $p \times q$ graphs (without hypothesis on the μ-graphs) is trivial for $p \leq 2$ and has been done by J. I. Hall [7,8] for $p = 3$. Here we settle the first unsolved case by finding all locally 4×4 graphs. The result is surprising in that there are besides the expected graphs $\binom{8}{4}$ and $\frac{1}{2}\binom{8}{4}$ (the quotient of $\binom{8}{4}$ obtained by identifying complementary 4-sets of the 8-set), two more examples on 40 vertices, one very regular graph (its automorphism group is transitive on points, edges, triangles, 4-c1iques, and 5-c1iques) described in Section 3, and a twisted version of the previous, described in Section 4.

It will be clear to those who know the language of Buekenhout-Tits diagrams that what we consider here are geometries belonging to the diagram



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Related work (characterizing certain classes of locally polar graphs) was done by Buekenhout and Hubaut [4]. There are two possible ways for a generalized quadrangle to be thin (and thus difficult to handle geometrically) — one is the case with two lines on each point, the case we consider here, and the other is the case with two points on each line. But in the latter case we have a locally $K_{p,q}$ graph, and it is trivial to check that such graphs do not exist for $p \neq q$ while the only locally $K_{p,p}$ graph is $K_{p,p,p}$. (In contrast to the difficulty of characterizing locally $p \times q$ graphs, it is completely trivial to characterize locally $\overline{p \times q}$ graphs (where the bar denotes complementation) in all interesting cases — cf. Buset [5].) We shall use $\sim$ to denote adjacency.

1 Characterization of the Johnson graphs

Let us first state the results we shall prove in this section.

Theorem 1. *Let Γ be connected and locally grid, and assume that the connected components of each μ-graph are 4-cycles. Then there are integers m, n such that Γ is locally $m \times (n-m)$, and either $\Gamma \cong \binom{n}{m}$ or $n = 2m$ and Γ is obtained from $\binom{2m}{m}$ by identifying each m-set with the image of its complement under σ, where σ is a permutation of the $2m$ symbols satisfying $\sigma^2 = 1$ and having at least 8 fixed points.*

Remark. For $m \leq 4$ we must have $\sigma = 1$ and $m = 4$ so that the only possible quotient is $\frac{1}{2}\binom{8}{4}$. (The quotients $\frac{1}{2}\binom{4}{2}$ and $\frac{1}{2}\binom{6}{3}$ are complete graphs K_3 and K_{10}.) In general, the number of transpositions in the cycle representation of σ is an invariant of the quotient so that one obtains several nonisomorphic quotients.

When σ has at least 10 fixed points then each μ-graph of the quotient is a 4-cycle; when σ has precisely 8 fixed points then each μ-graph is the union of two 4-cycles.

Remark. Some version of the above theorem was communicated by the second author to J. I. Hall at the Pullman conference in honor of T. G. Ostrom. Hall subsequently showed (in a letter dated May 21, 1981) that this theorem is essentially equivalent to Theorem 2 in Sprague [10].

Under the weaker assumption that Γ is locally a graph on $m(n-m)$ vertices with valency $(m-1)+(n-m-1)$ and each μ-graph has at most 4 vertices, one can prove that $\Gamma \cong \binom{n}{m}$ provided n is large enough. (T. A. Dowling [6] proved this for $n > 2m(m-1)+4$ and A. E. Brouwer [2] for $n \geq \max(6m-1, m^2+2m-1)$.)

A. Moon [9] obtains a characterization for $n > 4m$, characterizing $\binom{n}{m}$ not as a graph but as an association scheme, where all the parameters p^i_{jk} of the association scheme are given.

In all cases the difficult part of the proof is to find cliques of the right size — thus (the analogue of) Theorem 1 is trivial in these contexts; only when $n = 2m$ does one have to be a little careful.

Corollary. *Let Γ be connected and locally $2 \times q$. Then Γ is the triangular graph $\binom{q+2}{2}$.*

Lemma. *Let Γ be connected and locally grid. Then there are integers p, q such that Γ is locally $p \times q$. Each μ-graph is a union of cycles of even length. If C*

is a maximal clique and $x \notin C$, then x is adjacent to either 0 or 2 points of C. Each edge is in precisely two maximal cliques: one of size $p+1$ and one of size $q+1$. Each triangle is in a unique maximal clique.

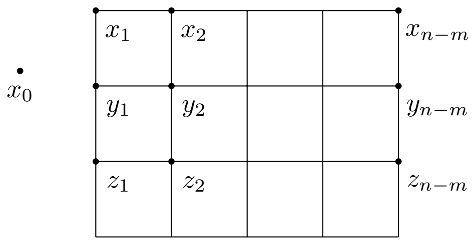
Proof of the Lemma. Suppose that the neighborhood of x is isomorphic to $p \times q$ and that $x \sim y$. Then the edge xy is in cliques of size $(p+1)$ and $(q+1)$, and hence the neighborhood of y is also isomorphic to $p \times q$. By connectedness of Γ, the graph is locally $p \times q$.

If $d(u,v) = 2$ and $a \in \mu(u,v)$, then looking at the neighbors of a we find that u, v, a have two common neighbors, so $\mu(u,v)$ is regular of valency 2. Since $\mu(u,v) \subset \Gamma(u)$, the edges of a component of $\mu(u,v)$ are alternatingly "horizontal" and "vertical," i.e., each component of $\mu(u,v)$ is an even cycle. $\square$

Proof of the Corollary. The only union of cycles of even length that occurs as subgraph of $2 \times q$ is a 4-cycle. Thus the theorem applies. $\square$

Proof of the Theorem. By the lemma, Γ is locally $m \times (n-m)$ and we may assume $m \leq n-m$. The proof is by induction on m. For $m = 0$, Γ has valency zero and hence is a single point, indeed $\Gamma \cong \binom{n}{0}$ for all $n \geq 0$. Consider the graph Γ^* that has the $(n-m+1)$-cliques of Γ as vertices and where two such cliques are adjacent when they have precisely one point in common. We want to show that each connected component of Γ^* satisfies the hypotheses of the theorem and is locally $(m-1) \times (n-m+1)$.

