Nonexistence of a distance-regular graph

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Abstract

We show that there is no partial linear space of girth 5 with valency 7 on 36 points. It follows that there is no distance-regular graph with intersection array $\{36, 28, 4; 1, 2, 24\}$ (on 625 vertices).

1 Introduction

For the concept of distance-regular graph, and any unexplained notation, cf. [1]. In the tables at the end of that book, the existence of a distance-regular graph with intersection array $\{36, 28, 4; 1, 2, 24\}$ is shown as undecided. Much happened since the publication of that book, and [3] is a survey of all that is new. But the tables at the end of this survey show no progress on this existence question. In this note we show that there is no such graph.

The fact that $\mu = 2$, implies that any such graph Γ is locally a partial linear space of girth (at least) 5. Now $k = 36$ and $\lambda = 7$ mean that that partial linear space has 36 points, and its collinearity graph is regular of valency 7. In this note we show that there is no such partial linear space.

The argument resembles the argument given in [2] to show that in this situation, if the partial linear space is connected then $k \geq \frac{1}{2}\lambda(\lambda + 3)$. The present case just escapes this bound, but some additional ad hoc arguments eliminate it.

The distance-2 graph of Γ would be strongly regular, and in fact strongly regular graphs with such parameters are known.

2 Nonexistence of a partial linear space

A *partial linear space* is a geometry with points and lines, where two lines meet in at most one point. Let xIL denote that the point x is incident with the line L. The *girth* g of a partial linear space is the length of a shortest cycle $x_0IL_1Ix_1IL_2...L_gIx_0$ of mutually distinct points and lines. Thus, girth at least 5 means that the partial linear space does not contain triangles or quadrangles.

Consider a partial linear space of girth at least 5 on 36 vertices, where each point is collinear with 7 other points. We shall derive a contradiction.

†Celebrated today, happens two weeks from now.

Fix a point x_0, and let there be m_i lines of size i (say, i-lines, $i \geq 2$) on x_0. The valency condition says $\sum m_i(i-1) = 7$. At distance 2 from x_0 there are $\sum m_i(i-1)(8-i)$ vertices, so this expression is not larger than $36 - 1 - 7 = 28$. Let us call a line *long* if it has size at least 5. If x_0 is not on any long lines, then we can bound $8 - i \geq 4$ and find $28 = 4 \sum m_i(i-1) \leq \sum m_i(i-1)(8-i) \leq 28$. It follows that equality holds everywhere and x_0 is on 4-lines only. But 3 does not divide 7, this is impossible.

We proved that each point is on a (unique) long line, so that the vertex set is partitioned by the long lines.

Consider a long line L of size m, $m = 5, 6, 7$. Then L has $m(8-m)$ neighbours, each on a different long line. So $5m(8-m) + m \leq 36$, a contradiction. It follows that each point is on an 8-line. But 8 does not divide 36, and we reach the final contradiction.

References

[1] A. E. Brouwer, A. M. Cohen & A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, 1989.

[2] A. E. Brouwer & A. Neumaier, *A remark on partial linear spaces of girth 5 with an application to strongly regular graphs*, Combinatorica **8** (1988) 57–61.

[3] E. R. van Dam, J. H. Koolen & H. Tanaka, *Distance-regular graphs*, Electr. J. Combin. (2016) #DS22.