A Hilton-Milner theorem for vector spaces

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Abstract

We show for $k \geq 2$ that if $q \geq 3$, $n \geq 2k+1$ or $q = 2$, $n \geq 2k+2$, then any intersecting family $\mathcal{F}$ of k-subspaces of an n-dimensional vector space over $GF(q)$ with $\bigcap_{F \in \mathcal{F}} F = 0$ has size at most $\left[{n-1 \atop k-1}\right] - q^{k(k-1)}\left[{n-k-1 \atop k-1}\right] + q^k$. This bound is sharp as is shown by Hilton-Milner type families. As an application of this result, we determine the chromatic number of the corresponding q-Kneser graphs.

1 Introduction

1.1 Sets

In 1961, Erdős, Ko and Rado [4] proved that if $\mathcal{F}$ is a k-uniform intersecting family of subsets of an n-element set X, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ when $2k \leq n$. Furthermore they proved that if $2k+1 \leq n$, then equality holds if and only if $\mathcal{F}$ is the family of all subsets containing a fixed element $x \in X$.

For any family $\mathcal{F}$ of sets the *covering number* $\tau(\mathcal{F})$ is the minimum size of a set that meets all $F \in \mathcal{F}$. The result of Erdős, Ko and Rado states that to obtain an intersecting family of maximum size, one has to consider a family with $\tau(\mathcal{F}) = 1$ when $2k+1 \leq n$.

Hilton and Milner [13] determined the maximum size of an intersecting family with $\tau(\mathcal{F}) \geq 2$.

Theorem 1.1 (Hilton & Milner [13]) *Let $\mathcal{F} \subset \binom{X}{k}$ be an intersecting family with $k \geq 2$, $n \geq 2k+1$ and $\tau(\mathcal{F}) \geq 2$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$. The families achieving that size are*

(i) for any k-subset F and $x \in X \setminus F$ the family

$$\{F\} \cup \{G \in \tbinom{X}{k} : x \in G,\ F \cap G \neq \emptyset\},$$

(ii) if $k = 3$, then for any 3-subset S the family

$$\{F \in \tbinom{X}{3} : |F \cap S| \geq 2\}.$$

In this paper we will be interested in the q-analogue of Theorem 1.1.

1.2 Vector spaces

The q-analogue of questions about sets and subsets are questions about vector spaces and subspaces. For a prime power q, and an n-dimensional vector space V over $GF(q)$, let $\begin{bmatrix} V \\ k \end{bmatrix}$ denote the family of k-subspaces of V.

In 1975, Hsieh [14] proved the q-analogue of the theorem of Erdős, Ko and Rado for $2k+1 \leq n$. Greene and Kleitman [12] found an elegant proof for the case where $k \mid n$, settling the missing $n = 2k$ case.

A family $\mathcal{F}$ of k-subspaces of V is called *t-intersecting* if $\dim(F_1 \cap F_2) \geq t$ for any $F_1, F_2 \in \mathcal{F}$. In 1986, Frankl and Wilson [9] proved the following result giving the maximum size of a t-intersecting family of k-spaces for $2k - t \leq n$.

Theorem 1.2 (Frankl & Wilson [9]) *Let V be a vector space over $GF(q)$ of dimension n. For any t-intersecting family $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$ we have*

$|\mathcal{F}| \leq \begin{bmatrix} n-t \\ k-t \end{bmatrix}$ *if* $2k \leq n$,

and

$|\mathcal{F}| \leq \begin{bmatrix} 2k-t \\ k \end{bmatrix}$ *if* $2k - t \leq n \leq 2k$.

These bounds are best possible.

Let the *covering number* $\tau(\mathcal{F})$ of a family $\mathcal{F}$ of subspaces of V be defined as the minimal dimension of a subspace of V meeting all elements of $\mathcal{F}$ nontrivially.

Already Hsieh's proof showed that if $t = 1$ and $n \geq 2k + 1$ then only *point-pencils*, that is, families $\mathcal{F}$ with $\tau(\mathcal{F}) = 1$, can achieve the bound in Theorem 1.2. We will prove a q-analogue of Theorem 1.1 for intersecting families of subspaces with $\tau(\mathcal{F}) \geq 2$.

Let us first remark that for a fixed 1-subspace $E \leqslant V$ and a k-subspace U with $E \nleqslant U$ the family $\mathcal{F}_{E,U} = \{U\} \cup \{W \in \begin{bmatrix} V \\ k \end{bmatrix} : E \leqslant W,\ \dim(W \cap U) \geq 1\}$ is not maximal as we can add all subspaces in $\begin{bmatrix} E+U \\ k \end{bmatrix}$. We will say that $\mathcal{F}$ is an *HM-type family* if

$$\mathcal{F} = \left\{W \in \begin{bmatrix} V \\ k \end{bmatrix} : E \leqslant W,\ \dim(W \cap U) \geq 1\right\} \cup \begin{bmatrix} E+U \\ k \end{bmatrix}$$

for some fixed $E \in \begin{bmatrix} V \\ 1 \end{bmatrix}$ and $U \in \begin{bmatrix} V \\ k \end{bmatrix}$ with $E \nleqslant U$. Note that the size of an HM-type family is

$$|\mathcal{F}| = f(n,k,q) := \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} - q^{k(k-1)} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} + q^k. \tag{1.1}$$

The main result of the paper is the following theorem.

