Cocliques in the Kneser graph on the point-hyperplane flags of a projective space

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Dedicated to the memory of András Gács

Abstract

We prove an Erdős-Ko-Rado-type theorem for the Kneser graph on the point-hyperplane flags in a finite projective space.

1 Introduction

Let V be a vector space of finite dimension $n > 0$ over $\mathbf{F}_q$, and consider in the associated projective space PV the flags (P, H) consisting of an incident point-hyperplane pair (that is, with $P \subseteq H$), and call two such flags (P, H) and (P', H') adjacent when $P \not\subseteq H'$ and $P' \not\subseteq H$. This defines the Kneser graph $\Gamma(V)$. We study maximal cocliques in this graph.

Let $F = (X_1, ..., X_{n-1})$ be a maximal flag (chamber) in PV, that is, X_i is a subspace of V of vector space dimension i for all i, and $X_i \subseteq X_j$ for $i < j$. Using F we define the coclique $C_F = \{(P, H) \mid \exists i : P \subseteq X_i \subseteq H\}$. This coclique is maximal, of size $|C_F| = 1 + 2q + 3q^2 + ... + (n-1)q^{n-2}$.

For a coclique C in $\Gamma(V)$, define $Z(C) = \{P \mid (P, H) \in C \text{ for some } H\}$ (the points involved in C) and $Z^D(C) = \{H \mid (P, H) \in C \text{ for some } P\}$.

Theorem 1 *Let C be a coclique in $\Gamma(V)$ and let $Z = Z(C)$. Let $f(n) := 1 + 2q + 3q^2 + \ldots + (n-1)q^{n-2}$ and $g(n) := 1 + q + q^2 + \ldots + q^{n-2}$. Then*

(i) $|C| \leq f(n)$, and equality holds iff $C = C_F$ for some chamber F, and

(ii) $|Z| \leq g(n)$, and for $q > 2$ equality holds iff Z is a hyperplane.

Note that $g(n)$ is the number of points in a hyperplane, and $f(n) = qf(n-1)+g(n)$. Part (i) was conjectured, and partial results were obtained, in Mussche [2]. Both inequalities are sharp, and the theorem characterises equality in (i). What about equality in (ii)? In examples of type C_F the set Z is a hyperplane ($Z = X_{n-1}$), and equality holds. When $q = 2$ there are further examples of equality ($|Z| = g(n) = 2^{n-1} - 1$), described below.

Example 1. ($n = 3$) In the plane, take non collinear points P_i ($i = 1, 2, 3$). Let C consist of the flags (P_i, P_i+P_{i+1}) (indices mod 3). Then C is a coclique, and $|Z| = 3 = 2^{3-1} - 1$.

(This example also arises from the trivial one point coclique for $n = 2$ by the construction in Example 3.)

Example 2. ($n = 4$) Consider a plane π and a point ∞ outside π. For each point P in π, let P' be the third point of the line $P + \infty$. Label the points of π with the integers mod 7, so that the lines become $\{i + 1, i + 2, i + 4\}$ (mod 7). The seven flags $(i', \langle i', i + 1, i + 2, i + 4\rangle)$ form a coclique C, and $|Z| = 7 = 2^{4-1} - 1$.

(This is an honest sporadic example.)

Example 3. Let H be a hyperplane in V, and let D be a coclique in $\Gamma(H)$ with $|Z(D)| = 2^{n-2} - 1$. Let G be a hyperplane of H, such that G is disjoint from $Z(D)$. Let Q be a point outside H. Let C consist of the flags (P, H) with $P \subseteq G$, the flags $(P, N + Q)$ with $(P, N) \in D$, and $(Q, G + Q)$. Then C is a coclique, and $Z(C) = Z(D) \cup G \cup \{Q\}$, so that $|Z| = 2^{n-1} - 1$.

(We'll see later that this only gives something for $n \leq 5$.)

