Exercise

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2013-10-12

1 Exercise

Let G be a finite abelian group of size g. Consider submultisets of G with zero sum. Claim: the number of such submultisets of size $< g$ equals the number of such submultisets of size g.

For example: let $G = \{0, a, b, c\}$ be elementary abelian. Zero sum multisets are $\emptyset$, 0, 00, aa, bb, cc, 000, 0aa, 0bb, 0cc, abc: 11 of size less than 4, and 0000, 00aa, 00bb, 00cc, 0abc, $aaaa$, $bbbb$, $cccc$, $aabb$, $aacc$, $bbcc$, also 11 of size 4.

Or $G = \{0, 1, 2, 3\}$ cyclic. Now zero sum multisets are $\emptyset$, 0, 00, 13, 22, 000, 013, 022, 112, 233: 10 of size less than 4, and 0000, 0013, 0022, 0112, 0233, 1111, 1133, 1223, 2222, 3333, also 10 of size 4.

Question 1: Proof?

Question 2: Bijective proof?

2 A bijection

For the case of a cyclic group G, Rob Eggermont provided a bijection:

Let G be cyclic of order g, written additively, say with generator 1. Write a multiset s as a vector $(x_0, \ldots, x_{g-1})$, where x_i is the number of occurrences of i in s. Then the size of s is $\sum x_i$, and s is zero-sum when $\sum ix_i \equiv 0 \pmod{g}$.

Map a zero-sum set s of size less than g to the zero-sum set t of size g via

$$(x_0, \ldots, x_{g-1}) \mapsto [x_0, x_0 + x_1, \ldots, x_0 + x_1 + \ldots + x_{g-1}],$$

where [...] denotes a multiset. The size of this multiset is clearly g. The sum of its elements is $gx_0 + (g-1)x_1 + (g-2)x_2 + \ldots + x_{g-1} = -x_1 - 2x_2 - \ldots - (g-1)x_{g-1} \equiv 0 \pmod{g}$, as desired.

Conversely, given the zero-sum multiset $t = [y_0, \ldots, y_{g-1}]$ of size g, we can view the y_i as elements of $\{0, 1, \ldots, g-1\}$ and assume $y_0 \leq y_1 \leq \ldots \leq y_{g-1}$. Now map t to s via

$$[y_0, \ldots, y_{g-1}] \mapsto (y_0, y_1 - y_0, y_2 - y_1, \ldots, y_{g-1} - y_{g-2}).$$

Then s has size $y_{g-1} < g$, and sum $(y_1 - y_0) + 2(y_2 - y_1) + \ldots + (g-1)(y_{g-1} - y_{g-2}) = -y_0 - y_1 - y_2 - \ldots - y_{g-2} + (g-1)y_{g-1} = -y_0 - y_1 - y_2 - \ldots - y_{g-1} \equiv 0 \pmod{g}$.

Both maps are each other's inverses, and we constructed the desired bijection in the case of a cyclic group.

3 Invariants

Let G be a finite group acting (say, via a representation ρ) on a vector space V of finite dimension n. This action induces an action on the algebra $k[V]$ of polynomials on V. Let $k[V]^G$ denote the subalgebra of G-invariant polynomials.

Since $k[V]$ is graded by polynomial degree, we have a Poincaré series $P(t) = \sum a_i t^i$, where a_i is the dimension of the i-homogeneous part of $k[V]^G$.

Molien gave the following explicit formula:

$$P(t) = \frac{1}{|G|} \sum_g \frac{1}{\det(I - t\rho(g))}$$

Now let V be the right regular representation of G. Then Molien's formula simplifies to

$$P(t) = \frac{1}{|G|} \sum_d \frac{N_d}{(1 - t^d)^{|G|/d}},$$

where N_d is the number of elements of G of order d.

4 Invariants of abelian groups

Let G be abelian, and let V be the regular representation. Now G permutes the coordinate functions x_g, but is diagonalized on the basis of V^* consisting of the $x_\chi = \sum \chi(g) x_g$. Indeed, if h sends x_g to x_{gh}, then h sends x_χ to $\sum \chi(g) x_{gh} = \chi(h)^{-1} x_\chi$. Thus, all monomials in this new basis are invariant up to a constant, and the invariant polynomials have as basis the set of monomials for which this constant factor is 1. Thus, the invariant monomials are the $x_{\chi_1} \ldots x_{\chi_m}$ for which $\chi_1 \ldots \chi_m$ is identically 1. Since the character group of G is isomorphic to G, such monomials correspond to zero-sum submultisets of G. We proved: the number of zero-sum submultisets of size m of G equals a_m, the dimension of the m-homogeneous part of $k[V]^G$.

5 Lemma

Let $d \mid g$. In the power series $\sum c_i t^i = (1 - t^d)^{-g/d}$ one has $c_g = \sum_{0 \le i \le g-1} c_i$.

Indeed, since $(1 - t)^{-n} = \sum \binom{-n}{i} (-t)^i$, this claims that $\sum_{i=0}^{n-1} \binom{n+i-1}{i} = \binom{2n-1}{n}$, and that is clear, since the left hand sum telescopes.

Applying this to Molien's formula, we see that $a_g = \sum_{0 \le i \le g-1} a_i$, proving our starting claim. It remains to find a bijective proof in the non-cyclic case.