Theorem 1.3 *Let V be an n-dimensional vector space over $GF(q)$, and let $k \geq 3$. If $q \geq 3$ and $n \geq 2k+1$ or $q = 2$ and $n \geq 2k+2$, then for any intersecting family $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$ with $\tau(\mathcal{F}) \geq 2$ we have $|\mathcal{F}| \leq f(n,k,q)$ (with $f(n,k,q)$ as in (1.1)). When equality holds, either $\mathcal{F}$ is an HM-type family, or $k = 3$ and*

$$\mathcal{F} = \mathcal{F}_3 = \{F \in \begin{bmatrix} V \\ k \end{bmatrix} : \dim(S \cap F) \geq 2\}$$

for some $S \in \begin{bmatrix} V \\ 3 \end{bmatrix}$.

Furthermore, if $k \geq 4$, then there exists an $\epsilon > 0$ (independent of n, k, q) such that if $|\mathcal{F}| \geq (1-\epsilon) f(n,k,q)$, then $\mathcal{F}$ is a subfamily of an HM-type family.

If $k = 2$, then a maximal intersecting family $\mathcal{F}$ of k-spaces with $\tau(\mathcal{F}) > 1$ is the family of all lines in a plane, and the conclusion of the theorem holds.

After proving the above theorem in Section 2, we apply this result to determine the chromatic number of q-Kneser graphs. The vertex set of the q-Kneser graph $qK_{n:k}$ is $\begin{bmatrix} V \\ k \end{bmatrix}$, where V is an n-dimensional vector space over $GF(q)$. Two vertices of $qK_{n:k}$ are adjacent if and only if the corresponding k-subspaces are disjoint (i.e., meet in 0). Section 4 contains the proof of the following theorem.

Theorem 1.4 *If $k \geq 3$ and $q \geq 3$, $n \geq 2k+1$ or $q = 2$, $n \geq 2k+2$, then for the chromatic number of the q-Kneser graph we have $\chi(qK_{n:k}) = \begin{bmatrix} n-k+1 \\ 1 \end{bmatrix}$. Moreover, each color class of a minimum coloring is a point-pencil and the points determining a color are the points of an $(n-k+1)$-dimensional subspace.*

In Section 5 we prove the non-uniform version of the Erdős-Ko-Rado theorem.

Theorem 1.5 *Let $\mathcal{F}$ be an intersecting family of subspaces of a vector space V of dimension n. Then*

(i) if n is odd, then

$$|\mathcal{F}| \leq \sum_{i>n/2} \begin{bmatrix} n \\ i \end{bmatrix},$$

(ii) if n is even, then

$$|\mathcal{F}| \leq \begin{bmatrix} n-1 \\ n/2-1 \end{bmatrix} + \sum_{i>n/2} \begin{bmatrix} n \\ i \end{bmatrix}.$$

For odd n equality holds only if $\mathcal{F} = \begin{bmatrix} V \\ >n/2 \end{bmatrix}$. For even n equality holds only if $\mathcal{F} = \begin{bmatrix} V \\ >n/2 \end{bmatrix} \cup \{F \in \begin{bmatrix} V \\ n/2 \end{bmatrix} : E \leqslant F\}$ for some $E \in \begin{bmatrix} V \\ 1 \end{bmatrix}$, or if $\mathcal{F} = \begin{bmatrix} V \\ >n/2 \end{bmatrix} \cup \begin{bmatrix} U \\ n/2 \end{bmatrix}$ for some $U \in \begin{bmatrix} V \\ n-1 \end{bmatrix}$.

Note that Theorem 1.5 follows from the profile polytope of intersecting families which was determined implicitly by Bey [1] and explicitly by Gerbner and Patkós [10], but the proof we present in Section 5 is direct and very simple.

2 Proof of Theorem 1.3

This section contains the proof of Theorem 1.3 which we divide into two cases.

2.1 The case $\tau(\mathcal{F}) = 2$

For any $A \leqslant V$ and $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$ let $\mathcal{F}_A = \{F \in \mathcal{F} : A \leqslant F\}$.

Before starting with the proof let us state some easy technical lemmas.

Lemma 2.1 *Let $a \geq 0$ and $n \geq k \geq a+1$ and $q \geq 2$. Then*

$$\begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-a-1 \\ k-a-1 \end{bmatrix} < \frac{1}{(q-1)q^{n-2k}} \begin{bmatrix} n-a \\ k-a \end{bmatrix}.$$

Proof. The inequality to be proved simplifies to

$$(q^{k-a}-1)(q^k-1)q^{n-2k} < q^{n-a} - 1. \qquad \square$$

Lemma 2.2 *Let $E \in \begin{bmatrix} V \\ 1 \end{bmatrix}$. If $E \not\leqslant L \leq V$, where L is an l-subspace, then the number of k-subspaces of V containing E and intersecting L is at least $\begin{bmatrix} l \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} - q \begin{bmatrix} l \\ 2 \end{bmatrix} \begin{bmatrix} n-3 \\ k-3 \end{bmatrix}$ (with equality for $l = 2$), and at most $\begin{bmatrix} l \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}$.*

Proof. The k-spaces containing E and intersecting L in a 1-dimensional space are counted exactly once in the first term. Those subspaces that intersect L in a 2-dimensional space are counted $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = q+1$ times in the first term and $-q$ times in the second term, thus once overall. If a subspace intersects L in a subspace of dimension $i \geq 3$, then it is counted $\begin{bmatrix} i \\ 1 \end{bmatrix}$ times in the first term and $-q\begin{bmatrix} i \\ 2 \end{bmatrix}$ times in the second term, thus a negative number of times overall. $\square$

Our next lemma gives bounds on the size of an HM-type family that are easier to work with than the precise formula mentioned in the introduction.