To this end first observe that if $C = \{x_0, \ldots, x_{n-m}\}$ is a maximal clique of Γ, then $\Gamma^*(C)$ has a partition into $n-m+1$ cliques of size $m-1$, namely the sets of points in $\Gamma^*(C)$ containing some fixed x_i $(0 \leq i \leq n-m)$.



Next assume that C_1 and C_2 are points of Γ^* such that $C_1 \cap C_2 = \{x_0\}$, say $C_1 = \{x_0, x_1, x_2, \ldots, x_{n-m}\}$, $C_2 = \{x_0, y_1, y_2, \ldots, y_{n-m}\}$, where $x_i \sim y_i$ $(1 \leq i \leq n-m)$.

Let D_i be the maximal clique containing x_i, y_i but not x_0 $(1 \leq i \leq n-m)$. Then $\{C_1, C_2, D_1, \ldots, D_{n-m}\}$ is a clique in Γ^*.

[For: let $i \neq j$. The point x_i has two neighbors in D_j, one is x_j and the other is a point not in $\Gamma(x_0)$, say a. Now $\mu(x_0, a)$ contains x_i, x_j, y_j and hence by our hypothesis on μ-graphs also y_i. Thus D_i is the clique determined by a, x_i, y_i and D_i is adjacent to D_j in Γ^*.]

Thus Γ^* is locally $(m-1) \times (n-m+1)$.

Next we check that the μ-graphs in Γ^* are unions of 4-cycles. Let C_1 and C_2 be as before, and let D meet C_2 in the point y_1, where $|C_1 \cap D| \neq 1$. Then $|C_1 \cap D| = 0$. The point x_0 has two neighbors on D, y_1 and say, z_1. Looking at $\Gamma(x_0)$ we see that $x_1 \sim z_1$, and $\mu(C_1, D)$ contains the 4-cycle (C_2, $D(x_1, y_1)$, $D(x_1, z_1)$, C_3), where $D(p,q)$ is the maximal clique on p, q not containing x_0, and $C_3 = \{x_0, z_1, \ldots, z_{n-m}\}$ is the maximal clique on x_0, z_1 not containing x_1, y_1.

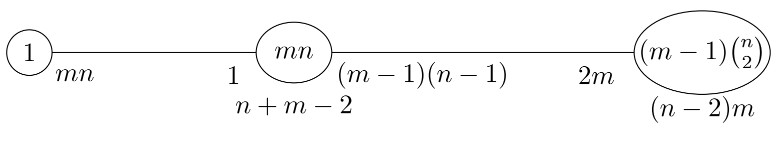
Thus, by induction, each component of Γ^* is the Johnson graph $\binom{n}{m-1}$. First assume that $n > 2m$; then each edge is in a unique $(n-m+1)$-clique and Γ^* is connected. Label the points of Γ^* with $(m-1)$-sets; next label each point x of Γ with the union of the labels of the $(n-m+1)$-cliques on x. Then x is labeled with an m-set (since the cliques incident with x form a small maximal clique in Γ^*) and clearly two points of Γ are adjacent iff they are together in a clique, i.e., iff their labels have an $(m-1)$-set in common. Thus $\Gamma \cong \binom{n}{m}$. (Note that Γ has the right number of vertices and no label can be repeated.) Next assume $n = 2m$. Now each edge is in precisely two $(m+1)$-cliques, and Γ^* has either one or two components. If Γ^* has two components, then use the labels of one of these components to label Γ and we find $\Gamma \cong \binom{n}{m}$ again. So assume that Γ^* is connected. Now each point x of Γ determines two m-cliques in Γ^*, and taking the union of the labels of the members of each m-clique, we find two labels for x. Both labels are m-sets, and if $x \sim y$ then each of the labels of x has an $(m-1)$-set in common with a unique label of y. Conversely, if a label of x and a label of y have an $(m-1)$-set in common then $x \sim y$ and the other labels also have an $(m-1)$-set in common. Thus Γ is a quotient of $\binom{2m}{m}$ under an automorphism σ of order 2. Now the automorphism group of $\binom{2m}{m}$ is the direct product $\mathrm{Sym}(2m) \times \mathbb{Z}_2$, where the $\mathbb{Z}_2$, sends an m-set to its complement. Also, if σ interchanges the m-sets, M, N, then the Johnson distance between M and N is at least 4 (otherwise the quotient will not be locally $m \times m$), i.e., $|\sigma(M) \cap M| \leq m-4$. If σ is an involution in $\mathrm{Sym}(2m)$ then there are m-sets M such that $|\sigma(M) \cap M| \geq m-1|$. Hence σ must be complementation followed by an involution σ_0 in $\mathrm{Sym}(2m)$. Now the requirement $|\sigma(M) \cap M| \leq m-4$ means $|\sigma_0(M) \cap M| \geq 4$, i.e., σ_0 has at least 8 fixed points. □

2 Locally $p \times q$ graphs with maximal μ

Lemma. *Let Γ be locally $p \times q$ and suppose $|\mu(x,y)| = 2p$ for vertices x, y of Γ with $d(x,y) = 2$. Then no neighbor of y has distance 3 to x. In particular, if $|\mu(x,y)| = 2p$ for all pairs x, y with $d(x,y) = 2$ then Γ has diameter (at most) 2.*

Proof. Obvious. □

Assume that Γ is locally $m \times n$ with $n \geq m \geq 2$, where each μ-graph has $\mu = 2m$ vertices. Then Γ is strongly regular with diagram and parameters



$$v = 1 + mn + (m-1)\binom{n}{2} = (n+1)(1 + \frac{1}{2}n(m-1))$$

vertices, eigenvalues

$$k = mn\ , \quad r = n-2\ , \quad s = -m$$

with multiplicities

$$1 \ , \quad f = \frac{m(m-1)n(n+1)}{2(n+m-2)} \ , \quad g = v - 1 - f \ .$$

(For definition and properties of strongly regular graphs and partial geometries see e.g. [3].)

Since f is an integer, we see that for fixed $m > 3$ only finitely many n are possible. For example, when $m = 4$ then $(n+2) \mid 12$, so that $n = 4$ or $n = 10$.

