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(19 Nov 1984 – 27 Nov 2013)

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The association scheme on the points off a quadric

F. Vanhove
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Abstract

The parameters of the association scheme on the points off a quadric are computed. This corrects a mistake in the literature.

In [BCN, Theorem 12.1.1], the existence of a certain association scheme is claimed, and details are given for $n = 3$. Here we correct the statements given there for odd $n \geq 5$.

Let q be a power of 2, and $n \geq 3$. Let V be an n-dimensional vector space over $\mathbb{F}_q$ provided with a nondegenerate quadratic form Q. Let B be the associated symmetric bilinear form, given by $B(x,y) = Q(x+y) - Q(x) - Q(y)$. If n is odd, there will be a nucleus $N = V^{\perp}$.

We construct an association scheme with point set X, where X is the set of projective points not on the quadric defined by Q and (for odd n) distinct from N. For $n = 3$ and for even n, the relations will be R_0, R_1, R_2, R_3 where

$$\begin{aligned} R_0 &= \{(x,x) \mid x \in X\}, \text{ the identity relation;} \\ R_1 &= \{(x,y) \mid x+y \text{ is a hyperbolic line (secant)}\}; \\ R_2 &= \{(x,y) \mid x+y \text{ is an elliptic line (exterior line)}\}; \\ R_3 &= \{(x,y) \mid x+y \text{ is a tangent}\}. \end{aligned}$$

For odd n, $n \geq 5$, it is necessary to distinguish R_{3a} and R_{3n}, defined by

$$\begin{aligned} R_{3a} &= \{(x,y) \mid x+y \text{ is a tangent not on } N\}; \\ R_{3n} &= \{(x,y) \mid x+y \text{ is a tangent on } N\}. \end{aligned}$$

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Note that every line on N is a tangent, and that for $n = 3$ there are no other tangents, so that R_{3a} is empty. For $q = 2$ a hyperbolic line contains only one nonisotropic point, and a tangent on N contains only one nonisotropic point distinct from N, so that R_1 and R_{3n} are empty.

We show that if $n = 3$ or n is even, then $(X, \{R_0, R_1, R_2, R_3\})$ is an association scheme. Also that if n is odd, $n \geq 5$, then $(X, \{R_0, R_1, R_2, R_{3a}, R_{3n}\})$ is an association scheme. We give the parameters p^i_{jk} and the eigenmatrix P in both cases.

1 Quadric size

The number M of isotropic projective points on a nonisotropic quadric in V, where V has vector space dimension n equals

$$M = \begin{cases} (q^{2m}-1)/(q-1) & \text{if } n = 2m+1 \\ (q^m-\varepsilon)(q^{m-1}+\varepsilon)/(q-1) & \text{if } n = 2m. \end{cases}$$

Equivalently,

$$M = (q^{n-1}-1)/(q-1) + \varepsilon q^{n/2-1}$$

with $\varepsilon = \pm 1$ if n is even, and $\varepsilon = 0$ if n is odd.

2 $n = 3$

Suppose first that $n = 3$. The parameters (p^i_{jk}) were given in [BCN], p. 375. Let us call them (a^i_{jk}) here in the special case $n = 3$.

$$(a^i_{0j})_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ (a^i_{1j})_{ij} = \begin{pmatrix} 0 & \frac{1}{2}q(q-2) & 0 & 0 \\ 1 & \frac{1}{4}(q-2)^2 & \frac{1}{4}q(q-2) & \frac{1}{2}q-2 \\ 0 & \frac{1}{4}(q-2)^2 & \frac{1}{4}q(q-2) & \frac{1}{2}q-1 \\ 0 & \frac{1}{4}q(q-4) & \frac{1}{4}q^2 & 0 \end{pmatrix},$$

$$(a^i_{2j})_{ij} = \begin{pmatrix} 0 & 0 & \frac{1}{2}q^2 & 0 \\ 0 & \frac{1}{4}q(q-2) & \frac{1}{4}q^2 & \frac{1}{2}q \\ 1 & \frac{1}{4}q(q-2) & \frac{1}{4}q^2 & \frac{1}{2}q-1 \\ 0 & \frac{1}{4}q^2 & \frac{1}{4}q^2 & 0 \end{pmatrix}, \ (a^i_{3j})_{ij} = \begin{pmatrix} 0 & 0 & 0 & q-2 \\ 0 & \frac{1}{2}q-2 & \frac{1}{2}q & 0 \\ 0 & \frac{1}{2}q-1 & \frac{1}{2}q-1 & 0 \\ 1 & 0 & 0 & q-3 \end{pmatrix}.$$

