

**stichting  
mathematisch  
centrum**



---

DEPARTMENT OF PURE MATHEMATICS

ZN 63/76 FEBRUARY

A.E. BROUWER

A NOTE ON THE COVERING OF ALL TRIPLES ON  
7 POINTS WITH STEINER TRIPLE SYSTEMS

---

**2e boerhaavestraat 49 amsterdam**

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

---

AMS(MOS) subject classification scheme (1970): 05B30, 05B40

---

A note on the covering of all triples on 7 points with Steiner triple systems

by

A.E. Brouwer

ABSTRACT

We give a partition of the thirty Steiner triple systems on seven points into three sets of ten, each set covering all triples exactly twice. In the language of designs, considering the 35 triples on seven points as points, we look for block designs with the restriction that each block is a Steiner triple system. In particular  $k = 7$  and  $b = 5\lambda$ . It is well known since Cayley that such a design does not exist for  $\lambda = 1$ , while it was known for  $\lambda = 3$ . Here we give it for  $\lambda = 2$  thus proving the existence of the design for each  $\lambda > 1$ .

KEY WORDS & PHRASES: *Steiner triple system*

## 0. INTRODUCTION

If  $E$  is a design (i.e. a collection of subsets of a finite set) and  $\mathcal{D}$  is a collection of designs then a  $\lambda$ -cover of  $E$  by  $\mathcal{D}$ , called a  $C_\lambda(E, \mathcal{D})$ , is a collection  $\{\mathcal{D}_j \mid j \in J\}$  of designs such that

- (i) each  $\mathcal{D}_j$  is isomorphic to an element of  $\mathcal{D}$
- (ii)  $\cup\{\mathcal{D}_j \mid j \in J\} = E$
- (iii) each  $E \in E$  occurs in exactly  $\lambda$  of the  $\mathcal{D}_j$ .

If  $\lambda = 1$  we will drop the subscript.

Specializing  $\mathcal{D}$  and  $E$  we get all kinds of familiar designs some of which are needed below.

1. Let  $I_n$  be a fixed set of size  $n$ , and let  $S_k(n)$  be the collection of all  $k$ -subsets of  $I_n$ . Then a  $C_\lambda(S_2(v), \{S_2(k) \mid k \in K\})$  is a pairwise balanced design with block sizes in  $K$ : a PBD( $K, \lambda; v$ ). In particular a  $C_\lambda(S_2(v), \{S_2(k)\})$  is a BIBD( $b, v, r, k, \lambda$ ). More generally, a  $C_\lambda(S_t(v), \{S_t(k)\})$  is a  $t$ -design.

2. Let  $\mathcal{D}$  be the collection of all Steiner triple systems on  $v$  points:  $\mathcal{D} := S_v := \text{STS}(v)$  and  $E$  the collection of all triples on  $v$  points:  $E = S_3(v)$ , then an  $A_\lambda(v) := C_\lambda(E, \mathcal{D})$  is a  $\lambda$ -fold cover of all triples with Steiner triple systems.

In particular an  $A_1(v)$  is a partition of all triples in disjoint  $\text{STS}(v)$ 's. Such partitions have been found among others by T.P. Kirkman [3], R.H.F. Denniston [2], A. Rosa [5] and L. Teirlinck [6]; it is conjectured that they exist for each  $v \neq 7$  (for which an  $\text{STS}(v)$  exists, that is, for  $v \equiv 1$  or  $3 \pmod{6}$ ), and they are known for all  $v < 100$  except  $v = 37, 85, 97$ . For  $v = 7$  however the maximum number of pairwise disjoint STSs is 2 (Cayley [1]), so it is impossible to cover  $S_3(7)$  with 5 disjoint  $\text{STS}(7)$ . Lindner & Rosa [4] showed the existence of an  $A_3(7)$ , and in this note we will show the existence of exactly two different  $A_2(7)$  so that for each  $\lambda > 1$  an  $A_\lambda(7)$  exists.

3. Finally, take  $\mathcal{D} = A_\lambda(7)$  and let  $S_7$  be the collection of all  $\text{STS}(7)$ . Note that  $|A| = 5\lambda$  for each  $A \in A_\lambda(7)$  and  $|S_7| = 30$ . Obviously a  $C(S_7, A_\lambda(7))$  can exist only if  $5\lambda \mid 30$  i.e. if  $\lambda \in \{1, 2, 3, 6\}$ . For  $\lambda = 6$

we have the trivial  $C(S_7, A_6(7))$  obtained by taking  $\{S_7\}$ . Lindner & Rosa showed the existence of a  $C(S_7, A_3(7))$  i.e. a partition of the 30 STS(7) in two  $A_3(7)$ 's, and here we will show the existence of exactly two  $C(S_7, A_3(7))$  one containing  $A_2(7)$ 's of different types while the other contains three isomorphic  $A_2(7)$ . Finally, since  $A_1(7)$  does not exist neither does  $C(S_7, A_1(7))$ . This settles all cases.