The Hoffman bound for the maximum size of a clique in a strongly regular graph with these parameters yields $|C| \le n+1$ and tells us that when $|C| = n+1$ and $x \notin C$ then x is adjacent to precisely two points of C. (It is also easy to see this directly.)

This means that if $n > m$ we have a partial geometry pg$(K = n+1, R = m, T = 2)$, while if $n = m$ we have a Zara graph with parameters $K = n+1$, $e = 2$. (For definition and properties of Zara graphs see F. Zara [11] and A. Blokhuis [1].) For $n > m$ one easily sees that the line graph of the partial geometry has the property that each of its μ-graphs is the disjoint union of $\frac{1}{2}(n+1)$ 4-cycles. It follows that n must be odd. (But it is possible that $n = m$ for even n.)

This observation immediately eliminates $(m, n) = (4, 10)$. (Another way to dispose of this parameter set is to observe that the line graph of pg$(11, 4, 2)$ violates the absolute bound and hence does not exist; in general, the absolute bound for tbe line graph states

$$V \le \frac{1}{2}G(G+3) \quad \text{for} \quad V = m\left(1 + (m-1)\cdot\frac{1}{2}n\right)$$
$$\text{and} \quad G = m - 1 + \frac{m(m-1)(m-3)n}{2(n+m-2)} \ ,$$

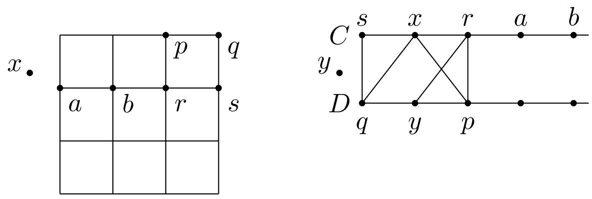
which yields a restriction of the form $n \le \frac{1}{4}m^4$.)

Thus, when $m = 4$ then necessarily $n = 4$, and there is a unique graph in this case:

Proposition. *There is a unique Zara graph with $K = 5$, $e = 2$, on $v = 35$ vertices. It is $\frac{1}{2}\binom{8}{4}$.*

Proof. Clearly such a graph must be locally 4×4. In order to apply the theorem of the previous section we have to prove that the μ-graphs are unions of 4-cycles. To this end, consider two disjoint 5-cliques C, D. The edges joining a point from C to a point from D form a bipartite graph of valency 2, i.e., a union of (even) cycles; the only possibilities are (i) a union of a 4-cycle and a 6-cycle, and (ii) a 10-cycle. We shall show that the former always is the case. Suppose $C = \{a_1, a_2, a_3, a_4, a_5\}$ and let C_{ij} be the 5-clique on a_i, and a_j distinct from C $(1 \le i < j \le 5)$. Then C_{12} and C_{34} are disjoint and we see a 4-cycle (1324) between C_{12} and C_{34} so the pair (C_{12}, C_{34}) is of type (i). Counting all such pairs we find $56 \cdot 10 \cdot 3$ ordered pairs (C', C'') of type (i). On the other hand, each clique C meets itself in 5 points, 10 others in 2 points, and 15 others in 1 point, and the remaining $(56 - 1 - 10 - 15) = 30$ in 0 points. Thus the number of pairs of disjoint cliques equals the number of such pairs of type (i) and type (ii) does not occur.

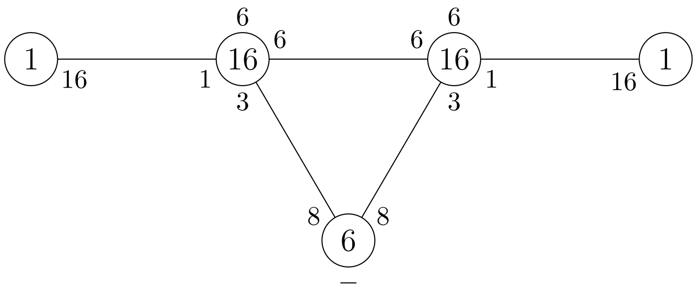
Now let x, y be two points at distance 2, and let $r \sim p \sim q$ be a path of length 3 in $\mu(x, y)$. Let C be the 5-clique on x, r, s, where s is the point completing the 4-cycle $(p\ q\ s\ r)$ in $\Gamma(x)$. Let D be the 5-clique on y, p, q.



Between (the disjoint 5-cliques) C and D we see the path $s \sim q \sim x \sim p \sim r \sim y$ and this must complete to a 6-cycle. Hence $y \sim s$ so that $\mu(x, y)$ contains the 4-cycle $(p\ q\ s\ r)$. $\square$

3 An interesting graph on 40 vertices

There exists a unique locally 4×4 graph Γ with diagram



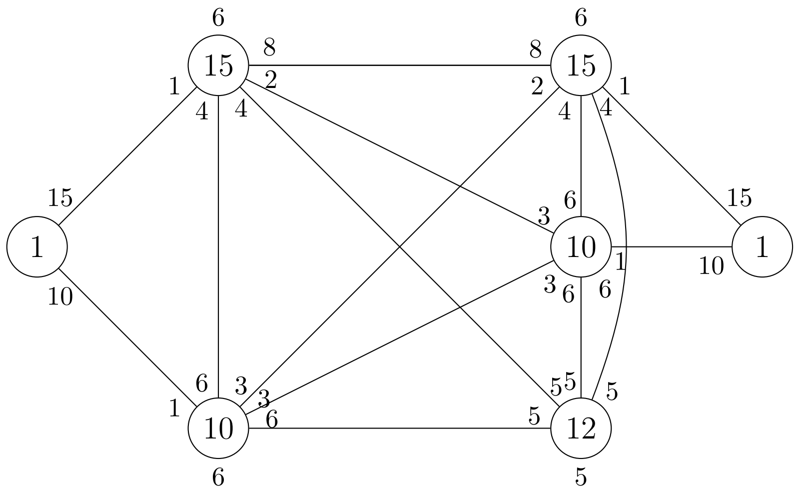
with respect to each of its points.