The P matrix has in column h the eigenvalues of $(p^i_{hj})_{ij}$. The rows correspond to eigenspaces. We find

$$P = \begin{pmatrix} 1 & q(q-2)/2 & q^2/2 & q-2 \\ 1 & q/2 & -q/2 & -1 \\ 1 & -q/2+1 & -q/2 & q-2 \\ 1 & -q/2 & q/2 & -1 \end{pmatrix}.$$

We see that R_3 is an equivalence relation (and the equivalence classes are the tangent lines, that is, the lines on N). We also see that R_2 has only three distinct eigenvalues, and hence defines a strongly regular graph.

Now suppose that $\dim V = 3$ but the quadratic form Q on V is degenerate in such a way that $N := V^{\perp}$ is a (single) isotropic point. Then the space is a cone over a hyperbolic or elliptic line. We have $v = |X| = q^2 - \varepsilon q$ and the valencies are $k_0 = 1, k_3 = q - 1$ and $k_1 = q^2 - 2q, k_2 = 0$ if $\varepsilon = 1, k_1 = 0, k_2 = q^2$ if $\varepsilon = -1$. Call the corresponding parameters (h^i_{jk}) and (e^i_{jk}), respectively. Then

$$(h^i_{1j})_{ij} = \begin{pmatrix} 0 & q^2-2q & 0 & 0 \\ 1 & q^2-3q & 0 & q-1 \\ * & * & * & * \\ 0 & q^2-2q & 0 & 0 \end{pmatrix},\ (h^i_{3j})_{ij} = \begin{pmatrix} 0 & 0 & 0 & q-1 \\ 0 & q-1 & 0 & 0 \\ * & * & * & * \\ 1 & 0 & 0 & q-2 \end{pmatrix},$$

$$(e^i_{2j})_{ij} = \begin{pmatrix} 0 & 0 & q^2 & 0 \\ * & * & * & * \\ 1 & 0 & q^2-q & q-1 \\ 0 & 0 & q^2 & 0 \end{pmatrix},\ (e^i_{3j})_{ij} = \begin{pmatrix} 0 & 0 & 0 & q-1 \\ * & * & * & * \\ 0 & 0 & q-1 & 0 \\ 1 & 0 & 0 & q-2 \end{pmatrix}.$$

(with undefined * since relation R_2 (resp. R_1) does not occur).

Finally, suppose that $\dim V = 3$ and the quadratic form Q on V is a double line (that is, B vanishes identically, Q is the square of a linear form). Now $k_0 = 1$, $k_1 = k_2 = 0, k_3 = q^2 - 1$. Call the corresponding parameters (z^i_{jk}). Then

$$(z^i_{3j})_{ij} = \begin{pmatrix} 0 & 0 & 0 & q^2-1 \\ * & * & * & * \\ * & * & * & * \\ 1 & 0 & 0 & q^2-2 \end{pmatrix}.$$

3 n even

Now let n be even, say $n = 2m$, where $m \geq 2$. Let the form have type ε, with $\varepsilon = 1$ for a hyperbolic and $\varepsilon = -1$ for an elliptic quadric.

The number of points of the scheme equals $v = |X| = q^{2m-1} - \varepsilon q^{m-1}$.

For the valencies k_i of the relations R_i we find

$$\begin{aligned} k_0 &= 1 \\ k_1 &= (q-2)q^{m-1}(q^{m-1}+\varepsilon)/2 \\ k_2 &= q^m(q^{m-1}-\varepsilon)/2 \\ k_3 &= q^{2m-2}-1 \end{aligned}$$

If $n = 2, m = 1$, then only one type of line occurs (since all of V is just a line), and $P = \begin{pmatrix} 1 & q-2 \\ 1 & -1 \end{pmatrix}$ if $\varepsilon = 1$, and $P = \begin{pmatrix} 1 & q \\ 1 & -1 \end{pmatrix}$ if $\varepsilon = -1$.

Let $n \geq 4, m \geq 2$. If $(x, y) \in R_h$ for a certain $h \in \{1, 2, 3\}$ then for each plane on the line $x + y$ we find the same relation, and a contribution as just computed for the case $n = 3$. In the plane we did not count the nucleus, but here that nucleus contributes 1 to p^h_{33} for $h \neq 3$. If $h = 3$ then x or y might itself be the nucleus of a nondegenerate plane on $x + y$. The details follow.