## 1. TRIPLES

On 7 points there are  $\binom{7}{3} = 35$  triples. These triples can be divided into five cyclic 1-designs: (represent a triple by a characteristic vector of length 7)

A: circ(1110000)  
 B: circ(1001100)  
 C: circ(1010100)  
 D: circ(1011000)  
 E: circ(1101000)

In this way each triple gets a name:  $A_0 = (1110000)$ ,  $A_1 = (0111000)$ ,  $C_4 = (0100101)$  etc.

## 2. STEINER TRIPLE SYSTEMS

On 7 points there are 30 Steiner triple systems (all isomorphic), partitioned into six orbits (with sizes 7,7,7,7,1,1) under the action of a cyclic shift.

We give from each of the orbits X the element  $X_0$ :

$I_0$ :	1110000	i.e.	$A_0$	$IV_0$ :	0001101	$E_3$
	1001100		$B_0$		0110001	$D_6$
	1000011		$A_5$		1000011	$A_5$
	0101010		$C_1$		0011010	$E_2$
	0100101		$C_4$		0100110	$B_1$
	0011001		$B_6$		1010100	$C_0$
	0010110		$D_2$		1101000	$E_0$

II <sub>0</sub> :	0000111	A <sub>4</sub>	V:	1101000	E <sub>0</sub>
	0011001	B <sub>6</sub>		0110100	E <sub>1</sub>
	1100001	A <sub>6</sub>		0011010	E <sub>2</sub>
	0101010	C <sub>1</sub>		0001101	E <sub>3</sub>
	1010010	C <sub>5</sub>		1000110	E <sub>4</sub>
	1001100	B <sub>0</sub>		0100011	E <sub>5</sub>
	0110100	E <sub>1</sub>		1010001	E <sub>6</sub>
III <sub>0</sub> :	1011000	D <sub>0</sub>	VI:	1011000	D <sub>0</sub>
	1000110	E <sub>4</sub>		0101100	D <sub>1</sub>
	1100001	A <sub>6</sub>		0010110	D <sub>2</sub>
	0101100	D <sub>1</sub>		0001011	D <sub>3</sub>
	0110010	B <sub>5</sub>		1000101	D <sub>4</sub>
	0010101	C <sub>2</sub>		1100010	D <sub>5</sub>
	0001011	D <sub>3</sub>		0110001	D <sub>6</sub>

That there are no more Steiner triple systems can be seen for instance from the fact that V has an automorphism group of order  $168 = 7 \cdot 6 \cdot 4$  (and the fact that all STS(7) are isomorphic, which is readily seen by looking at I<sub>0</sub>).

Each triple occurs in six STS(7), i.e. we have the trivial  $A_6(7) = S_7$  and therefore a unique  $C(S_7, A_6(7))$ .

Incidences between triples and STSs can be read off the following table:

	I	II	III	IV	V	VI
A	05	46	6	5	-	-
B	06	06	5	1	-	-
C	14	15	2	0	-	-
D	2	-	013	6	-	0-6
E	-	1	4	023	0-6	-

Here the entry 05 in row A, column I means that I<sub>0</sub> contains the triples A<sub>0</sub> and A<sub>5</sub> (and hence I<sub>1</sub> contains A<sub>1</sub> and A<sub>6</sub> etc.).

Intersections between STSs: two STS(7) have either 0 or 1 or 3 triples in common. If we take all STS(7) with intersection number 1 with a given STS(7), then it is found that these STS(7) also have mutual intersection number 1. In this

Given a pair of intersecting triples

there are exactly 2  $STS(7)$ 's containing both  
and these have intersection  $\lambda = 3$ , hence are in different  $\mathcal{P}_3(7)$  sets.