Example 4. Let T be a t-space in V, $0 < t < n$. Let D be a coclique in $\Gamma(T)$. Let E be a coclique in $\Gamma(V/T)$ such that $Z(E)$ has maximal size $2^{n-1-t} - 1$. Construct a coclique C in $\Gamma(V)$ by taking (i) all flags (P, H) with $P \subseteq T \subseteq H$, (ii) the flags (P, K) where $(P, K \cap T) \in D$, and (iii) the flags (Q, H) where $(Q + T, H) \in E$. Then $|Z(C)| = (2^t - 1) + 2^t(2^{n-1-t} - 1) = 2^{n-1} - 1$. The case that Z is a hyperplane corresponds to $t = n - 1$.

Note that the cocliques C_F are of this form (for every T in F).

The examples above describe all cases of equality:

Proposition 2 *Let V be an n-dimensional vector space over $\mathbf{F}_2$, and let C be a coclique in the graph $\Gamma(V)$. Then $|Z(C)| \leq g(n) = 2^{n-1} - 1$ with equality if and only if C arises by the construction of Example 2, 3, or 4.*

1.1 Rank 1 matrices

The graph $\Gamma(V)$ can also be described in terms of rank 1 matrices. Represent points by column vectors p and hyperplanes by row vectors h, then a point-hyperplane pair can be represented by the rank 1 matrix ph. If the point is contained in the hyperplane then $hp = 0$, that is $\operatorname{tr} ph = 0$. Rank 1 matrices that differ by a constant factor represent the same point-hyperplane pair.

Two incident point-hyperplane pairs $x = ph$ and $x' = p'h'$ are nonadjacent when $(hp')(h'p) = 0$, i.e., when $\operatorname{tr} xx' = 0$. Thus, finding the maximal cocliques in $\Gamma(V)$ is equivalent to finding the intersections of the maximal totally isotropic subspaces for the symmetric bilinear form $(x, y) = \operatorname{tr} xy$ on the space $M_n(\mathbf{F}_q)$ (of matrices of order n over $\mathbf{F}_q$) with the space of trace 0 rank-1 matrices. The extremal example C_F is conjugate to the example of all rank-1 strictly upper-triangular matrices.

1.2 The thin case

There is a thin analog ($q = 1$ version) of our problem. Consider for an n-set V the pairs (P, H), where $|P| = 1$, $|H| = n - 1$, and $P \subseteq H \subseteq V$. Call (P, H) and (P', H') adjacent when $P \not\subseteq H'$ and $P' \not\subseteq H$, that is, when $(P', H') = (V \setminus H, V \setminus P)$. Here the graph is the union of $\binom{n}{2}$ components K_2. A maximal coclique is obtained by taking a single vertex from each K_2, so that $|C| \leq \binom{n}{2} = f(n)$ and $|Z| \leq n$. Here $g(n) = n - 1$, and $|Z| \leq g(n)$ only holds for $n = 1, 2$.

2 Maximum-size cocliques

Proof of Theorem 1. The problem is self-dual, so all that is proved for points and hyperplanes, also holds for hyperplanes and points. In particular, $g(n)$ will also be an upper bound for the number $|Z^D(C)|$ of hyperplanes involved in C.

We may assume that C is maximal. It follows that C has a certain linear structure:

Lemma 3 *Let C be a maximal coclique in $\Gamma(V)$, and let $H \in Z^D(C)$. Then H has a subspace $S(H)$ such that $(P, H) \in C$ if and only if $P \subseteq S(H)$. If $H, K \in Z^D(C)$ then $S(H) \subseteq K$ or $S(K) \subseteq H$ (or both).*

Proof. If $(P,H),(Q,H) \in C$ and R is a point on the line $P+Q$, then (R,H) is non adjacent to every (O,K) in the coclique: if $O \in H$ then this is clear, if $O \notin H$ then $P,Q \in K$, but then also R. So also $(R,H) \in C$ since C is maximal. If P is in $S(H)$ but not in K, and Q is in $S(K)$ but not in H, then (P,H) and (Q,K) are adjacent in $\Gamma(V)$, impossible. □

We may of course extend the definition of $S(H)$ to all hyperplanes by putting $S(H) = \{\mathbf{0}\}$ for hyperplanes not in $Z^D(C)$

In the proof we will use induction on $n = \dim V$. If T is a subspace of V, then $C_T := \{(P, H \cap T) \mid (P,H) \in C,\ P \subseteq T,\ T \not\subseteq H\}$ is a coclique in the graph $\Gamma(T)$. Also $C^T := \{(P+T, H) \mid (P,H) \in C,\ P \not\subseteq T,\ T \subseteq H\}$ is a coclique in the graph $\Gamma(V/T)$ (we have already met these cocliques in the description of Example 4).