Lemma 2.3 *Let $n \geq 2k+1$, $k \geq 3$ and $q \geq 2$. If $\mathcal{F}$ is an HM-type family, then $(1 - \frac{1}{q^3-q}) \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} < \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} - q \begin{bmatrix} k \\ 2 \end{bmatrix} \begin{bmatrix} n-3 \\ k-3 \end{bmatrix} \leq f(n,k,q) = |\mathcal{F}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}$.*

Proof. The first inequality follows immediately from Lemma 2.1 by noting that $q\begin{bmatrix} k \\ 2 \end{bmatrix} = \begin{bmatrix} k \\ 1 \end{bmatrix}(\begin{bmatrix} k \\ 1 \end{bmatrix} - 1)/(q+1)$ and $n \geq 2k+1$. $\square$

Lemma 2.4 *If a subspace S does not intersect each element of $\mathcal{F}$, then there is a subspace $T > S$ with $\dim T = \dim S + 1$ and $|\mathcal{F}_T| \geq |\mathcal{F}_S| / \begin{bmatrix} k \\ 1 \end{bmatrix}$.*

Proof. There is an $F \in \mathcal{F}$ such that $S \cap F = 0$. Average over all $T = S + E$ where E is a 1-subspace of F. $\square$

Lemma 2.5 *If an s-dimensional subspace S does not intersect each element of $\mathcal{F}$, then $|\mathcal{F}_S| \leq \left[{k \atop 1}\right]\left[{n-s-1 \atop k-s-1}\right]$.*

Proof. There is an $(s+1)$-space T with $\left[{n-s-1 \atop k-s-1}\right] \geq |\mathcal{F}_T| \geq |\mathcal{F}_S|/\left[{k \atop 1}\right]$. $\square$

Corollary 2.6 *Let $\mathcal{F} \subseteq \left[{V \atop k}\right]$ be an intersecting family with $\tau(\mathcal{F}) \geq s$. Then for any i-space $L \leqslant V$ with $i \leq s$ we have $|\mathcal{F}_L| \leq \left[{k \atop 1}\right]^{s-i}\left[{n-s \atop k-s}\right]$.* $\square$

Proof. By $\tau(\mathcal{F}) \geq s$ we know that for any j-space A, $j < s$, there exists an $F \in \mathcal{F}$ disjoint from A. Now apply Lemma 2.4 $s - i$ times. $\square$

Before proving the q-analogue of the theorem of Hilton-Milner we describe the essential part of maximal intersecting families with $\tau(\mathcal{F}) = 2$. Let us define $\mathcal{T}$ to be the family of 2-spaces of V that intersect all subspaces in $\mathcal{F}$.

Proposition 2.7 *Let $\mathcal{F}$ be a maximal intersecting family with $\tau(\mathcal{F}) = 2$. Then $\mathcal{F}$ contains all k-spaces containing an element of $\mathcal{T}$ and we have one of the following three possibilities:*

(i) $|\mathcal{T}| = 1$ *and* $\left[{n-2 \atop k-2}\right] < |\mathcal{F}| < \left[{n-2 \atop k-2}\right] + (q+1)\left(\left[{k \atop 1}\right] - 1\right)\left[{k \atop 1}\right]\left[{n-3 \atop k-3}\right]$;

(ii) $|\mathcal{T}| > 1$, $\tau(\mathcal{T}) = 1$, *and there is an* $(l+1)$*-space* W *(with* $2 \leq l \leq k$*) and a* 1*-space* $E \leqslant W$ *so that* $\mathcal{T} = \{M : E \leqslant M \leqslant W,\ \dim M = 2\}$. *In this case*
$$\left[{l \atop 1}\right]\left[{n-2 \atop k-2}\right] - q\left[{l \atop 2}\right]\left[{n-3 \atop k-3}\right] \leq |\mathcal{F}| \leq \left[{l \atop 1}\right]\left[{n-2 \atop k-2}\right] + \left[{k \atop 1}\right]\left(\left[{k \atop 1}\right] - \left[{l \atop 1}\right]\right)\left[{n-3 \atop k-3}\right] + q^l\left[{n-l \atop k-l}\right].$$
For $l = 2$ the upper bound here can be strengthened to
$$|\mathcal{F}| \leq (q+1)\left[{n-2 \atop k-2}\right] - q\left[{n-3 \atop k-3}\right] + \left[{k \atop 1}\right]\left(\left[{k \atop 1}\right] - \left[{2 \atop 1}\right]\right)\left[{n-3 \atop k-3}\right] + q^2\left[{k \atop 1}\right]\left[{n-3 \atop k-3}\right];$$

(iii) $\mathcal{T} = \left[{A \atop 2}\right]$ *for some* 3*-subspace* A *and* $\mathcal{F} = \{U \in \left[{V \atop k}\right] : \dim(U \cap A) \geq 2\}$ *and* $|\mathcal{F}| = (q^2+q+1)\left(\left[{n-2 \atop k-2}\right] - \left[{n-3 \atop k-3}\right]\right) + \left[{n-3 \atop k-3}\right]$.

In case (ii) there is a 1-space E and an l-space L such that $\mathcal{F}$ contains the set $\mathcal{F}_{E,L}$ of all k-spaces containing E and intersecting L. The last two terms of the upper bound for $|\mathcal{F}|$ in (ii) give an upper bound on $|\mathcal{F} \setminus \mathcal{F}_{E,L}|$.

Proof. Let $\mathcal{F}$ be a maximal intersecting family with $\tau(\mathcal{F}) = 2$. Since $\mathcal{F}$ is maximal, it contains all k-spaces containing a $T \in \mathcal{T}$. Since $n \geq 2k$ and $k \geq 2$ two disjoint elements of $\mathcal{T}$ would be contained in disjoint elements of $\mathcal{F}$, which is impossible. So $\mathcal{T}$ is intersecting.