If we let the  $STS(7)$ 's of the set  $\{I_{0-6}, IV_{0-6}, VI\}$  be points ( $v=15$ )  
and the triples be lines ( $b=35$ ), then each point is incident with 7 lines ( $r=7$ )  
and each line contains three points ( $k=3$ ), and two points determine a line ( $\lambda=1$ ).  
This yields a representation of  $PG(3,2)$  (an  $STS(15)$ ).

way we get the two sets  $\{I_{0-6}, IV_{0-6}, VI\}$  and  $\{II_{0-6}, III_{0-6}, V\}$ ; where two STS(7) have exactly one triple in common iff they are in the same set.

Each of these two sets covers all triples exactly thrice, that is, we have here two isomorphic  $A_3(7)$  covering  $S_7$ : an  $C(S_7, A_3(7))$ .

If two STS(7) have intersectionnr 3 then the three triples they have in common intersect in a singleton; for instance  $I_0 \cap II_0 = \{B_0, B_6, C_1\}$  and

$$B_0 \cap B_6 = B_0 \cap C_1 = B_6 \cap C_1 = B_0 \cap B_6 \cap C_1 = \{3\}$$

- $B_0 = (1001100)$
- $B_6 = (0011001)$
- $C_1 = (0101010)$
- $\{3\} = (0001000)$

Conversely, given a singleton, then it is associated in this way with 90 pairs of STS(7) with intersectionnr 3.

The following table gives for each singleton and a representative from each orbit of STS(7)'s the uniquely determined STS(7) such that the singleton is associated with this pair of triple systems.

	0	1	2	3	4	5	6
$I_0$	$II_1$	$II_3$	$III_1$	$II_0$	$III_2$	$III_6$	$II_6$
$II_0$	$I_1$	$IV_1$	$IV_5$	$I_0$	$IV_6$	$I_4$	$I_6$
$III_0$	$IV_1$	$I_6$	$I_5$	$VI$	$IV_2$	$IV_4$	$I_1$
$IV_0$	$II_6$	$III_3$	$III_5$	$V$	$II_2$	$II_1$	$III_6$
$V$	$IV_4$	$IV_5$	$IV_6$	$IV_0$	$IV_1$	$IV_2$	$IV_3$
$VI$	$III_4$	$III_5$	$III_6$	$III_0$	$III_1$	$III_2$	$III_3$

Finally, each STS(7) is disjoint from (the remaining) 8 systems; for example  $I_0$  is disjoint from  $II_2, II_4, II_5, III_0, III_3, III_4, III_5, V$ . Since these 8 systems have always mutual intersectionnr 1, it follows that there are no three mutually disjoint STS(7). In particular it is not possible to cover all triples with 5 mutually disjoint STS(7), i.e. an  $A_1(7)$  does not exist.



gebruik de volgende Johnson bound:

$$A(n, z, w) \leq \left\lfloor \frac{\delta n}{w^2 w n + \delta n} \right\rfloor \quad (\text{mits noemer positief})$$

dan

$$A(35, 12, 7) \leq \frac{6.35}{\lfloor 6.35 - 7 \cdot (35 - 7) \rfloor} = \frac{6.5}{\lfloor 6.5 - 28 \rfloor} = 5.$$

Dit STS(15) is in feite PG(3,2).

### 3. NOTE ON THE $A_3(7)$ .

The  $A_3(7)$  indicated above is an interesting system:

Firstly, if we view the 35 triples as points, then the 15 STS(7) are vectors of length 35 and weight 7 and mutual distance 12. But by the Johnson bound the maximum number of vectors with these properties is 15, i.e.  $A_3(7)$  is an optimal constant weight code showing that  $A(35,12,7) = 15$ .

Secondly, if we view the 15 STS(7) as points then each triple determines a triple of STS(7) [since the 15 STS(7) cover all triples thrice], while each pair of STS(7) determines one triple [their intersection].

Therefore we have a block design with  $v = 15$ ,  $b = 35$ ,  $k = 3$ ,  $r = 7$ ,  $\lambda = 1$  in other words, an STS(15).

It is easy to see that there are other  $A_3(7)$ 's besides the one indicated above. In fact if one fixes a triple (say  $D_6$ ) then the three STS(7) from  $M_1 := \{I_{0-6}, IV_{0-6}, VI\}$  containing  $D_6$  (sc.  $I_4, IV_0, VI$ ) and the three STS(7) from  $M_2 := \{II_{0-6}, III_{0-6}, V\}$  containing  $D_6$  (sc.  $III_3, III_5, III_6$ ) contain the same triples, namely besides  $D_6$  both sets of three STS(7) cover exactly those triples which intersect  $D_6$  in a single point. Therefore one might exchange  $\{I_4, IV_0, VI\}$  and  $\{III_3, III_5, III_6\}$ , thus obtaining two new  $A_3(7)$ :  $M'_1$  and  $M'_2$  (which of course together form a  $C(S_7, A_3(7))$ ).

