The equivalence of two inequalities for quasisymmetric designs

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It has been an open problem whether Hobart's inequality on the parameters of a quasisymmetric 2-design is independent of earlier known restrictions. In this note we show that it is equivalent to inequalities found by Neumaier and Calderbank.

1 Quasisymmetric designs

A *design* is a finite set called the point set, provided with a collection of subsets called blocks. A t-(v,k,λ) *design* is a design with v points, where all blocks have size k and any t distinct points are in precisely λ blocks.

A *quasisymmetric design* with intersection numbers x, y, is a design where distinct blocks meet in either x or y points, where x, y are distinct and both occur.

A *strongly regular graph* with parameters (v, k, λ, μ) is a finite undirected graph without loops, having both edges and nonedges, with v vertices, regular of valency k, where two distinct adjacent (resp. nonadjacent) vertices have precisely λ (resp. μ) common neighbours. In this note we shall write (V, K, Λ, M) for the parameters of a strongly regular graph, to avoid a clash with design parameters.

Let $(X, \mathcal{B})$ be a quasisymmetric 2-(v, k, λ) design with intersection numbers x, y, where $1 < k < v$. The number of blocks on each point is $r = \lambda(v-1)/(k-1)$ and the total number of blocks is $b = vr/k$.

Let N be the point-block incidence matrix. Let A be the 0-1 matrix indexed by the blocks with (B, C)-entry 1 precisely when $|B \cap C| = x$. Then $NN^\top = rI + \lambda(J-I)$ and $N^\top N = kI + xA + y(J-I-A)$. Now A is the adjacency matrix of a strongly regular graph. Indeed, $NN^\top$ has two eigenvalues $r-\lambda$ and kr, so $N^\top N$ has three eigenvalues 0, $r-\lambda$ and kr, and also $A = \frac{1}{x-y}(N^\top N - (k-y)I - yJ)$ has three eigenvalues, namely $K = \frac{(r-1)k-(b-1)y}{x-y}$, $R = \frac{r-\lambda-k+y}{x-y}$ and $S = -\frac{k-y}{x-y}$ with multiplicities 1, $v-1$, and $b-v$, respectively.

We see that the intersction-x graph of $(X, \mathcal{B})$ with vertex set $\mathcal{B}$, where $B \sim C$ when $|B \cap C| = x$, is strongly regular with parameters (V, K, Λ, M) and eigenvalues K, R, S, where $V = b$, and K, R, S are as above, and Λ, M are determined by $RS = M - K$ and $R + S = \Lambda - M$.

Many examples are known. For example, the Steiner system $S(4,7,23)$ is a quasisymmetric 2-$(23,7,21)$ design with intersection numbers 1 and 3. Its intersection-3 graph is strongly regular with parameters $(V,K,\Lambda,M) = (253,140,87,65)$ with spectrum $140^1\ 25^{22}\ (-3)^{230}$ where multiplicities are written as exponents.

Blokhuis & Haemers [3] constructed an infinite family of examples with parameters $v=q^3$, $k=\frac{1}{2}q^2(q-1)$, $\lambda=\frac{1}{4}q(q^3-q^2-2)$, $x=\frac{1}{2}k$, $y=x-\frac{1}{4}q^2$ where q is a power of two.

1.1 Complement

Given a quasisymmetric 2-(v,k,λ) design $(X,\mathcal{B})$, with b blocks, r on each point, and intersection numbers x,y, the *complementary design* is $(X,\mathcal{B}')$, where $\mathcal{B}' = \{X \setminus B \mid B \in \mathcal{B}\}$. It has parameters $v'=v$, $k'=v-k$, $\lambda'=b-2r+\lambda$, $b'=b$, $r'=b-r$, $x'=v-2k+x$, $y'=v-2k+y$.

2 Inequalities

2.1 The Calderbank-Cowen inequality

The following result allows one to express the number of blocks b of a quasisymmetric 2-design in terms of the parameters v,k,x,y.

Proposition 2.1 (Calderbank [4]) *Every 1-(v,k,r) design with b blocks, and two block intersection numbers x,y, satisfies*

$$1-\frac{1}{b} \le \frac{k(v-k)}{v(v-1)}\left(\frac{(v-1)(2k-x-y)-k(v-k)}{(k-x)(k-y)}\right)$$

with equality if and only if the design is a 2-design. □

2.2 Neumaier's inequality

Let Γ be a strongly regular graph. A proper nonempty subset Y of its vertex set is called a *regular set* with *degree* d and *nexus* e when each vertex inside (resp. outside) Y has d (resp. e) neighbours in Y.