The following observation is immediate: if $A, B \in \mathcal{T}$ and $A \cap B < C < A + B$, then $C \in \mathcal{T}$. As an intersecting family of 2-spaces is either a family of 2-spaces containing some fixed 1-space E or a set of 2-subspaces of a 3-space, we get the following:

$(*)$: $\mathcal{T}$ is either a family of all 2-subspaces in a given $(l+1)$-space containing some fixed 1-space E (and $k \geq l \geq 1$), or $\mathcal{T}$ is the set of all 2-subspaces of a 3-space.

(i) : If $|\mathcal{T}| = 1$, then let S denote the only 2-space in $\mathcal{T}$ and let $E \leqslant S$ be any 1-space. Since $\tau(\mathcal{F}) > 1$ there exists an $F \in \mathcal{F}$ with $E \nleqslant F$, for which we must have $\dim(F \cap S) = 1$. Since S is the only element of $\mathcal{T}$, for any 1-subspace E' of F different from $F \cap S$, $\mathcal{F}_{E+E'} \leq \left[{k \atop 1}\right]\left[{n-3 \atop k-3}\right]$ by Lemma 2.5, hence the number of subspaces containing E but not containing S is at most $(\left[{k \atop 1}\right] - 1)\left[{k \atop 1}\right]\left[{n-3 \atop k-3}\right]$. This gives the upper bound.

(ii) : Assume that $\tau(\mathcal{T}) = 1$ and $|\mathcal{T}| > 1$. By $(*)$, $\mathcal{T}$ is the set of 2-spaces in an $(l+1)$-space W (with $l \geq 2$) containing some fixed 1-space E. Every $F \in \mathcal{F} \setminus \mathcal{F}_E$ intersects W in a hyperplane. Let L be a hyperplane in W not on E. Then $\mathcal{F}$ contains all k-spaces on E that intersect L. Hence the lower bound and the first term in the upper bound come from Lemma 2.2. The second term comes from counting the k-spaces of $\mathcal{F}$ that contain E and intersect a given $F \in \mathcal{F}$ (not containing E) in a point of $F \setminus W$. Here Lemma 2.5 is used. If $l \geq 3$, then there are q^l hyperplanes in W not containing E and there are $\left[{n-l \atop k-l}\right]$ k-spaces through such a hyperplane. For $l = 2$ there are q^2 hyperplanes in W and they cannot be in $\mathcal{T}$. Using Lemma 2.5 gives the bound.

(iii) is immediate. $\square$

Corollary 2.8 *Let $\mathcal{F}$ be a maximal intersecting family with $\tau(\mathcal{F}) = 2$. If $\mathcal{F}$ is at least as large as an HM-type family, and either $q \geq 3$, $n \geq 2k+1$, $k \geq 3$ or $q = 2$, $n \geq 2k+2$, $k \geq 3$, then $\mathcal{F}$ is an HM-type family, or, in case $k = 3$, an $\mathcal{F}_3$-type family.*

There exists an $\epsilon > 0$ (independent of n, k, q) such that if $k \geq 4$ and either $q \geq 3$, $n \geq 2k+1$ or $q = 2$, $n \geq 2k+2$, and $|\mathcal{F}|$ is at least $(1-\epsilon)$ times the size of an HM-type family, then $\mathcal{F}$ is a subfamily of an HM-type family.

Proof. Apply Proposition 2.7. Note that the Hilton-Milner families are precisely those from case (ii) with $k = l$.

Let $n \geq 2k + a$ where $a \geq 1$. In case (i) of Proposition 2.7 we have $|\mathcal{F}|/\begin{bmatrix} n-2 \\ k-2 \end{bmatrix} < 1 + \frac{q+1}{(q-1)q^a}\begin{bmatrix} k \\ 1 \end{bmatrix}$ by Lemma 2.1. In case (ii) we find for $l < k$ that $|\mathcal{F}|/\begin{bmatrix} n-2 \\ k-2 \end{bmatrix} < (\frac{1}{q} + \frac{1}{(q-1)q^a})\begin{bmatrix} k \\ 1 \end{bmatrix} + \frac{q^2}{(q-1)q^a}$. In both cases, for $q \geq 3$, $k \geq 3$, or $q = 2$, $k \geq 4$, $a \geq 2$, this is less than $(1 - \epsilon)$ times the lower bound on the size of an HM-type family given in Lemma 2.3. Using the stronger estimate in Lemma 2.3 we find the same conclusion for $q = 2$, $k = 3$, $a \geq 2$.

In case (iii) $|\mathcal{F}_3| = \begin{bmatrix} 3 \\ 2 \end{bmatrix}\begin{bmatrix} n-2 \\ k-2 \end{bmatrix} - \frac{q^3-q}{q-1}\begin{bmatrix} n-3 \\ k-3 \end{bmatrix}$. For $k \geq 4$, this is much smaller than the size of the HM-type families. For $k = 3$, the two families have the same size. □

Proposition 2.9 *Suppose that $k \geq 3$ and $n \geq 2k$. Let $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$ be an intersecting family with $\tau(\mathcal{F}) \geq 2$. Let $3 \leq l \leq k$. If $\mathcal{F}$ has an intersecting l-space, and*

$$|\mathcal{F}| > \begin{bmatrix} l \\ 1 \end{bmatrix}\begin{bmatrix} k \\ 1 \end{bmatrix}^{l-1}\begin{bmatrix} n-l \\ k-l \end{bmatrix} \tag{2.2}$$

then $\mathcal{F}$ has an intersecting $(l-1)$-space.