This process may be repeated, exchanging three STS(7) between  $M'_1$  and  $M'_2$  giving  $M''_1$  and  $M''_2$ . In this way we obtain at least three non-isomorphic types of  $A_3(7)$  namely  $M_1, M'_1$  and  $M''_1$ . I do not know whether there are any other types besides these.

### 4. DESCRIPTION OF THE TWO TYPES OF $A_2(7)$ .

In order to examine all possibilities for an  $A_2(7)$  we first need to know how many different (i.e. non-isomorphic) pairs of STS(7) there are. Of course the cardinality of the intersection of the pair is an invariant; on the other hand, by examining a few permutations it is readily seen that this is the only invariant: two pairs of STS(7) are isomorphic iff they have the same intersectionnr.

Now considering the number of triples of each of the types A, B, C, D, and E contained in STSs of each of the orbits it is found that an  $A_2(7)$  must contain two disjoint STS(7), say V and VI. Now it follows in the same way that we need 3 systems of types I and II each and 1 system of types III and IV each. [Note that an  $A_2(7)$  contains 10 STS(7).] By using an appropriate cyclic shift we may ensure the occurrence of  $III_0$ . The collection  $\{V, VI, III_0\}$  can be completed to an  $A_2(7)$  in four ways, giving the systems

1. V, VI,  $III_0$ ,  $I_0$ ,  $I_2$ ,  $I_3$ ,  $II_0$ ,  $II_4$ ,  $II_5$ ,  $IV_0$
2. V, VI,  $III_0$ ,  $I_0$ ,  $I_2$ ,  $I_4$ ,  $II_2$ ,  $II_4$ ,  $II_6$ ,  $IV_6$
3. V, VI,  $III_0$ ,  $I_0$ ,  $I_3$ ,  $I_4$ ,  $II_1$ ,  $II_2$ ,  $II_5$ ,  $IV_5$
4. V, VI,  $III_0$ ,  $I_2$ ,  $I_3$ ,  $I_4$ ,  $II_0$ ,  $II_1$ ,  $II_6$ ,  $IV_3$

The permutation (016)(245) maps 2. onto 3. and (015)(246) maps 3. onto 4., hence the last three systems are isomorphic. The first two systems however are not isomorphic as can be seen as follows:

Both systems can be partitioned in five pairs of systems with intersectionnr 3 (we write the point associated with such a pair in front of it):

1. 3: ( $I_0, II_0$ ), 3: ( $I_2, II_5$ ), 3: ( $I_3, II_4$ ), 3: (V,  $IV_0$ ), 3: (VI,  $III_0$ )
2. 6: ( $I_0, II_6$ ), 5: ( $I_2, II_2$ ), 0: ( $I_4, II_4$ ), 2: (V,  $IV_6$ ), 3: (VI,  $III_0$ )

That is, the first type of  $A_2(7)$  has associated with it a unique point (in the above case the point {3}) while the second type of  $A_2(7)$  has associated with it a pair of points (those not occurring as point associated with a pair with intersectionnr 3, in the above case the pair {1,4}).

The structure of an  $A_2(7)$  of the first kind - called a centered  $A_2(7)$  - can be described as follows:

1. Colour of the edges of a  $K_6$  with five colours (three edges of each colour).
2. Add a seventh point  $\{\infty\}$  to each of these edges; this gives 15 triples divided in five groups of three.
3. Each group of three triples can be completed to an STS(7) in exactly

two ways. This gives five pairs of STS(7).

It is easily seen that these ten STS(7) form a centered  $A_2(7)$ , and since all centered  $A_2(7)$  are isomorphic each  $A_2(7)$  is obtained in this way. Since there are 6 ways to colour a  $K_6$  there are 6  $A_2(7)$  with a given center, and 42  $A_2(7)$  in all.

Two centered  $A_2(7)$  with the same center have exactly one pair (with intersectionnr 3) in common; given an  $A_2(7)$  with center  $\{i\}$  and a point  $j \neq i$  then there is exactly one  $A_2(7)$  with center  $\{j\}$  disjoint from the given one. Two such disjoint centered  $A_2(7)$  determine a unique  $C(S_7, A_2(7))$  the third  $A_2(7)$  simply consisting of all STS(7) not occurring in the first two. This third one is of the second kind, with associated pair  $\{i, j\}$ . Therefore there are 6  $A_2(7)$  of the second kind associated with a given pair  $\{i, j\}$ , and 126  $A_2(7)$  in all.