Let the maximal dimension of $S(H)$ (over all H) be s, and let H be a hyperplane with $\dim S(H) = s$. We have $1 \leq s \leq n-1$.

Lemma 4 *If $s = n-1$, then $|C| \leq qf(n-1) + g(n) = f(n)$ and $|Z| = g(n)$, and $Z = H$.*

Proof. If $s = n-1$ then $S(H) = H$, so that $S(H) \not\subseteq K$ for $K \neq H$ and therefore $S(K) \subseteq H$ for all K, i.e., $Z = H$ and $|Z| = g(n)$. The coclique C consists of the $g(n)$ flags (P,H) together with at most $qf(n-1)$ flags (P,K) with $P \subseteq H$, $K \neq H$, so that $|C| \leq qf(n-1) + g(n) = f(n)$, as desired. □

Lemma 5 *If $s < n-1$, then $|C| < qf(n-1) + g(n) = f(n)$.*

Proof. Count the three types of elements in the coclique: a: flags involving H, b: flags (P,M) with $P \subseteq H \neq M$, c: flags (Q,K) with $Q \not\subseteq H$ (and hence $S(H) \subseteq K$).

a: $\dfrac{q^s-1}{q-1}$; b: at most $qf(n-1)$; c: at most $g(n-s)q^{s-1}$.

a: This is because $\dim S(H) = s$. b: By induction C_H has at most size $f(n-1)$, and each hyperplane intersection $M \cap H$ occurs for at most q hyperplanes M. c: For a flag $(Q,K) \in C$ with $Q \not\subseteq H$, the hyperplane K contains $S(H)$; the flags $(Q+S(H), K)$ form the coclique $C^{S(H)}$ in $\Gamma(V/S(H))$, so by induction there are at most $g(n-s)$ such hyperplanes K. For each K there are at most q^{s-1} points Q outside H with $(Q,K) \in C$.

It follows that $|C| \leq qf(n-1) + 2 \cdot \dfrac{q^{n-2}-1}{q-1} < f(n)$, as desired. □

Since there is strict inequality here, equality $|C| = f(n)$ only occurs for $s = n-1$, where there is a hyperplane H such that $(P, H) \in C$ for all points $P \subseteq H$, and all other elements of C restrict to a coclique with equality in $\Gamma(H)$. By induction it follows that equality implies that $C = C_F$ for some flag F.

Lemma 6 *Let $q > 2$ and $s < n-1$. Then $|Z| < g(n)$.*

Proof. We estimate $|Z|$ using the same three types of flags as above, and doing essentially the same counting.
a: This gives the $(q^s - 1)/(q-1)$ points of $S(H)$.
b: These flags give the points involved in the coclique C_H, so by induction at most $g(n-1)$.
c: Here again we use the bound for the number of hyperplanes involved in $C^{S(H)}$ to estimate the number of points (outside H) involved: $g(n-s)q^{s-1}$.
Adding all up gives

$$|Z| \leq \frac{q^s - 1}{q-1} + g(n-1) + g(n-s) \cdot q^{s-1} < g(n).$$

□

For $q = 2$ and $s > 1$ more work is required, because now the above estimate of $|Z|$ is $2^{s-1} - 1$ larger than the desired upper bound $g(n)$. For $q = 2$ and $s = 1$ we do get $|Z| \leq g(n)$, but we cannot conclude that Z is a hyperplane.

As observed, there are at most $g(n-s)$ hyperplanes K involved in flags $(P, K) \in C$ with $P \not\subseteq S(H)$ (and hence $S(H) \subseteq K$). If the number of such K is strictly smaller than this, or if $S(K) \subseteq H$ for one of them, then the upper bound on $|Z|$ improves by 2^{s-1}, and $|Z| < g(n)$. If for at least two of them $\dim S(K) < s$, then the same holds. We may therefore assume that there are precisely $g(n-s)$ such hyperplanes K, for none of them $S(K) \subseteq H$, and for all of them with at most one exception $\dim S(K) = s$.