Let L be a hyperbolic line, and consider the $(q^{n-2} - 1)/(q - 1)$ planes on L. A degenerate plane must be the span $L + z$ of L and an isotropic point z in $L^\perp$. Now $L^\perp$ has the same type ε as V and dimension $n - 2$, so has $a := (q^{2m-3} - 1)/(q - 1) + \varepsilon q^{m-2}$ isotropic points. Hence L is on a degenerate planes $L + z$, and on $(q^{n-2} - 1)/(q - 1) - a = q^{n-3} - \varepsilon q^{m-2}$ nondegenerate planes. All parameters p^1_{jk} follow by summing such parameters of these two types of planes: If $(x, y) \in R_1$, then $L = x + y$ is a hyperbolic line that contributes $q - 3$ to p^1_{11} and nothing to p^1_{jk} for $\{j, k\} \not\subseteq \{0, 1\}$. A degenerate plane on L is a cone over a hyperbolic line, and contributes h^1_{jk}. Thus

$$p^1_{11} = q - 3 + (q^{n-3} - \varepsilon q^{m-2})(a^1_{11} - q + 3) + a(h^1_{11} - q + 3)$$

and

$$p^1_{33} = (q^{n-3} - \varepsilon q^{m-2})(a^1_{33} + 1) + a h^1_{33}$$

and

$$p^1_{jk} = (q^{n-3} - \varepsilon q^{m-2}) a^1_{jk} + a h^1_{jk}$$

for nonzero j, k not both 1 or both 3.

Let L be an elliptic line, and consider planes on L. This time $L^\perp$ has the opposite type, so has $b := (q^{2m-3} - 1)/(q - 1) - \varepsilon q^{m-2}$ isotropic points, and L is on $(q^{n-2} - 1)/(q - 1) - b = q^{n-3} + \varepsilon q^{m-2}$ nondegenerate planes. We find

$$p^2_{22} = q - 1 + (q^{n-3} + \varepsilon q^{m-2})(a^2_{22} - q + 1) + b(e^2_{22} - q + 1)$$

and

$$p^2_{33} = (q^{n-3} + \varepsilon q^{m-2})(a^2_{33} + 1) + b e^2_{33}$$

and

$$p^2_{jk} = (q^{n-3} + \varepsilon q^{m-2}) a^2_{jk} + b e^2_{jk}$$

for nonzero j, k not both 2 or both 3.

Let L be a tangent, with isotropic point z. Then $L^\perp$ is an $(n - 2)$-space containing L. The line L is on q^{n-3} nondegenerate planes (where Q is a conic, L a tangent to the conic, and the nucleus of the plane is a nonisotropic point of L), namely those not contained in $z^\perp$. The line L is on $(q^{n-4} - 1)/(q - 1)$ planes contained in $L^\perp$ (on which the symplectic form vanishes identically, and the quadratic form is a double line). The line L is on q^{n-4} degenerate planes with radical z (contained in $z^\perp$ but not in $L^\perp$). The space $z^\perp/z$ is a nondegenerate $(n - 2)$-space of the same type ε in which L is a nonisotropic point. The quadric in that space has size $(q^{n-3} - 1)/(q - 1) + \varepsilon q^{m-2}$, and through the point L there are $(q^{n-4} - 1)/(q - 1)$ tangents, and $(q^{n-4} + \varepsilon q^{m-2})/2$ hyperbolic lines, and $(q^{n-4} - \varepsilon q^{m-2})/2$ elliptic lines. Consequently, of the q^{n-4} degenerate planes π on L with radical z, for $(q^{n-4} + \varepsilon q^{m-2})/2$ the quotient π/z is hyperbolic, and for $(q^{n-4} - \varepsilon q^{m-2})/2$ elliptic. Each of the q nonisotropic points of L is nucleus of q^{n-4} nondegenerate planes. For the computation of p^3_{3k} starting with two points x, y where $L = x + y$

is a tangent, the q^{n-4} nondegenerate planes in which x is nucleus each contribute $\frac{1}{2}q(q-2)$ for $k=1$ and $\frac{1}{2}q^2$ for $k=2$. There are $q^{n-4}(q-2)$ such planes where none of x,y is nucleus. Altogether, we find

$$p^3_{jk} = q^{n-4}(q-2)a^3_{jk} + \tfrac{1}{2}(q^{n-4}+\varepsilon q^{m-2})h^3_{jk} + \tfrac{1}{2}(q^{n-4}-\varepsilon q^{m-2})e^3_{jk}$$

for $j,k \neq 0,3$, and

$$\begin{aligned} p^3_{31} &= \tfrac{1}{2}q^{n-3}(q-2),\\ p^3_{32} &= \tfrac{1}{2}q^{n-2},\\ p^3_{33} &= q-2+\frac{q^{n-4}-1}{q-1}(z^3_{33}-q+2). \end{aligned}$$