Let Γ be the strongly regular graph on the blocks of a quasi-symmetric 2-(v,k,λ) design $(X,\mathcal{B})$ with block intersection numbers x,y, where blocks are adjacent if they meet in x points. Let $r=\lambda(v-1)/(k-1)$ be the replication number (number of blocks on any point).

Proposition 2.2 (Neumaier [8]) *The sets of all blocks $S(x)$ containing a fixed point x are regular sets in Γ of size r, degree $d=\frac{(\lambda-1)(k-1)-(r-1)(y-1)}{x-y}$ and nexus $e=\frac{\lambda k-ry}{x-y}$.*

Proof. Clearly, $|S(x)|=r$. For $B \in S(x)$, with d_B neighbours in $S(x)$, count the number of pairs (y,C) with $y \ne x$ and $C \ne B$ and $x,y \in C$ and $y \in B$. This number is $(k-1)(\lambda-1)$ and also $d_B(x-1)+(r-d_B-1)(y-1)$ so that $d=d_B$ does not depend on B and has the stated value. Similarly, for $B \notin S(x)$, with e_B neighbours in $S(x)$, we find $k\lambda = e_B x + (r - e_B)y$, so that e_B does not depend on B and has the stated value. $\square$

Proposition 2.3 (NEUMAIER [8]) *The parameters of* $(X, \mathcal{B})$ *satisfy*

$$B(B - A) \leq AC, \tag{N}$$

where

$$A = (v-1)(v-2), \quad B = r(k-1)(k-2)$$

$$C = rd(x-1)(x-2) + r(r-1-d)(y-1)(y-2).$$

Equality holds if and only if $(X, \mathcal{B})$ *is a 3-design.*

Proof. For distinct points x, y, z, let λ_{xyz} denote the number of blocks containing these three points. Fix x and sum over all ordered pairs y, z with x, y, z distinct. One obtains $\sum 1 = A$, $\sum \lambda_{xyz} = B$, $\sum \lambda_{xyz}(\lambda_{xyz} - 1) = C$. Now $0 \leq \sum(\lambda_{xyz} - \frac{B}{A})^2 = B + C - \frac{B^2}{A}$. $\square$

One may check that Neumaier's inequality (N) for a design is equivalent to the inequality for the complementary design.

2.3 The Calderbank and Hobart inequalities

Proposition 2.4 (CALDERBANK [4]) *Let* $\bar{x} = k - x$ *and* $\bar{y} = k - y$*. Then*

$$(v-1)(v-2)\bar{x}\bar{y} - k(v-k)(v-2)(\bar{x} + \bar{y}) + k(v-k)(k(v-k) - 1) \geq 0, \tag{C}$$

with equality if and only if the design is a 3*-design.* $\square$

Clearly, inequality (C) for a design is equivalent to this inequality for the complementary design. Calderbank observes that (C) is equivalent to (N).

The following inequality was derived by Hobart as a consequence of inequalities for coherent configurations.

Proposition 2.5 (HOBART [7]) *The parameters of a quasisymmetric* 2-(v, k, λ) *design with intersection numbers* x, y*, where* $k > x > y$*, with strongly regular intersection-*x *graph with eigenvalues* K, R, S*, where* $K > R > S$*, satisfy*

$$\frac{v-2}{v}\left(1 + \frac{R^3}{K^2} - \frac{(R+1)^3}{(b-K-1)^2}\right) - \frac{(v-2k)^2\lambda}{k^2(k-1)(v-k)} \geq 0. \tag{H}$$

This can also be formulated as $Q_{11}^1 \geq \frac{(v-2k)^2(v-1)}{k(v-k)(v-2)}$, where Q_{11}^1 is the obvious Krein parameter of the strongly regular graph.

Since the strongly regular graph (for the largest intersection size) is the same for a quasisymmetric design and the complementary design, we see that inequaliy (H) for a design is equivalent to this inequality for the complementary design.

In the next section we show the equivalence of (C) and (H).

3 Proof of Hobart's inequality

Let $A = 1 + \frac{R^3}{K^2} - \frac{(R+1)^3}{(b-K-1)^2}$ be the parenthetical part of the inequality (H). Substitute $b = V$ and $V = \frac{(K-R)(K-S)}{M}$ and $M = K + RS$ to get $A = -\frac{(K-R)(KR+R^2-2KS+2R^2S-KS^2-RS^2)}{K^2(S+1)^2}$. Now (H) says

$$-\frac{v-2}{v}\,\frac{(K-R)(KR+R^2-2KS+2R^2S-KS^2-RS^2)}{K^2(S+1)^2} - \frac{(v-2k)^2\lambda}{k^2(k-1)(v-k)} \geq 0.$$