Let $E(H)$ be the set of points P in $S(H)$ that occur in a single flag only, namely in (P, H). Let $e(H) = |E(H)|$. The (first) term $2^s - 1$ in the bound can in fact be replaced by $e(H)$, because the points of $S(H) \setminus E(H)$ are also points in flags of type b, that are counted by the term $g(n-1)$. If $e(H) \leq 2^{s-1}$ then $|Z| \leq g(n)$ as desired. So, we may assume that $e(H) \geq 2^{s-1} + 1$.

First consider the case $s = n-2$. We have $g(n-s) = g(2) = 1$, and $S(H)$ is a hyperplane in the unique K. Now if $S(K)$ has dimension $t (\leq s)$, then $S(H)$ and $S(K)$ have $2^{t-1}-1$ points in common, $S(K)$ contributes at most 2^{t-1} points outside H to $|Z|$ and $e(H) \leq 2^s - 2^{t-1}$, proving the bound. So, we may assume $2 \leq s \leq n-3$.

Let $\mathcal{H}$ be the collection of all hyperplanes H such that $\dim S(H) = s$. All we have said about H above holds for all $H \in \mathcal{H}$. Make a directed graph Δ with vertex set $\mathcal{H}$ and arrows $H \to K$ when $S(H) \subseteq K$. (As we have seen, this now implies $K \not\to H$, so that Δ is a tournament.)

The outdegree of Δ is $g(n-s)$ or $g(n-s)-1$ at each vertex. It follows that the indegree at some vertex is at least $g(n-s)-1$.

First suppose that the indegree of H is at least $g(n-s)$. The number of points P involved in flags (P, M) where $M \to H$ is bounded above by $g(n-1)$. On the other hand it is bounded from below by $(g(n-s)-1)(2^{s-1}+1)+(2^s-1) > g(n-1)$ since $e(M) \geq 2^{s-1}+1$ for each such M. This is a contradiction. It follows that all outdegrees and all indegrees of Δ are precisely $g(n-s)-1$.

Our estimate now becomes $|Z| \leq e(H)+g(n-1)+(g(n-s)-1)2^{s-1}+2^{s-2}$ and $|Z| \leq g(n)$ will follow if $e(H) \leq 3 \cdot 2^{s-2}$. So, we may assume that $e(H) \geq 3 \cdot 2^{s-2}+1$ for all $H \in \mathcal{H}$. Again we bound the number of points P from below. We find the contradiction $(g(n-s)-2)(3 \cdot 2^{s-2}+1)+(2^s-1) > g(n-1)$ if $s \leq n-4$. So, we may assume that $s = n-3$.

Now $S(H)$ has codimension 2 in each K, so codimension at most 2 in each $S(K)$, so that $\dim(S(H) \cap S(K)) \geq s-2$. It follows that $e(H) \leq 3 \cdot 2^{s-2}$, as desired.

This finally proves part (ii) of the theorem. □

3 Maximum number of points

Proof of Proposition 2. The inequality $|Z| \leq 2^{n-1}-1$ was shown already, so assume we have equality. We recall the counting in the above proof. Given a hyperplane H, consider the three types of flags: (a) flags (P, H), (b) flags (P, M) with $P \subseteq H$ and $M \neq H$, (c) flags (Q, K) with $Q \not\subseteq H$, Correspondingly we get three contributions to $|Z|$: the points P from flags of type (b), the points Q from flags of type (c), and the points P from flags (P, H) that were not counted in (b), i.e., that occur in flags (P, H) only: the set $E(H)$ of size $e(H) \leq 2^s - 1$.