Since we could compute all p^i_{jk}, this proves that we have an association scheme. Let us substitute the values of a^i_{jk}, h^i_{jk}, e^i_{jk} and z^i_{jk} and compute the eigenmatrix P of the scheme. In order to save space, we abbreviate $r := q-2$.

For $(p^i_{1j})_{ij}$ one finds

$$\begin{pmatrix} 0 & \frac{1}{2}q^{m-1}(q^{m-1}+\varepsilon)r & 0 & 0 \\ 1 & \frac{1}{4}q^{n-3}r^2+\varepsilon q^{m-2}(\frac{3}{4}q^2-2q-1) & \frac{1}{4}q^{m-1}(q^{m-1}-\varepsilon)r & \frac{1}{2}(q^{m-1}-\varepsilon)(q^{m-2}r+2\varepsilon) \\ 0 & \frac{1}{4}q^{m-2}(q^{m-1}+\varepsilon)r^2 & \frac{1}{4}q^{m-1}(q^{m-1}+\varepsilon)r & \frac{1}{2}q^{m-2}(q^{m-1}+\varepsilon)r \\ 0 & \frac{1}{4}q^{m-1}(q^{m-2}r+2\varepsilon)r & \frac{1}{4}q^{n-2}r & \frac{1}{2}q^{n-3}r \end{pmatrix}$$

with eigenvalues $\frac{1}{2}q^{m-1}(q^{m-1}+\varepsilon)(q-2)$, $\frac{1}{2}\varepsilon q^{m-2}(q+1)(q-2)$, $-\varepsilon q^{m-1}$, 0.

For $(p^i_{2j})_{ij}$ one finds

$$\begin{pmatrix} 0 & 0 & \frac{1}{2}q^{m}(q^{m-1}-\varepsilon) & 0 \\ 0 & \frac{1}{4}q^{m-1}(q^{m-1}-\varepsilon)r & \frac{1}{4}q^{m}(q^{m-1}-\varepsilon) & \frac{1}{2}q^{m-1}(q^{m-1}-\varepsilon) \\ 1 & \frac{1}{4}q^{m-1}(q^{m-1}+\varepsilon)r & \frac{1}{4}q^{n-1}-\varepsilon q^{m-1}(\frac{3}{4}q-1) & \frac{1}{2}(q^{m-1}+\varepsilon)(q^{m-1}-2\varepsilon) \\ 0 & \frac{1}{4}q^{n-2}r & \frac{1}{4}q^{m}(q^{m-1}-2\varepsilon) & \frac{1}{2}q^{n-2} \end{pmatrix}$$

with eigenvalues $\frac{1}{2}q^m(q^{m-1}-\varepsilon)$, εq^{m-1}, $-\frac{1}{2}\varepsilon q^{m-1}(q-1)$, 0.

For $(p^i_{3j})_{ij}$ one finds

$$\begin{pmatrix} 0 & 0 & 0 & q^{n-2}-1 \\ 0 & \frac{1}{2}(q^{m-1}-\varepsilon)(q^{m-2}r+2\varepsilon) & \frac{1}{2}q^{m-1}(q^{m-1}-\varepsilon) & q^{m-2}(q^{m-1}-\varepsilon) \\ 0 & \frac{1}{2}q^{m-2}(q^{m-1}+\varepsilon)r & \frac{1}{2}(q^{m-1}+\varepsilon)(q^{m-1}-2\varepsilon) & q^{m-2}(q^{m-1}+\varepsilon) \\ 1 & \frac{1}{2}q^{n-3}r & \frac{1}{2}q^{n-2} & q^{n-3}-2 \end{pmatrix}$$

with eigenvalues $q^{n-2}-1$, $q^{m-1}-1$, $-q^{m-1}-1$, $\varepsilon q^{m-2}-1$.