If $S = -1$, then $x = k$ and the design is a multiple of a square (or symmetric) design, a case that was excluded. Hence $S < -1$. Multiply by $vK^2(S+1)^2$ and substitute $R = \frac{r-\lambda-k+y}{x-y}$ and $S = -\frac{k-y}{x-y}$ and $K = \frac{(r-1)k-(b-1)y}{x-y}$ and multiply by $(x-y)^4$ and substitute $\lambda = \frac{r(k-1)}{v-1}$ and $r = \frac{bk}{v}$ and multiply by $\frac{(v-1)^3}{b^3}$ and substitute the value of b found from equality in Proposition 2.1. Since we have $e > 0$ in Proposition 2.2, it follows that $k\lambda \neq ry$, that is, $k^2 - k - vy + y \neq 0$. Divide by $(k^2-k-vy+y)^2$. We see that (H) says

$$(v-1)(v-2)xy + k^2(k-1)(k-3) + 2k(k-1)(x+y) - k(k-1)v(x+y-1) \geq 0$$

but this is precisely inequality (C).

In the same way one sees that Calderbank's inequality (C) is equivalent to Neumaier's inequality (N).

4 On the Blokhuis-Calderbank conditions

Additional nonexistence results were given by Bagchi [1] and Blokhuis & Calderbank [2]. The methods and results are rather similar, but the results are not equivalent: the latter paper eliminates several parameter sets that survive other tests. We repeat the table from [2], p. 203.

v	k	λ	y	x	comment
1090	540	2646	243	270	fails [2], Theorem 5.1
1101	495	2223	198	225	
1266	396	1422	99	126	fails [2], Lemma 5.5
1443	624	2136	246	273	fails [2], Theorem 5.1
2704	544	1086	85	112	
2976	528	1023	69	96	fails [2], Theorem 5.1 for complement
5292	378	29	0	27	fails [9], Theorem 3

In [2] it is said that Theorem 5.1 summarizes the earlier results, but that theorem does not rule out the third parameter set, while Lemma 5.5 does (but the theorem rules out the complementary parameter set).

The last parameter set here is that of an $ARD(14,2)$, where an affine resolvable design $ARD(n,t)$ is a 2-(v,k,λ) design with parameters $v = nk = n^2((n-1)t+1)$, $b = nr = n(n^2t+n+1)$, $\lambda = nt+1$ where there is a resolution into r parallel classes, and any two blocks from different classes have $k^2/v = (n-1)t+1$ points in common. Using the Hasse invariant Shrikhande [9] shows that no $ARD(n,t)$ exists when $n \equiv 2 \pmod 4$ and the square-free part of n contains a prime $\equiv 3 \pmod 4$.

On the other hand, several far smaller parameter sets are ruled out.

v	k	λ	y	x	r	b	comment
77	33	24	12	15	57	133	fails [1] and [2]
101	21	21	3	6	105	505	fails [1] and [2]
137	40	195	10	15	680	2329	fails [1] and [2]
145	70	161	28	35	336	696	fails [1] and [2]
163	64	672	22	28	1728	4401	fails [2]
172	28	63	4	10	399	2451	fails [2]
176	50	49	8	15	175	616	fails [2]

In the first four cases, the complementary design violates [1], Theorem 1.

References

[1] B. Bagchi, *On quasi-symmetric designs*, Des. Codes Cryptogr. **2** (1992) 69–79.

[2] A. Blokhuis & A. R. Calderbank, *Quasi-symmetric designs and the Smith normal form*, Des. Codes Cryptogr. **2** (1992) 189–206.

[3] A. Blokhuis & W. H. Haemers, *An infinite family of quasi-symmetric designs*, J. Stat. Plann. Infer. **95** (2001) 117–119.

[4] A. R. Calderbank, *Inequalities for quasi-symmetric designs*, J. Combin. Th. (A) **48** (1988) 53–64.

[5] J.-M. Goethals & J. J. Seidel, *Quasisymmetric block designs*, pp. 111–116 in: Combinatorial Structures and their Applications (Proc. Calgary 1969), Gordon and Breach, New York, 1970.

[6] J.-M. Goethals & J. J. Seidel, *Strongly regular graphs derived from combinatorial designs*, Canad. J. Math. **22** (1970) 597–614.

[7] S. A. Hobart, *Krein conditions for coherent configurations*, Lin. Alg. Appl. **226–228** (1995) 499–508.

[8] A. Neumaier, *Regular sets and quasi-symmetric 2-designs* pp. 258–275 in: Combinatorial Theory, D. Jungnickel & K. Vedder (eds.), Springer Lecture Notes in Math. **969**, 1982.

[9] S. S. Shrikhande, *The non-existence of certain affine resolvable balanced incomplete block designs*, Canad. J. Math. **5** (1953) 413–420.