Let H be a hyperplane with $\dim S(H) = s$ maximal. We find $|Z| \leq e(H) + g(n-1) + g(n-s)2^{s-1} \leq 2^{n-1} + 2^{s-1} - 2$. This estimate is precisely

$2^{s-1}-1$ too large, so it cannot be improved by 2^{s-1}. It follows that there are precisely $g(n-s)$ hyperplanes $(H \neq) K \in Z^D(C)$ on $S(H)$, and all except at most one have $\dim S(K) = s$. None of these K satisfies $S(K) \subseteq H$. It also follows that $e(H) \geq 2^{s-1}$. By Lemma 4 we may suppose that $1 \leq s \leq n-2$.

Lemma 7 *If $s = 1$, then $n \leq 4$.*

Proof. For $s = 1$ the counting $g(n) = |Z| \leq 2^s - 1 + g(n-1) + g(n-s)2^{s-1}$ holds with equality. This means that not only P is the only point in H, for $(P, H) \in C$ (since $s = 1$), but also that H is the only such hyperplane on P, because the counting is exact. Hence C involves equally many points as hyperplanes, $g(n)$. Above we saw that if $S(H) \subseteq K$, then $S(K) \not\subseteq H$, which in this case means that for two flags (P, H) and (Q, K) exactly one of $P \subseteq K$ and $Q \subseteq H$ holds. Consider the point-hyperplane nonincidence matrix A of $PG(n-1, 2)$. The 2-rank of this matrix is n, because for a suitable labeling of points and hyperplanes we have $A = BB^T$, where B is the $(2^n - 1) \times n$ matrix whose rows are the nonzero vectors in V. The submatrix M (of rank at most n of course) corresponding to the points and hyperplanes in C, again suitably labeled, has the property that $M + M^T = J - I$ of (almost full) 2-rank $2^{n-1} - 2$. It follows that $2^{n-1} - 2 \leq 2n$, so $n \leq 4$. □

As before let Δ be the directed graph on the set $\mathcal{H}$ of hyperplanes H with $\dim S(H) = s$, defined by $H \to K$ if $S(H) \subseteq K$.

Lemma 8 *The digraph Δ is a tournament, and we have one of three cases, where $k = g(n-s) = 2^{n-s-1} - 1$ and $v = |\mathcal{H}|$ is the number of vertices:*

a) All indegrees and all outdegrees are equal to k and $v = 2k + 1$,

b) All indegrees and all outdegrees are equal to $k - 1$ and $v = 2k - 1$,

c) All indegrees are $k - 1$ or k (and both occur) and all outdegrees are $k - 1$ or k (and both occur) and $v = 2k$.

Proof. We have seen already that Δ is a tournament, and that each vertex has outdegree k or $k - 1$, where $k = g(n-s)$. If Δ has v vertices, then at each vertex the indegree equals $v - 1 - k$ or $v - k$, since Δ is a tournament. Since the average indegree equals the average outdegree, we have one of the stated cases. □

Lemma 9 *If H has outdegree $k - 1$, so that (exactly) one hyperplane K on $S(H)$ has $\dim S(K) = t < s$, then $e(H) \geq 2^s - 2^{t-1} \geq 3.2^{s-2}$.*

Proof. The contribution to $|Z|$ from hyperplanes K is now at most $a := (g(n-s)-1)2^{s-1} + 2^{t-1}$, so $e(H) \geq |Z| - g(n-1) - a = 2^s - 2^{t-1}$. $\square$

Lemma 10 *If $k = 1$, and H is a vertex of Δ with indegree 1, then $e(H) = 2^{s-1}$, and H also has outdegree 1.*

Proof. If H has indegree 1, say $M \to H$, then $S(M)$ is a hyperplane in H (since $s = n-2$) and covers at least $2^{s-1}-1$ of the points of $S(H)$. It follows that $e(H) = 2^{s-1}$, and H also has outdegree 1 by Lemma 9. $\square$

Lemma 11 *If H is a vertex of Δ with indegree k, then each of its inneighbours has outdegree k. If $k > 1$ then $e(M) = 2^{s-1}$ for each inneighbour M, and all inneighbours have the same $2^{s-1} - 1$ nonunique points.*