The P-matrix is

$$P = \begin{pmatrix} 1 & \frac{1}{2}q^{m-1}(q^{m-1}+\varepsilon)(q-2) & \frac{1}{2}q^{m}(q^{m-1}-\varepsilon) & q^{2m-2}-1 \\ 1 & \frac{1}{2}\varepsilon q^{m-2}(q+1)(q-2) & -\frac{1}{2}\varepsilon q^{m-1}(q-1) & \varepsilon q^{m-2}-1 \\ 1 & 0 & \varepsilon q^{m-1} & -\varepsilon q^{m-1}-1 \\ 1 & -\varepsilon q^{m-1} & 0 & \varepsilon q^{m-1}-1 \end{pmatrix}.$$

The multiplicities (in the order of the rows of P) are 1, $q^2(q^{n-2}-1)/(q^2-1)$, $\frac{1}{2}q(q^{m-1}-\varepsilon)(q^m-\varepsilon)/(q+1)$, $\frac{1}{2}(q-2)(q^{m-1}+\varepsilon)(q^m-\varepsilon)/(q-1)$.

4 n odd

Now let n be odd, say $n = 2m+1$, where $m \geq 2$. Let Q be a nondegenerate quadric, and let N be its nucleus. We compute the p^i_{jk} as before, this time splitting relation R_3 (being joined by a tangent) into the two relations R_{3a} and R_{3n}, depending on whether the tangent does not or does pass through N.

The number of points of the scheme equals $v = |X| = q^{n-1} - 1$.

For the valencies k_i of the relations R_i we find

$$\begin{aligned} k_0 &= 1 \\ k_1 &= \tfrac{1}{2}q^{n-2}(q-2) \\ k_2 &= \tfrac{1}{2}q^{n-1} \\ k_{3a} &= q^{n-2} - q \\ k_{3n} &= q-2 \end{aligned}$$

The number of planes on a line L is $(q^{n-2}-1)/(q-1)$. If L is hyperbolic or elliptic, then a degenerate plane must be the span $L+z$ of L and an isotropic point z in $L^\perp$. Now $L^\perp$ is a nondegenerate $(n-2)$-space, and has $(q^{n-3}-1)/(q-1)$ isotropic points, so there are q^{n-3} nondegenerate planes, and $(q^{n-3}-1)/(q-1)$ degenerate planes on L. We find for $i = 1,2$ that

$$p^i_{jk} = q^{n-3}(a^i_{jk} - c) + \frac{q^{n-3}-1}{q-1}(\mathrm{x}^i_{jk} - c) + c$$

with $\mathrm{x} = h$ for $i = 1$ and $\mathrm{x} = e$ for $i = 2$, and $c = q-3$ if $i = j = k = 1$, $c = q-1$ if $i = j = k = 2$ and $c = 0$ otherwise.

If L is a tangent on N, with isotropic point z, then the q^{n-3} nondegenerate planes on L are the planes not in $z^\perp$. The remaining $(q^{n-3}-1)/(q-1)$ planes on L are contained in $L^\perp$, and the form induces a double line on these. Hence

$$p^i_{jk} = q^{n-3}a^3_{jk}$$

for $i = 3n$ when not $\{j,k\} \subseteq \{0, 3a, 3n\}$.

If L is a tangent not on N, with isotropic point z, then the q^{n-3} nondegenerate planes on L are the planes not in $z^\perp$. Each nonisotropic point of L is the nucleus of q^{n-4} of these planes. There are $(q^{n-4}-1)/(q-1)$ planes on L contained in $L^\perp$, where the form induces a double line. The remaining planes are degenerate, cones over a hyperbolic or elliptic line, $\frac{1}{2}q^{n-4}$ of each.

Relation R_{3n} is an equivalence relation with equivalence classes of size $q-1$. If L does not pass through N, then it is on a unique plane $L+N$ on N, and the points that have relation R_{4n} with x or y live in that plane. We find $p^1_{1,3n} = \frac{1}{2}q - 2$, $p^1_{2,3n} = \frac{1}{2}q$, $p^2_{1,3n} = p^2_{2,3n} = \frac{1}{2}q - 1$.