Proof. By averaging there is a 1-space P with $|\mathcal{F}_P| \geq |\mathcal{F}|/\begin{bmatrix} l \\ 1 \end{bmatrix}$. If $\tau(\mathcal{F}) = l$, then by Corollary 2.6 $|\mathcal{F}| \leq \begin{bmatrix} l \\ 1 \end{bmatrix}\begin{bmatrix} k \\ 1 \end{bmatrix}^{l-1}\begin{bmatrix} n-l \\ k-l \end{bmatrix}$, contradicting the hypothesis. □

Corollary 2.10 *Let $k \geq 3$ and $n \geq 2k + 1$ and $n \geq 2k + 2$ if $q = 2$. If $|\mathcal{F}| > \begin{bmatrix} 3 \\ 1 \end{bmatrix}\begin{bmatrix} k \\ 1 \end{bmatrix}^2\begin{bmatrix} n-3 \\ k-3 \end{bmatrix}$, then $\tau(\mathcal{F}) = 2$, that is, $\mathcal{F}$ is contained in one of the systems described in Proposition 2.7 satisfying the bound on $|\mathcal{F}|$.*

Proof. Since the right hand side of (2.2) is decreasing in l for $3 \leq l \leq k$ (this uses $n \geq 2k + 1$ and $n \geq 2k + 2$ for $q = 2$), we can find a hitting 2-space if the condition (2.2) holds for $l = 3$, and it does by the assumption on $|\mathcal{F}|$. □

Remark 2.11 For $n \geq 3k$ all the systems described in Proposition 2.7 occur.

2.2 The case $\tau(\mathcal{F}) > 2$

Suppose that $\mathcal{F}$ is an intersecting family and $\tau(\mathcal{F}) = l > 2$. We shall derive a contradiction from $|\mathcal{F}| \geq f(n, k, q)$, and even from $|\mathcal{F}| \geq (1 - \epsilon)f(n, k, q)$ for some $\epsilon > 0$ (independent of n, k, q).

For each point P we have $|\mathcal{F}_P| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^{l-1}\begin{bmatrix} n-l \\ k-l \end{bmatrix}$, and for each line L we have $|\mathcal{F}_L| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^{l-2}\begin{bmatrix} n-l \\ k-l \end{bmatrix}$, by Corollary 2.6.

If there are two l-spaces that meet all $F \in \mathcal{F}$, and these meet in an m-space, where $0 \le m \le l-1$, then

$$|\mathcal{F}| \le \begin{bmatrix} m \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^{l-1} \begin{bmatrix} n-l \\ k-l \end{bmatrix} + \left(\begin{bmatrix} l \\ 1 \end{bmatrix} - \begin{bmatrix} m \\ 1 \end{bmatrix}\right)^2 \begin{bmatrix} k \\ 1 \end{bmatrix}^{l-2} \begin{bmatrix} n-l \\ k-l \end{bmatrix}. \tag{2.3}$$

2.2.1 The case $k = l$

First consider the case $k = l$. Then $|\mathcal{F}| \le \begin{bmatrix} k \\ 1 \end{bmatrix}^k$. On the other hand, $|\mathcal{F}| \ge (1 - \frac{1}{q^3-q}) \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} > (1 - \frac{1}{q^3-q}) \begin{bmatrix} k \\ 1 \end{bmatrix}^{k-1} ((q-1)q^{n-2k})^{k-2}$ by Lemma 2.3 and Lemma 2.1. If either $q > 2$, $n \ge 2k+1$ or $q = 2$, $n \ge 2k+2$, then either $k \le 3$ or $(n, k, q) = (9, 4, 3)$ or $(n, k, q) = (10, 4, 2)$. But if $(n, k, q) = (10, 4, 2)$, then $f(n, k, q) = 153171$, and $15^4 = 50625$, contradiction. And if $(n, k, q) = (9, 4, 3)$ then $f(n, k, q) = 3837721$, and $40^4 = 2560000$, contradiction. So $k = 3$. Now $|\mathcal{F}| \ge (1 - \frac{1}{q^3-q}) \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}$ gives a contradiction for $n \ge 8$, so $n = 7$. So, if we assume that $n \ge 2k+1$ and either $q > 2$, $(n, k) \ne (7, 3)$ or $q = 2$, $n \ge 2k+2$ then we are not in the case $k = l$.

It remains to settle the case $n = 7$, $k = l = 3$.

Pick a 1-space E such that $|\mathcal{F}_E| \ge |\mathcal{F}| / \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and a 2-space S on E such that $|\mathcal{F}_S| \ge |\mathcal{F}_E| / \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Then $|\mathcal{F}_S| > q+1$ since $|\mathcal{F}| > \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}^2$. Pick $F' \in \mathcal{F}$ disjoint from S. Put $H = S + F'$. Then all $F \in \mathcal{F}_S$ are contained in the 5-space H. But $|\mathcal{F}| > \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ so there is an $F_0 \in \mathcal{F}$ not contained in H. If $F_0 \cap S = 0$, then each $F \in \mathcal{F}_S$ is contained in $S + (H \cap F_0)$, so $|\mathcal{F}_S| \le q+1$, contradiction. Thus, all elements of $\mathcal{F}$ disjoint from S are in H.