Proof. Look at the coclique D in $\Gamma(H)$ consisting of the flags $(P, M \cap H)$ for $M \to H$. We have $|Z(D)| \leq g(n-1)$. Each M has $e(M) \geq 2^{s-1}$ unique points, and contributes $2^s - 1$ points altogether to $Z(D)$, so we find at least $k.2^{s-1} + (2^{s-1} - 1) = 2^{n-2} - 1 = g(n-1)$ points in $Z(D)$. Equality must hold, so if $k > 1$ all inneighbours M have $e(M) = 2^{s-1}$. By Lemma 9 they all have outdegree k. Obviously, if $k = 1$, an inneighbour cannot have outdegree $k - 1$. $\square$

Lemma 12 *If all indegrees and all outdegrees are k, then there is a subspace T of V of dimension $s-1$ such that for each $H \in \mathcal{H}$ we have $S(H) \setminus E(H) = T$.*

Proof. The counting of the previous lemma shows for $k > 1$ that each inneighbour M of H has the same set $S(M) \setminus E(M)$ of size $2^{s-1}-1$. Since this covers $S(H) \setminus E(H)$ which has the same size, we conclude that $S(M) \setminus E(M) = S(H) \setminus E(H)$ when there is an arrow $M \to H$. If $k = 1$, the same conclusion follows from Lemma 10.

This set of size $2^{s-1} - 1$ is the intersection of all $S(H)$, $H \in \mathcal{H}$, so it is a subspace. $\square$

Lemma 13 *If all indegrees and all outdegrees are k, and $s > 1$, then C arises by the construction of Example 4 with T of dimension $t = s - 1$.*

Proof. We have $|\mathcal{H}| = v = 2k + 1 = 2^{n-s} - 1$. Each $H \in \mathcal{H}$ contributes $e(H) = 2^{s-1}$ unique points to Z, and all vertices contribute the same set of $|T| = 2^{s-1} - 1$ common points, $2^{n-1} - 1 = |Z|$ points altogether. Now consider another flag $(P, H) \in C$ (with $\dim S(H) < s$). The point P cannot be one of the unique points, so $P \subseteq T$. This shows that C arises as in Example 4. □

Lemma 14 *If $s = n - 2$ then C arises as in Example 1, 3 or 4.*

Proof. If $s = n - 2$, then $k = 1$. Now Δ has at most 3 vertices. If $v = 3$, then it is a directed 3-cycle. By Lemma 13, either C arises by the construction of Example 4, or $s = 1$, $n = 3$, in which case we have Example 1.

By Lemma 10, if $v \neq 3$, then $v = 1$. Let K be the unique hyperplane in $Z^D(C)$ with $S(H) \subseteq K$, and suppose $\dim S(K) = t$. Then $1 \leq t < s$. If $t = 1$, then we have Example 3 (with $G = S(H)$ and $Q = S(K)$). Let $t > 1$.

The subspace $S(H)$ is a hyperplane in both H and K, so $S(H) = H \cap K$, and the $2^{t-1} - 1$ points in $H \cap S(K)$ are not unique in H. By Lemma 9 we have $e(H) = 2^{n-2} - 2^{t-1}$. The situation is tight again, so the flags (P, M) contribute precisely $2^{n-2} - 1$ points. Put $T := S(K) \cap S(H)$, so that $\dim T = t - 1$. Since $t > 1$, $T \neq 0$. Compare (P, M) and (Q, K). If $S(K) \not\subseteq M$, then $S(M) \subseteq K \cap H = S(H)$. Since $E(H) = S(H) \setminus T$, this means that $T \subseteq M$ or $S(M) \subseteq T$. This is Example 4 (applied to a coclique from Example 3). □

Now assume $1 \leq s \leq n - 3$, so that $k \geq 3$.