It has 40 vertices and 64 blocks (5-cliques). Its group of automorphisms is $2^6 \cdot S_5$, of order 7680 acting transitively on j-cliques for $0 \le j \le 5$. Switching antipodes is an automorphism. If we call two blocks adjacent when they meet in a unique point then the graph Γ^* on the blocks has 2 components of size 32. Each of the components has diagram

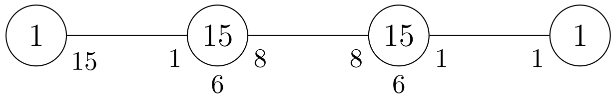
and is locally GQ$(2,2)$. (Uniqueness will be the subject of Section 5. Here we are concerned with the existence.) The nicest description we know is the following:

Let Δ_{01} be the graph with as vertices the 2×2 matrices with entries in $\mathbb{F}_4$, and trace 0 or 1, where two matrices are adjacent if their difference has rank 1.

Let $\overline{\Delta_{01}}$ be the quotient of Δ_{01} under identification of A and $A + I$ for all matrices A. Then Δ_{01} has 64 vertices and diagram



(The subgraph $\overline{\Delta_0}$ of $\overline{\Delta_{01}}$ consisting of the points with trace 0 has diagram



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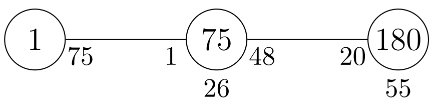
Let Γ be the graph with as vertices the 8-cliques of $\overline{\Delta_{01}}$, two cliques being adjacent when they have nonempty intersection; then Γ has the properties stated above (and the vertices of $\overline{\Delta_{01}}$ are the 5-cliques in Γ).

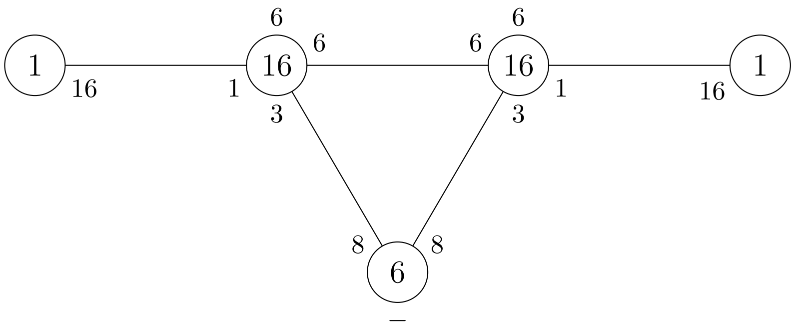
[From the description one immediately sees the group of automorphisms $2^6 \cdot \mathrm{P\Gamma L}(2,4)$, but $\mathrm{P\Gamma L}(2,4) \cong S_5$. The 8-cliques of $\overline{\Delta_{01}}$ have 4 points with trace 0 and 4 points with trace 1; in fact, the graph Δ on all 256 2×2 matrices over $\mathbb{F}_4$ is strongly regular with diagram



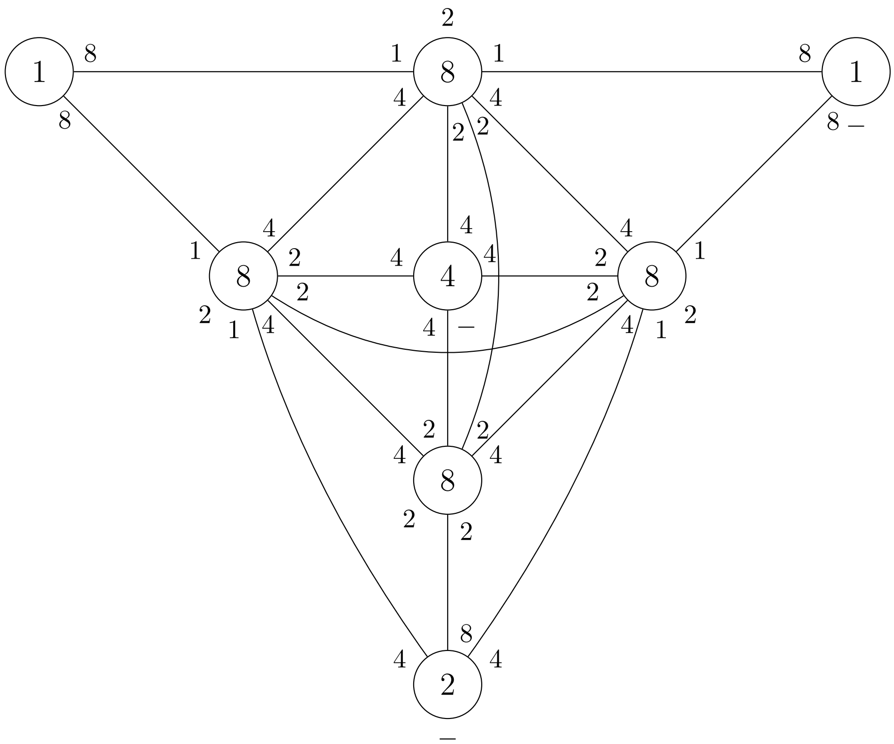
and the maximal cliques meeting the Hoffman bound are the two-dimensional affine subspaces such that any two matrices in such a subspace differ by a rank 1 matrix. From this it is not difficult to see that Γ will be locally 4×4. The automorphism switching antipodes is $A \mapsto A + \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}$. The partition of Γ into 5 cocliques of size 8 corresponds to the parallelism on Δ.]

4 Another graph on 40 vertices

Unfortunately, there is another graph on 40 points that is locally 4×4. It has diagram



with respect to 8 of its points, while it looks as follows around each of the remaining 32 points:



(thus, its diameter is 3, but for 32 points x one has $\Gamma_3(x) = \emptyset$). A direct description can be given as follows:

Let $X = \Omega \cup Y \cup Z$, where $\Omega = \{\infty_i, \overline{\infty_i} \mid i \in \mathbb{Z}_4\}$, Y and Z are both copies of $\mathbb{Z}_4 \times \mathbb{Z}_4$ — write (i,j) for elements of Y and $[i,j]$ for elements of Z. The graph will have $d(\infty_i, \overline{\infty_i}) = 3$ $(i \in \mathbb{Z}_4)$, while $\Gamma_3(x) = \emptyset$ for $x \in X \setminus \Omega$. The induced subgraph on Ω is a coclique, and on Y, Z we have the natural 4×4 grid. Each point in $Y \cup Z$ that is nonadjacent to ∞_i is adjacent to $\overline{\infty_i}$, so we need only specify the adjacencies of ∞_i $(i \in \mathbb{Z}_4)$.