Now F_0 must meet F' and S, so F_0 meets H in a 2-space S_0. Since $|\mathcal{F}_S| > q+1$, we can find two elements F_1, F_2 of $\mathcal{F}_S$ with the property that S_0 is not contained in the 4-space $F_1 + F_2$. Since any $F \in \mathcal{F}$ disjoint from S is contained in H and meets F_0, it must meet S_0 and also F_1 and F_2. Hence the number of such F's is at most q^5. Altogether $|\mathcal{F}| \le q^5 + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}^2$ (counting F disjoint from S or on a given $E < S$) which contradicts $|\mathcal{F}| \ge (1 - \frac{1}{q^3-q}) \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

2.2.2 l is small

The upper bound (2.3) is a quadratic in $x = \begin{bmatrix} m \\ 1 \end{bmatrix}$ and is largest at one of the extreme values $x = 0$ and $x = \begin{bmatrix} l-1 \\ 1 \end{bmatrix}$. The maximum is taken at $x = 0$ only when $\begin{bmatrix} l \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} k \\ 1 \end{bmatrix} > \frac{1}{2} \begin{bmatrix} l-1 \\ 1 \end{bmatrix}$, that is, when $k = l$. Since we just considered that case, we can assume that $l < k$ and then the upper bound in (2.3) is largest for $m = l - 1$. We find

$$|\mathcal{F}| \le \begin{bmatrix} l-1 \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^{l-1} \begin{bmatrix} n-l \\ k-l \end{bmatrix} + (\begin{bmatrix} l \\ 1 \end{bmatrix} - \begin{bmatrix} l-1 \\ 1 \end{bmatrix})^2 \begin{bmatrix} k \\ 1 \end{bmatrix}^{l-2} \begin{bmatrix} n-l \\ k-l \end{bmatrix}.$$

On the other hand,

$$|\mathcal{F}| \ge (1 - \tfrac{1}{q^3-q}) \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} > (1 - \tfrac{1}{q^3-q}) \begin{bmatrix} k \\ 1 \end{bmatrix}^{l-1} \begin{bmatrix} n-l \\ k-l \end{bmatrix} ((q-1)q^{n-2k})^{l-2}.$$

Comparing these, and using $k > l$, $n \ge 2k+1$, and $n \ge 2k+2$ if $q = 2$, we find either $(n,k,l,q) = (9,4,3,3)$ or $q = 2$, $n = 2k+2$, $l = 3$, $k \le 5$. But if $(n,k,l,q) = (9,4,3,3)$ then $f(n,k,q) = 3837721$, while the upper bound is 3508960, contradiction. And if $(n,k,l,q) = (12,5,3,2)$ then $f(n,k,q) = 183628563$, while the upper bound is 146766865, contradiction. And if $(n,k,l,q) = (10,4,3,2)$ then $f(n,k,q) = 153171$, while the upper bound is 116205, contradiction. So, under our assumptions the case $2 < l < k$ does not occur.

2.2.3 A unique l-space

The discussion so far assumed that there are two distinct l-spaces that meet all $F \in \mathcal{F}$. The alternative is that there is a unique l-space T that meets all $F \in \mathcal{F}$. We can pick a 1-space $E < T$ such that $|\mathcal{F}_E| \ge |\mathcal{F}| / \begin{bmatrix} l \\ 1 \end{bmatrix}$. Now there is some $F' \in \mathcal{F}$ not on E, so E is in $\begin{bmatrix} k \\ 1 \end{bmatrix}$ lines such that each $F \in \mathcal{F}_E$ contains at least one of these lines. If L is one of these lines and L does not lie in T, then we can enlarge L to an l-space that still does not meet all elements of $\mathcal{F}$, so $|\mathcal{F}_L| \le \begin{bmatrix} k \\ 1 \end{bmatrix}^{l-1} \begin{bmatrix} n-l-1 \\ k-l-1 \end{bmatrix}$. Otherwise we have $|\mathcal{F}_L| \le \begin{bmatrix} k \\ 1 \end{bmatrix}^{l-2} \begin{bmatrix} n-l \\ k-l \end{bmatrix}$. Altogether $|\mathcal{F}_E| \le \begin{bmatrix} l-1 \\ 1 \end{bmatrix} (\begin{bmatrix} k \\ 1 \end{bmatrix}^{l-2} \begin{bmatrix} n-l \\ k-l \end{bmatrix}) + (\begin{bmatrix} k \\ 1 \end{bmatrix} - \begin{bmatrix} l-1 \\ 1 \end{bmatrix})(\begin{bmatrix} k \\ 1 \end{bmatrix}^{l-1} \begin{bmatrix} n-l-1 \\ k-l-1 \end{bmatrix})$. On the other hand, $|\mathcal{F}| > (1 - \frac{1}{q^3-q})((q-1)q^{n-2k})^{l-2} \begin{bmatrix} k \\ 1 \end{bmatrix}^{l-1} \begin{bmatrix} n-l \\ k-l \end{bmatrix}$, so that

$$(1 - \tfrac{1}{q^3-q})((q-1)q^{n-2k})^{l-2} \begin{bmatrix} k \\ 1 \end{bmatrix} < \begin{bmatrix} l \\ 1 \end{bmatrix} (\begin{bmatrix} l-1 \\ 1 \end{bmatrix} + (\begin{bmatrix} k \\ 1 \end{bmatrix} - \begin{bmatrix} l-1 \\ 1 \end{bmatrix}) \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-l-1 \\ k-l-1 \end{bmatrix} / \begin{bmatrix} n-l \\ k-l \end{bmatrix}).$$

Under our standard assumptions $n \ge 2k+1$ and $n \ge 2k+2$ if $q = 2$, this implies $q = 2$, $n = 2k+2$, $l = 3$, and also that last case gives a contradiction. We showed: If $n \ge 2k+1$ and $n \ge 2k+2$ if $q = 2$, then $\tau(\mathcal{F}) \le 2$. Together with Corollary 2.8 this proves Theorem 1.3.