Lemma 15 *Case c) of Lemma 8 does not occur.*

Proof. In case c), suppose that we have $v = 2k$ vertices, b of which have outdegree $k - 1$ (and indegree k) and $v - b$ of which have outdegree k (and indegree $k - 1$). The total number of arrows is $vk - b = vk - (v - b)$, so that $b = v/2 = k$. Let A be the set of vertices with outdegree k, B that with outdegree $k - 1$. By Lemma 11 we have all arrows from A to B, and this fills up all outarrows in A. Then there are no arrows inside A, contradiction. So case c) does not occur. □

Lemma 16 *Case b) of Lemma 8 does not occur.*

Proof. In case b) we have $e(H) \geq 3.2^{s-2}$ for each $H \in \mathcal{H}$, by Lemma 9. The $k-1$ inneighbours of any vertex H contribute at least $3.2^{s-2}(k-1)+2^{s-2}-1 = 3.2^{n-3} - 5.2^{s-2} - 1$ points in H, so $3.2^{n-3} - 5.2^{s-2} - 1 \leq 2^{n-2} - 1$, that is, $2^{n-3} \leq 5.2^{s-2}$ and hence $s \geq n-3$. Since we are assuming $s < n-2$ this means $s = n-3$. Now $k = 3$ and all vertices have indegree and outdegree 2. The digraph Δ has 5 vertices. They can be labeled H_i, $i = 0, 1, 2, 3, 4$, such that $H_i \to H_j$ iff $j = i+1$ or $j = i+2 \pmod 5$. Now H_2 contains S_{H_1} and S_{H_0}, both of dimension $s = n-3$. Since H_1 contributes at least 2^{s-1} points Q, we have $S_{H_1} \neq S_{H_0}$. So $S_{H_0} + S_{H_1}$ is an $(n-2)$-space contained in the $(n-2)$-space $H_1 \cap H_2$. Hence $S_{H_0} + S_{H_1} = H_1 \cap H_2$. Since $S_{H_1} \not\subseteq H_0$, also $S_{H_0} = H_0 \cap H_1 \cap H_2$. Now S_{H_0} and S_{H_1} meet in an $(n-4)$-space, so at least $2^{n-4} - 1$ points are not unique in H_0. The standard counting gives $|Z| \leq (2^{n-2}-1)+(2.2^{s-1}+2^{s-2})+(2^s-2^{s-1}) = 2^{n-1}-1-2^{n-5}$, contradiction. So case b) does not occur. □

So we always have example 1, 3 or 4, except for the case $s = 1$, $n = 4$.

Lemma 17 *If $s = 1$, $n = 4$, then we have Example 2.*

Proof. Let $Z^D = Z^D(C)$. Here $|Z| = |Z^D| = 7$, and Δ is a tournament on seven vertices with all in- and outdegrees equal to 3. We first show that Z is a cap, that is, no three points in Z are collinear. Indeed, if $(P, H), (Q, K), (R, L) \in C$, and P, Q, R are collinear, and $K \to H$, then $Q \subseteq H$ but also $P \subseteq H$, hence $R \subseteq H$, that is $L \to H$. But now there can be no consistent arrow between K and L. So the points form a cap of size 7. The only maximal caps of $PG(3, 2)$ are the elliptic quadric (of size 5) and the complement of a plane (of size 8), cf. [1, Theorem 18.2.1], so there is a (hyper)plane π disjoint from Z and a point A not in Z and not in π. If $(P, H), (Q, K) \in C$ and $H \cap K$ is a line in π, then $(P, H), (Q, K)$ are adjacent, impossible. So the seven planes in Z^D meet π in seven different lines. The situation is self-dual, so there is a point not on any plane in Z^D, necessarily the point A. So, the seven points of Z are the points different from A outside π, and the seven planes of Z^D are the planes different from π not on A. One now quickly establishes that we have Example 2. □

This finishes the proof of Proposition 2. □

Remarks. The construction of Example 4 does not change the value of $n - s$. The hyperplane has $n - s = 1$, in Example 3 we have $n - s = 2$

and in Example 2 we have $n - s = 3$. So, all examples of $|Z| = g(n)$ have $1 \leq n - s \leq 3$.

Example 3 needs as ingredient a coclique D with a hyperplane disjoint from $Z(D)$. It follows that $Z(D)$ cannot contain linear subspaces of dimension larger than 1, so that the coclique D must have $s = 1$. Hence the construction of Example 3 applies only for (i) the hyperplane example for $n = 2$, (ii) Example 1, (iii) Example 2. The resulting examples are Example 1, and two further examples with $n = 4$, $s = 2$ and $n = 5$, $s = 3$.

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