Let $\Gamma(\infty_0) = Y$ (so that $\Gamma(\overline{\infty_0}) = Z$), and $\Gamma(\infty_i) = \{(a,a), (a,a+i), [a,a], [a,a-i] \mid a \in \mathbb{Z}_4\}$ for $i = \pm 1$, and $\Gamma(\infty_2) = \{(a,a), (a,a+2), [a,a+1], [a,a-1] \mid a \in \mathbb{Z}_4\}$.

Let the point $(ab) \in Y$ be adjacent to the 6 points $[a-\delta+i, b+\delta+j]$ for $i,j \in \mathbb{Z}_4$, $|\{0,i,-j\}| = 3$, $\delta = b-a$.

This defines a graph Γ, and by inspection it is locally 4×4. Its automorphism group has order 2^9, is transitive on the 64 5-cliques, and has two orbits of sizes 8, 32 on the 40 points.

The graph Γ^* with as vertices the 5-cliques in Γ, where $C \sim C'$ iff $|C \cap C'| = 1$, has two components of size 32.

5 The locally 4×4 graphs

Theorem. *There are, up to isomorphism, precisely four locally 4×4 graphs, namely*

(i) $\binom{8}{4}$ *(with $v = 70$ vertices, all μ-graphs are 4-cycles).*

(ii) $\frac{1}{2}\binom{8}{4})$ *(with $v = 35$ vertices, all μ-graphs are unions of two 4-cycles).*

(iii) G_{40} *the graph described in Section 3 (with $v = 40$ vertices, each μ-graph is either a 6-cycle or a union of two 4-cycles).*

(iv) G'_{40}, *the graph described in Section 4 (with $v = 40$ vertices, each μ-graph is eithcr a 4-cycle, a 6-cycle, an 8-cycle, or the union of two 4-cycles).*

The proof is split up into a series of lemmas. Let Γ be a locally 4×4 graph. Write $\Gamma_i(x)$ for the set of vertices at distance i from a vertex x.

Lemma 1.

(i) Γ has valency 16, each edge is in 6 triangles, and the graph $\lambda(p,q)$ on the common neighbors of two adjacent points p and q is the union of two triangles.

(ii) Each edge is in two 5-cliques; each triangle is in a unique 5-clique.

(iii) If Γ has v vertices and b 5-cliques, then $b = \frac{8}{5}v$. In particular, $5 \mid v$.

(iv) For $d(p,q) = 2$, the graph $\mu(p,q)$ is either a 4-cycle, a 6-cycle, an 8-cycle, or the union of two 4-cycles. If $|\mu(p,q)| = 4,6,8$, then q has at most $4,1,0$ neighbors in $\Gamma_3(p)$ (respectively).

Lemma 2. *If C is a 5-clique and $d(x,C) = 2$, then x has distance two from at least 3 points of C. Consequèntly, if $d(x,y) = 3$, then the number of neighbors of y at distance two from x is either 9 or at least 12.*

Proof. (i) Let $x \sim p$ with $d(p,C) = 1$. Then p has two neighbors on G, say y and z. $\mu(x,y)$ is not a clique so cannot be contained in $\lambda(y,z)$. It follows that $x \sim q$ where q has two neighbors on C but $q \not\sim z$.

(ii) Now assume $d(x,y) = 3$. Looking at the 4×4 grid that is the neighborhood of y we see on each line 0, 3, or 4 points at distance 2 from x. □

Lemma 3. *Equivalent are (i) $v \geq 70$, (ii)* diam $\Gamma \geq 4$*, and (iii)* $\Gamma \cong \binom{8}{4}$.

Proof. Estimating the number of points k_i at distance i from a given point x we find

$$k_0 = 1, \quad k_1 = 16, \quad k_2 \le \frac{16 \cdot 9}{4} = 36, \quad k_3 \le \frac{36 \cdot 4}{9} = 16,$$
$$k_4 \le \lfloor \frac{16 \cdot 1}{9} \rfloor = 1, \quad k_5 \le \lfloor \frac{1 \cdot 1}{9} \rfloor = 0.$$

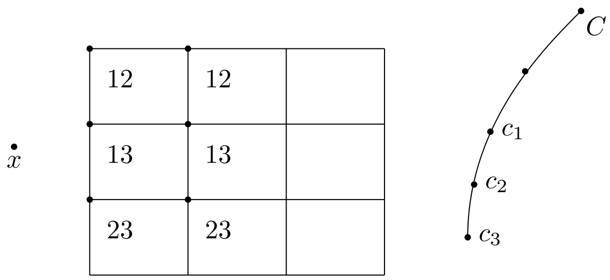
Hence $v \le 1+16+36+16+1 = 70$ with equality iff $|\mu(x,y)| = 4$ for all pairs x, y with $d(x,y) = 2$. Also, if diam $\Gamma = 4$ then $k_4 = 1$, and the 16 neighbors of a point at distance 4 from x must have distance 3 from x so that $k_3 = 16$ and we have equality everywhere so that $v = 70$. Now apply the theorem in Section 1. $\square$

From now on we may assume that diam $\Gamma \le 3$.

Lemma 4. *Let $d(x,C) = 2$ and $|\Gamma_2(x) \cap C| = 3$. Then $|\mu(x,c)| = 4$ for all $c \in \Gamma_2(x) \cap C$.*

Proof. Follows immediately from Lemma 1(iv). $\square$

Remark. In the case of this lemma one necessarily has the following situation:



Here the points in $\Gamma(x)$ labeled ij are adjacent to c_i and c_j. That is, the points in $\Gamma(x)$ at distance 1 from C lie on a 2×3 grid.