For (p^i_{1j}) one finds

$$\begin{pmatrix} 0 & \frac{1}{2}q^{n-2}(q-2) & 0 & 0 & 0 \\ 1 & \frac{1}{4}q^{n-3}(q-2)^2 & \frac{1}{4}q^{n-2}(q-2) & \frac{1}{2}(q^{n-3}-1)(q-2) & \frac{1}{2}q-2 \\ 0 & \frac{1}{4}q^{n-3}(q-2)^2 & \frac{1}{4}q^{n-2}(q-2) & \frac{1}{2}(q^{n-3}-1)(q-2) & \frac{1}{2}q-1 \\ 0 & \frac{1}{4}q^{n-3}(q-2)^2 & \frac{1}{4}q^{n-2}(q-2) & \frac{1}{2}q^{n-3}(q-2) & 0 \\ 0 & \frac{1}{4}q^{n-2}(q-4) & \frac{1}{4}q^{n-1} & 0 & 0 \end{pmatrix}$$

with eigenvalues $\frac{1}{2}q^{2m-1}(q-2)$, $\pm\frac{1}{2}q^{m-1}(q-2)$, $\pm\frac{1}{2}q^m$.

For (p^i_{2j}) one finds

$$\begin{pmatrix} 0 & 0 & \frac{1}{2}q^{n-1} & 0 & 0 \\ 0 & \frac{1}{4}q^{n-2}(q-2) & \frac{1}{4}q^{n-1} & \frac{1}{2}q(q^{n-3}-1) & \frac{1}{2}q \\ 1 & \frac{1}{4}q^{n-2}(q-2) & \frac{1}{4}q^{n-1} & \frac{1}{2}q(q^{n-3}-1) & \frac{1}{2}q-1 \\ 0 & \frac{1}{4}q^{n-2}(q-2) & \frac{1}{4}q^{n-1} & \frac{1}{2}q^{n-2} & 0 \\ 0 & \frac{1}{4}q^{n-1} & \frac{1}{4}q^{n-1} & 0 & 0 \end{pmatrix}$$

with eigenvalues $\frac{1}{2}q^{2m}$, $\pm\frac{1}{2}q^m$ (each twice).

For $(p^i_{3a,j})$ one finds

$$\begin{pmatrix} 0 & 0 & 0 & q(q^{n-3}-1) & 0 \\ 0 & \frac{1}{2}(q^{n-3}-1)(q-2) & \frac{1}{2}q(q^{n-3}-1) & q^{n-3}-1 & 0 \\ 0 & \frac{1}{2}(q^{n-3}-1)(q-2) & \frac{1}{2}q(q^{n-3}-1) & q^{n-3}-1 & 0 \\ 1 & \frac{1}{2}q^{n-3}(q-2) & \frac{1}{2}q^{n-2} & q^{n-3}-2q+1 & q-2 \\ 0 & 0 & 0 & q(q^{n-3}-1) & 0 \end{pmatrix}$$

with eigenvalues $q(q^{2m-2}-1)$, $(q^{m-1}-1)(q-1)$, $-(q^{m-1}+1)(q-1)$, 0 (twice).

For $(p^i_{3n,j})$ one finds

$$\begin{pmatrix} 0 & 0 & 0 & 0 & q-2 \\ 0 & \frac{1}{2}q-2 & \frac{1}{2}q & 0 & 0 \\ 0 & \frac{1}{2}q-1 & \frac{1}{2}q-1 & 0 & 0 \\ 0 & 0 & 0 & q-2 & 0 \\ 1 & 0 & 0 & 0 & q-3 \end{pmatrix}$$

with eigenvalues $q-2$ (three times) and -1 (twice).

Since we could compute all p^i_{jk}, this is indeed an association scheme.

The P-matrix is

$$P = \begin{pmatrix} 1 & \frac{1}{2}q^{2m-1}(q-2) & \frac{1}{2}q^{2m} & q(q^{2m-2}-1) & q-2 \\ 1 & \frac{1}{2}q^{m-1}(q-2) & \frac{1}{2}q^m & -(q^{m-1}+1)(q-1) & q-2 \\ 1 & -\frac{1}{2}q^{m-1}(q-2) & -\frac{1}{2}q^m & (q^{m-1}-1)(q-1) & q-2 \\ 1 & \frac{1}{2}q^m & -\frac{1}{2}q^m & 0 & -1 \\ 1 & -\frac{1}{2}q^m & \frac{1}{2}q^m & 0 & -1 \end{pmatrix}$$

The multiplicities (in the order of the rows of P) are 1, $\frac{1}{2}q(q^m+1)(q^{m-1}-1)/(q-1)$, $\frac{1}{2}q(q^m-1)(q^{m-1}+1)/(q-1)$, $\frac{1}{2}(q-2)(q^{2m}-1)/(q-1)$ (twice).

5 Conclusion

F. Vanhove computed all p^i_{jk} and communicated both P matrices by email. This note was written by A. E. Brouwer, and confirms his results.

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