3 Critical families

A subspace will be called a *hitting subspace* (and we shall say that the subspace intersects $\mathcal{F}$), it it intersects each element of $\mathcal{F}$.

The previous results just used the parameter τ, so only the hitting subspaces of smallest dimension were taken into account. A more precise description is possible if we make the intersecting system of subspaces critical.

Definition 3.1 An intersecting family $\mathcal{F}$ of subspaces of V is *critical* if for any two distinct $F, F' \in \mathcal{F}$ we have $F \not\subset F'$, and moreover for any hitting subspace G there is a $F \in \mathcal{F}$ with $F \subset G$.

Lemma 3.2 *For every non-extendable intersecting family $\mathcal{F}$ of k-spaces there exists some critical family $\mathcal{G}$ such that*

$$\mathcal{F} = \{F \in \begin{bmatrix} V \\ k \end{bmatrix} : \exists G \in \mathcal{G},\ G \subseteq F\}.$$

Proof. Extend $\mathcal{F}$ to a maximal intersecting family $\mathcal{H}$ of subspaces of V, and take for $\mathcal{G}$ the minimal elements of $\mathcal{H}$. $\square$

The following construction and result are an adaptation of the corresponding results from Erdős and Lovász [5]:

Construction 3.3 *Let $A_1, \ldots, A_k$ be subspaces of V such that* $\dim A_i = i$ *and* $\dim(A_1 + \cdots + A_k) = \binom{k+1}{2}$. *Define*

$$\mathcal{F}_i = \{F \in \begin{bmatrix} V \\ k \end{bmatrix} : A_i \subseteq F,\ \ \dim A_j \cap F = 1 \textit{ for } j > i\}.$$

Then $\mathcal{F} = \mathcal{F}_1 \cup \ldots \cup \mathcal{F}_k$ is a critical, non-extendable, intersecting family of k-spaces, and $|\mathcal{F}_i| = \begin{bmatrix} i+1 \\ 1 \end{bmatrix} \begin{bmatrix} i+2 \\ 1 \end{bmatrix} \cdots \begin{bmatrix} k \\ 1 \end{bmatrix}$ for $1 \leq i \leq k$.

For subsets Erdős and Lovász proved that a critical, non-extendable, intersecting family of k-sets cannot have more than k^k members. They conjectured that the above construction is best possible but this was disproved by Frankl, Ota and Tokushige [8]. Here we prove the following analogous result.

Theorem 3.4 *Let $\mathcal{F}$ be a critical, intersecting family of subspaces of V of dimension at most k. Then $|\mathcal{F}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^k$.*

Proof. Suppose that $|\mathcal{F}| > \begin{bmatrix} k \\ 1 \end{bmatrix}^k$. By induction on i, $0 \le i \le k$, we find an i-dimensional subspace A_i of V such that $|\mathcal{F}_{A_i}| > \begin{bmatrix} k \\ 1 \end{bmatrix}^{k-i}$. Indeed, since by induction $|\mathcal{F}_{A_i}| > 1$ and $\mathcal{F}$ is critical, the subspace A_i is not hitting, and there is an $F \in \mathcal{F}$ disjoint from A_i. Now all elements of $\mathcal{F}_{A_i}$ meet F, and we find $A_{i+1} > A_i$ with $|\mathcal{F}_{A_{i+1}}| > |\mathcal{F}_{A_i}| / \begin{bmatrix} k \\ 1 \end{bmatrix}$. For $i = k$ this is a contradiction. $\square$

Remark 3.5 For $l \le k$ this argument shows that there are not more than $\begin{bmatrix} l \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^{l-1}$ l-spaces in $\mathcal{F}$.

If $l = 3$ and $\tau > 2$ then for the size of $\mathcal{F}$ the previous remark essentially gives $\begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^2 \begin{bmatrix} n-3 \\ k-3 \end{bmatrix}$, which is the bound in Corollary 2.10.

Modifying the Erdős-Lovász construction (see Frankl [6]), one can get intersecting families with many l-spaces in the corresponding critical family.

Construction 3.6 *Let $A_1, \ldots, A_l$ be subspaces with* $\dim A_1 = 1$, $\dim A_i = k+i-l$ *for* $i \ge 2$. *Define* $\mathcal{F}_i = \{F \in \begin{bmatrix} V \\ k \end{bmatrix} : A_i \le F, \dim(F \cap A_j) \ge 1$ *for* $j > i\}$. *Then $\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_l$ is intersecting and the corresponding critical family has at least $\begin{bmatrix} k-l+2 \\ 1 \end{bmatrix} \cdots \begin{bmatrix} k \\ 1 \end{bmatrix}$ l-spaces.*

For n large enough the Erdős-Ko-Rado theorem for vector spaces follows from the obvious fact that no critical, intersecting family can contain more than one 1-dimensional member. The Hilton-Milner theorem and the stability of the systems follow from $(*)$ which was used to describe the intersecting systems with $\tau = 2$. As remarked above, the fact that the critical family has to contain only spaces of dimension 3 or more limits its size to $O(\begin{bmatrix} n \\ k-3 \end{bmatrix})$, if k is fixed and n is large enough. Stronger and more general stability theorems can be found in Frankl [7] for the subset case.

4 Coloring q-Kneser graphs

In this section, we prove Theorem 1.4, that is, we show that $\chi(qK_{n:k}) = \begin{bmatrix} n-k+1 \\ 1 \end{bmatrix}$. The case $k = 2$ was proven in [3] and the general case for $q > q_k$ in [16]. We will need the following result of Bose and Burton and its extension by Metsch [15].