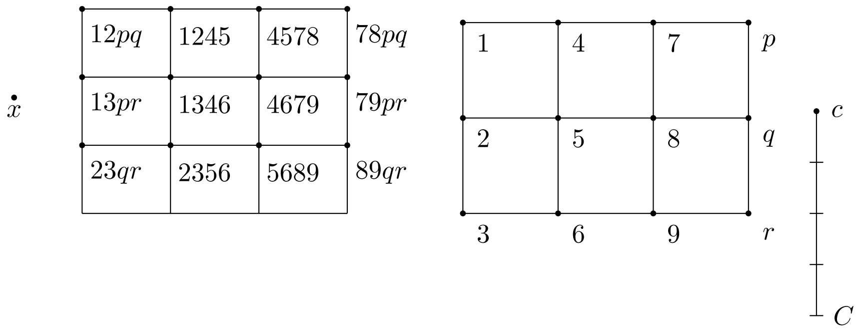
Lemma 5. *For no point x and 5-clique C do we have $C \subset \Gamma_3(x)$.*

Proof. Let $A = \{a \mid a \sim x, d(a,C) = 2\}$, $B = \{b \mid b \in \Gamma_2(x), d(b,C) = 1\}$. Let $C = \{c_1, c_2, c_3, c_4, c_5\}$.

(i) The neighbors of a point $c_i \in C$ lie on the four cliques C_{ij} on c_i, c_j, $(j \ne i)$ with $C_{ij} \ne C$. Each c_i has at least 9 neighbors in B, i.e there is at most one clique C_{ij} on c_i not meeting B. $|C|$ is odd, so there is a point $c \in C$, say $c = c_1$, such that all the cliques C_{ij} $(j \ne 1)$ meet B.

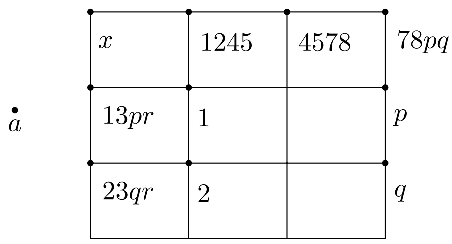
(ii) For $b \in B$ we have $|\mu(b,x)| = 4$ (cf. Lemma 1(iv)), so $\sum_{b \sim c} |\mu(b,x)| = 48$. But $48 = \sum_{b \sim c} |\mu(b,x)| = \sum_{\substack{a \sim x \\ d(a,c)=2}} |\mu(a,c)|$, and $|\mu(a,c)| \in \{4,6\}$ for $a \in A \cap \Gamma_2(c)$. If for some $a \in A \cap \Gamma_2(c)$ we have $|\mu(a,c)| = 6$, then $|A \cap \Gamma_2(c)| < 12$ so $|A \cap \Gamma_2(c)| = 9$ and $A \cap \Gamma_2(c)$ forms a 3×3 grid in $\Gamma(x)$. If D is a 5-clique on a and x then $|D \cap \Gamma_2(c)| = 3$ so $|\mu(a,c)| = 4$ by Lemma 4, a contradiction. Hence always $|\mu(a,c)| = 4$ and $|A \cap \Gamma_2(c)| = 12$.

Considering the adjacencies between $\Gamma(c)$ and A we see that one necessarily has



(where points in $A \cap \Gamma_2(c)$ are labeled with the 4 points of $B \cap \Gamma(c)$ they are adjacent to).

Let a be the point labeled "$12pq$" and consider the neighborhood of a:



Let D be the clique $\{7,\ 8,\ 9,\ e,\ d\}$. We see that $|\mu(a,7)| = |\mu(a,8)| = 6$, $|\mu(a,9)| = 0$ (since it is impossible to find a cycle in $\Gamma(a)$ avoiding the nonneighbors of the point 9), $|\mu(a,c)| = 4$, and $\mu(a,d) = 0$. But this contradicts the previous lemma. $\square$

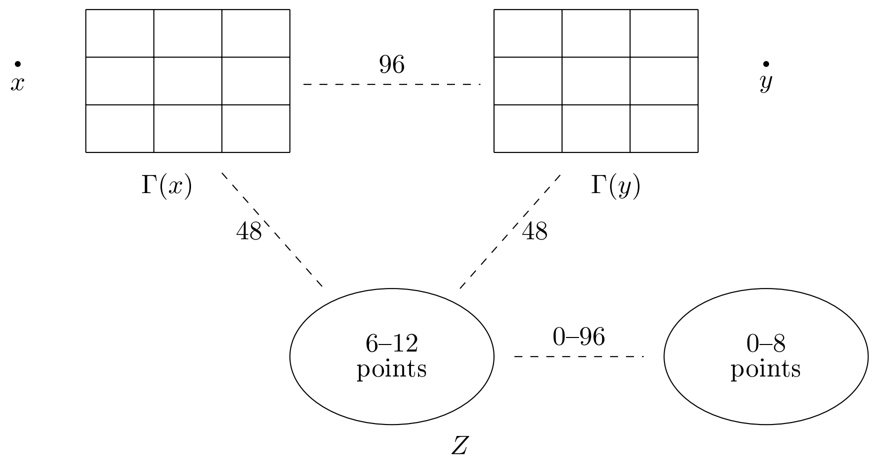
Corollary. *If $d(x,y) = 3$ then $|\Gamma(y) \cap \Gamma_2(x)| \geq 12$.*

Lemma 6. *Let C be a 5-clique, $d(x,C) = 2$ and $\Gamma_2(x) \cap C| = 4$. Then there is a $\mu \in \{4,6\}$ so that for all $c \in \Gamma_2(x) \cap C$ we have $|\mu(x,c)| = \mu$.*

Proof. If $c, c' \in \Gamma_2(x) \cap C$ then $\mu(x,c) \cap \mu(x,c')$ is either empty or an edge (for if $d \in \mu(x,c) \cap \mu(x,c')$ then x has two neighbors on the 5-clique on c, c', d). The edges of this form are pairwise disjoint and partition $\frac{1}{2}\sum_{c \in \Gamma_2(x) \cap C} \mu(x,c)$ points, so this latter number is even. This means that if $|\mu(x,c)|$ is not constant then it is twice 4 and twice 6. But one quickly checks that it is impossible to find two 4-cycles in $\Gamma(x)$, each having one edge in common with a 6-cycle in $\Gamma(x)$ (where these two edges are parallel). $\square$

Lemma 7. *Suppose $d(x,y) = 3$, $d(x,z) = 2$, $y \sim z$, $|\mu(x,z)| = 6$. Then $|\Gamma_3(x) = 1$ and $v = 40$ and $|\mu(x,p)| \in \{6,8\}$ for all $p \in \Gamma_2(x)$.*

Proof. Look at the neighborhood of y. Lemmas 4 and 6 state that if for one point z of a line in this 4×4 grid we have $d(x,z) = 2$ and $|\mu(x,z)| = 6$, then this holds for all points of this line. Thus $\Gamma(y) \subset \Gamma_2(x)$ and for each neighbor z of y we have $|\mu(x,z)| = 6$. This leads to the situation



From $\Gamma(x)$ there are 144 edges to $\Gamma_2(x)$; 96 end in $\Gamma(y)$, 48 in $\Gamma_2(x) \setminus \Gamma(y) =: Z$. [Clearly $6 = \frac{48}{8} \le |Z| \le \frac{48}{4} = 12$.] By Lemma 1(iv), the points in $\Gamma(y)$ have no neighbors in $\Gamma_3(x) \setminus \{y\}$, so any additional point in $\Gamma_3(x)$ has all its neighbors in $Z \cup \Gamma_3(x)$. We show that Z is a coclique: in fact, looking at the neighbors of a point $a \in \Gamma(x) \cup \Gamma(y)$, we see that the three neighbors of a in Z form a coclique. Now if we have an edge pq in Z then pq is contained in a 5-clique $C \subset Z \cup \Gamma_3(x)$. Let $x \sim r \sim p$. Then r has two neighbors in $C \cap Z$, a contradiction.

Now it follows that $\Gamma_3(x) = \{y\}$, since any additional point has at least 12 neighbors in Z, and Z could not be a coclique.

Since there emerge precisely 96 edges from Z we have $|Z| = 6$, $v = 40$. $\square$

Remark. Of course the situation of this lemma will yield conclusion (iii) of the theorem, but we shall first prove that, when we know that the situation is the same with respect to any point x of Γ.

Lemma 8. *Suppose $C \subset \Gamma(x)$ for some 5-clique C. Then $|\mu(c, x)| = 4$ for a unique point $c \in C$ and $|\mu(c, x)| = 6$ for all other points $c \in C$.*

Proof. As we saw in the proof of Lemma 6, the sets $\mu(c, x) \cap \mu(c', x)$ are either empty or an edge (for $c, c' \in C$), and the edges of this form partition the $\frac{1}{2} \sum_{c \in C} |\mu(c, x)|$ points in $\Gamma(x)$ at distance 1 to C. Thus the vector $\mu = (|\mu(c, x)|)_{c \in C}$ takes one of the values $(4, 4, 4, 4, 4)$, $(4, 4, 4, 6, 6)$, or $(4, 6, 6, 6, 6)$. In the first case we have five 4-cycles in $\Gamma(x)$, covering five parallel edges— impossible. In the second case we find a 6-cycle having an edge in common with each of two disjoint 4-cycles (these edges are parallel), and we saw already in the proof of Lemma 6 that this is impossible. So the third case, $\mu = (4, 6, 6, 6, 6)$ holds. $\square$

Lemma 9. *If for some pair of points x, z with $d(x, z) = 2$ we have $|\mu(x, z)| = 4$, then $\Gamma_3(x) = \emptyset$.*

Proof. Let $A = \{a \in \Gamma_2(x) \mid \Gamma(a) \cap \Gamma_3(x) = \emptyset\}$ and $B = \Gamma_2(x) \setminus A$. If for some $b \in B$ we have $|\mu(b, x)| = 6$, then we are in the situation of Lemma 7 and $|\mu(x, z)| = 4$ is impossible. Hence we have $|\mu(b, x)| = 4$ for each $b \in B$. Considering the neighborhood of a point $b \in B$ we see at least one point in $\Gamma_3(x)$, and for each $b' \in \Gamma(b) \cap \Gamma(y)$ we have $\mu(x, b')| = 4$, so by Lemma 8 no 5-c1ique on b is contained in $\Gamma_2(x)$: 4 of the 5-c1iques on b meet $\Gamma(x)$ and the remaining 4 meet $\Gamma_3(x)$. But this means that no point from B is adjacent to a point in A. Let C be a 5-c1ique meeting A.

Since $A \subset \Gamma_3(y)$ for $y \in \Gamma_3(x)$ we have $|C \cap A| \leq 2$. Also $|C \cap \Gamma(x)| \leq 2$. But there is nowhere the fifth point of C could go, so there is no such C and $A = \emptyset$. (Now $|\Gamma_2(x) = 36$.)

Let $y \in \Gamma_3(x)$. Counting paths of length 3 between x and y we see that $p := |\Gamma(y) \cap \Gamma_2(x)| = |\Gamma(x) \cap \Gamma_2(y)|$. If $p = 12$ then there is no room for a 2×3 grid in $\Gamma(x) \cap \Gamma_2(y)$ so each 5-clique on Y has 4 points in $\Gamma_2(x)$ and $p = 16$, a contradiction. Thus $p > 12$, and estimating $\Gamma_3(x)$ we find $5 = \lceil 2 \cdot 36 \rceil / 16 \leq |\Gamma_3(x)| \leq 4 \cdot 36 / \lfloor 13 \rfloor = 11$ so that $58 \leq v \leq 64$, and since $5 \mid v$, we must have $v = 60$, $|\Gamma_3(x)| = 7$.