Theorem 4.1 (Bose & Burton [2]) *If V is an n-dimensional vector space over $GF(q)$ and $\mathcal{E}$ is a family of 1-subspaces of V such that any k-subspace* *of V contains at least one element of $\mathcal{E}$, then $|\mathcal{E}| \geq \left[{n-k+1 \atop 1}\right]$. Furthermore, equality holds if and only if $\mathcal{E} = \left[{H \atop 1}\right]$ for some $(n-k+1)$-subspace H of V.*

Proposition 4.2 (Metsch [15]) *If V is an n-dimensional vector space over $GF(q)$ and $\mathcal{E}$ is a family of $\left[{n-k+1 \atop 1}\right] - \varepsilon$ 1-subspaces of V, then the number of k-subspaces of V that are disjoint from all $E \in \mathcal{E}$ is at least $\varepsilon q^{(k-1)(n-k)}$.*

Proof. For the proof (which uses an unpublished result by Szőnyi and Weiner), see [15]. A slightly weaker result, enough for most applications, has a very simple proof that we give here. We show that the number of k-subspaces of V that are disjoint from all $E \in \mathcal{E}$ is at least $\varepsilon q^{(k-1)(n-k+1)} / \left[{k \atop 1}\right]$. Induction on k. For $k = 1$ there is nothing to prove. Next, let $k > 1$ and count incident pairs (1-space, k-space), where the k-space is disjoint from all $E \in \mathcal{E}$:

$$N \begin{bmatrix} k \\ 1 \end{bmatrix} \geq \left(\begin{bmatrix} n \\ 1 \end{bmatrix} - \begin{bmatrix} n-k+1 \\ 1 \end{bmatrix} + \varepsilon \right) \varepsilon q^{(k-2)(n-k+1)} / \begin{bmatrix} k-1 \\ 1 \end{bmatrix} \geq \varepsilon q^{(k-1)(n-k+1)}.$$

□

Proof of Theorem 1.4. Suppose that we have a coloring with at most $\left[{n-k+1 \atop 1}\right]$ colors. Let G (the good colors) be the set of colors that are point-pencils and let B (the bad colors) be the remaining set of colors. Then $|G| + |B| \leq \left[{n-k+1 \atop 1}\right]$. Suppose $|B| = \varepsilon > 0$. By Proposition 4.2, the number of k-spaces with a color in B is at least $\varepsilon q^{(k-1)(n-k)}$, so that the average size of a bad color class is at least $q^{(k-1)(n-k)}$. This must be smaller than the size of a HM-type family. Thus, by Lemma 2.3,

$$q^{(k-1)(n-k)} \leq \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}.$$

For $k \geq 3$ and $q \geq 3$, $n \geq 2k+1$ or $q = 2$, $n \geq 2k+2$, this is a contradiction. (The weaker form of Proposition 4.2 suffices unless $q = 2$, $n = 2k+2$.)

If $|B| = 0$, then all color classes are point-pencils, and we are done by Theorem 4.1. □

5 Proof of Theorem 1.5

Let $a + b = n, a < b$ and let $\mathcal{F}_a = \mathcal{F} \cap \left[{V \atop a}\right]$ and $\mathcal{F}_b = \mathcal{F} \cap \left[{V \atop b}\right]$. We prove

$$|\mathcal{F}_a| + |\mathcal{F}_b| \leq \left[{n \atop b}\right] \tag{5.4}$$

with equality only if $\mathcal{F}_a = \emptyset$, $\mathcal{F}_b = \begin{bmatrix} V \\ b \end{bmatrix}$.

Adding up (5.4) for $n/2 < b \le n$ gives the bound on $|\mathcal{F}|$ in Theorem 1.5 if n is odd and adding the result of Greene and Kleitman [12] that states $|\mathcal{F}_{n/2}| \le \begin{bmatrix} n-1 \\ n/2-1 \end{bmatrix}$ proves it for n even. For the uniqueness part of Theorem 1.5 we only have to note that if n is even and $|\mathcal{F}_{n/2}| = \begin{bmatrix} n-1 \\ n/2-1 \end{bmatrix}$, then by results of Frankl and Wilson [9] and Godsil and Newman [11] we must have $\mathcal{F}_{n/2} = \{F \in \begin{bmatrix} V \\ n/2 \end{bmatrix} : E \leqslant F\}$ for some $E \in \begin{bmatrix} V \\ 1 \end{bmatrix}$ or $\mathcal{F}_{n/2} = \begin{bmatrix} U \\ n/2 \end{bmatrix}$ for some $U \in \begin{bmatrix} V \\ n-1 \end{bmatrix}$.

Now let us prove (5.4). Consider the bipartite graph with vertex set $\begin{bmatrix} V \\ a \end{bmatrix} \cup \begin{bmatrix} V \\ b \end{bmatrix}$ and join $A \in \begin{bmatrix} V \\ a \end{bmatrix}$ and $B \in \begin{bmatrix} V \\ b \end{bmatrix}$ if and only if $A \cap B = 0$. Clearly this graph is regular (with degree q^{ab}) and therefore any independent set (that corresponds to an intersecting subfamily of $\begin{bmatrix} V \\ a \end{bmatrix} \cup \begin{bmatrix} V \\ b \end{bmatrix}$) has size at most $\begin{bmatrix} n \\ b \end{bmatrix}$. Moreover, independent sets of that size can only be $\begin{bmatrix} V \\ a \end{bmatrix}$ or $\begin{bmatrix} V \\ b \end{bmatrix}$ but the former is not an intersecting family. This proves (5.4). $\square$

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