Now a counting argument will kill this situation: let s be the number of edges between $\Gamma_2(x)$ and $\Gamma_3(x)$, t the number of edges in $\Gamma_3(x)$, and a_i, $(2 \leq i \leq 4)$ the number of points $b \in \Gamma_2(x)$ with precisely i neighbors in $\Gamma_3(x)$. Then we have

$$\begin{aligned} a_2 + a_3 + a_4 &= 36 \\ 2a_2 + 3a_3 + 4a_4 &= s \\ a_3 + 2a_4 &= t \\ s + 2t &= 7 \cdot 16 = 112, \end{aligned}$$

so that $s = t + 72$, $3t = 40$, and t is not integral. □

Lemma 10. *If $\Gamma_3(x) = \emptyset$ for some x then $\Gamma \cong \frac{1}{2}\binom{8}{4}$ or $\Gamma \cong G'_{40}$.*

Proof. $18 = \frac{144}{8} \leq |\Gamma_2(x)| \leq \frac{144}{4} = 36$. If $|\Gamma_2(x)| = 18$ then $v = 35$ and each μ-graph has 8 points, so that Γ is strongly regular. By Lemma 8, Γ is a Zara graph with $K = 5$, $e = 2$, and now the proposition in Section 2 shows that $\Gamma \cong \frac{1}{2}\binom{8}{4}$. Now let $v > 35$, and count c1iques. There are 8 5-c1iques on x, 48 cliques not on x meeting $\Gamma(x)$ and hence $\frac{8}{5}(v - 35)$ cliques contained in $\Gamma_2(x)$. Let $M_\mu = \{b \in \Gamma_2(x) \mid |\mu(x, b)| = \mu\}$. By Lemma 8 each clique C in $\Gamma_2(x)$ has 4 points in M_6, and 1 point in M_4 so $|M_6| = \frac{16}{5}(v - 35)$, $|M_4| = \frac{2}{5}(v - 35)$, and $v = 1 + 16 + \frac{18}{5}(v - 35) + |M_8|$, $144 = 4|M_4| + 6|M_6| + 8|M_8| = \frac{104}{5}(v - 35) + 8|M_8|$. It follows that $v < 45$, and since $5 \mid v$, we have $v = 40$, $|M_8| = 5$, $|M_6| = 16$, $|M_4| = 2$. Now enough information has been gathered to define all adjacencies without loss of generality (a rather tedious task). One finds a unique solution, namely the graph described in Section 4. □

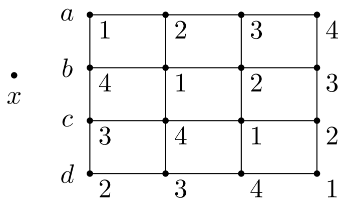
Lemma 11. *If $\Gamma_3(x) \neq \emptyset$ for all x, then $\Gamma \cong G_{40}$, the graph described in Section 3.*

Proof. Clearly it suffices to show that Γ is determined up to isomorphism.

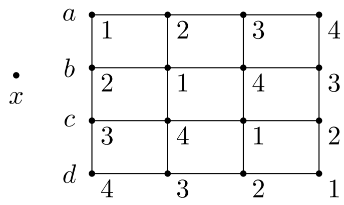
By Lemmas 1(iv), 7, 9, and 10 we have that for each $x \in \Gamma$ there is a unique antipode $\bar{x} \in \Gamma$ with $d(x, \bar{x}) = 3$. Clearly the antipodes of the points in $\Gamma(x)$ are in $\Gamma(\bar{x})$, so $x \mapsto \bar{x}$ is an automorphism of Γ.

Fix $x \in \Gamma$ and let $Z = \Gamma_2(x) \cap \Gamma_2(\bar{x})$ $(= M_8$, see Lemma 7). Then $Z = \{p, q, r, \bar{p}, \bar{q}, \bar{r}\}$. Thus given x we find a clique of size 8 $O_x = Z \cup \{x, \bar{x}\}$, and starting with another point, say $p \in O_x$, we find $O_p = O_x$. (Note that $\mu(x, p)$ and $\mu(x, \bar{p})$ partition $\Gamma(x)$.) Consequently, we have a partition of Γ in five 8-cocliques.

Consider the quotient $\bar{\Gamma}$ of Γ under identifying a with $\bar{a}$ for each $a \in \Gamma$. It has 20 points, a partition into five 4-cocliques and valency 16, i.e., it is the complete multipartite graph $K_{5\times4}$. There are two ways of distributing four 4-cocliques over $\Gamma(x)$, corresponding to the two Latin squares of order 4, namely

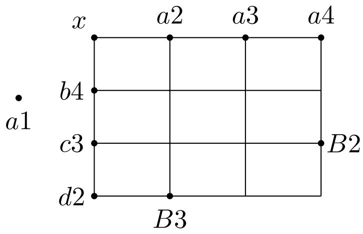


and



.

In the first case, label the 4 rows of $\Gamma(x)$ with a, b, c, d, and let al be the point in row a and coclique 1, etc. Let $A1 = \overline{a1}$, etc. Now we see locally at $a1$



but $B2 \sim B3$ contradiction.

Since $p \not\sim q$, $\mu(p,q) \cap \Gamma(x)$ is a coclique (and idem with $\bar{x}$ instead of x) so $|\mu(p,q) \cap \Gamma(x)| = 4$. Also, this remains a coclique in $\bar{\Gamma}$, so $\mu(p,q) \cap \Gamma(x)$ consists of four points labeled with the same digit. Since there are 4 points in Z distinct from p, $\bar{p}$ and each point in $\Gamma(x)$ has three neighbors in Z we see that w.l.o.g. the points p, $\bar{p}$, q, $\bar{q}$, r, $\bar{r}$ are adjacent to the points in $\Gamma(x)$ labeled 1 or 2, 3 or 4, 1 or 3, 2 or 4, 2 or 3, 1 or 4, respectively. Taking antipodes we find the edges between Z and $\Gamma(\bar{x})$. Also the edges between $\Gamma(x)$ and $\Gamma(\bar{x})$ are determined: e.g., the point $A1$ is not adjacent to a point in the same row or column as al and not adjacent to a point in coclique 1, so its neighbors in $\Gamma(x)$ must be $b4$, $b3$, $e4$, $e2$, $d3$, $d2$. $\square$

Thus we showed that for diam $\Gamma = 2, 3, 4$ we have $\Gamma \cong \frac{1}{2}\binom{8}{4}$, G_{40}, or G'_{40}, $\binom{8}{4}$, respectively. Clearly this proves the theorem.

Acknowledgement

We thank Prof. Timmesfeld for suggesting the way S_5, could act on a module 2^6 and thus providing us with the description of the graph G_{40}, given in Section 3.

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