Given an  $A_2(7)$  associated with the pair  $\{i, j\}$  then an  $A_2(7)$  of the second kind disjoint with it must be associated with a pair  $\{i, k\}$  or  $\{j, k\}$  with  $k \notin \{i, j\}$ . Indeed, given an  $A_2(7)$  associated with  $\{i, j\}$  and a point  $\{k\}$  there is exactly one  $A_2(7)$  associated with  $\{i, k\}$  disjoint with it. Two such disjoint  $A_2(7)$  of the second kind determine a unique  $C(S_7, A_2(7))$ , the third  $A_2(7)$  being associated with  $\{j, k\}$ . Therefore there are two different  $C(S_7, A_2(7))$ , the first containing 2 centered  $A_2(7)$  and the second containing no centered  $A_2(7)$ ; there are 126 of the first type and 210 of the second type.

#### REFERENCES

- [1] CAYLEY, A., *On the triadic arrangements on seven and fifteen things*, London, Edinburgh and Dublin Philos. mag. and J. Sci. (3) 37 (1850) pp 50-53 ("Collected Mathematical Papers", vol 1, pp 481-484).
- [2] DENNISTON, R.H.F., *Some packings with Steiner triple systems*, Discr. Math. 9 (1974) pp 213-227.
- [3] KIRKMAN, T.P., *Note on an unanswered prize question*, Cambridge and Dublin math. J. 5 (1850) pp 255-262.

- [4] LINDNER C.C. & A. ROSA, *Construction of large sets of almost disjoint Steiner triple systems*, *Canad. J. math.* 27 (1975) pp 256-260
- [5] ROSA, A., *A theorem on the maximum number of disjoint Steiner triple systems*, *Journ. Comb. Theory (A)* 18 (1975) pp 305-312
- [6] TEIRLINCK, L., *On the maximum number of disjoint Steiner triple systems*, *Discr. math.* 6 (1973) pp 299-300.

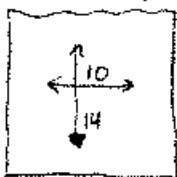
gegeven STS $\gamma$

intervalle	0	met	8	STS $\gamma$ 's.
	1		14	
	3		7	
	7		1	
			<u>30</u>	

paren STS $\gamma$ :

doorsn 0	:	120
1	:	210
3	:	105
		<u>435 = <math>\binom{30}{2}</math></u>

30 STS $\gamma$



42  
centered  
 $A_2(7)$ 's

tel paren STS $\gamma$  overdekt door een <sup>centered</sup>  $A_2(7)$ .

$$(i) 42 - \binom{10}{2} = 1890 \text{ paren}$$

(ii) als doorsnede paar  $i$  dan in  $X_i$  <sup>centered</sup>  $A_2(7)$ 's.

$$1890 = 120 X_0 + 210 X_1 + 105 X_3$$

$$126 = 8 X_0 + 14 X_1 + 7 X_3$$

alle centered  $A_2(7)$ 's zijn gelijk en overlappen

20 paren met doorsnede 1

5 paren met doorsnede 3

20 paren met doorsnede 0

$$\text{dus } 120 X_0 = 20 \cdot 42 \rightarrow X_0 = 7$$

$$210 X_1 = 20 \cdot 42 \rightarrow X_1 = 4$$

$$105 X_3 = 5 \cdot 42 \rightarrow X_3 = 2 \text{ klopt}$$

Conclusies: als doorsnede  $v$  paar 3 dan in 2 centered  $A_2(7)$ 's, die dus ten minste 2 STS $\gamma$ 's gemeen hebben (zij hebben zelfs precies dit paar gemeen).

tel paren <sup>centered</sup>  $A_2(7)$ 's overdekt door een STS $\gamma$ :

$$(i) 30 \cdot \binom{14}{2} = 13 \cdot 14 \cdot 15 \text{ paren}$$

(ii) zelfde center  $\rightarrow$  2 STS $\gamma$ 's gemeen

7 centers, 6  $A_2(7)$ 's elk:

$$7 \cdot \binom{6}{2} \cdot 2 = 7 \cdot 6 \cdot 5 \text{ paren}$$

verschillend center

$$13 \cdot 14 \cdot 15 - 7 \cdot 6 \cdot 5 = 7 \cdot 6 \cdot 5 (13 - 1) = 7 \cdot 6 \cdot 5 \cdot 12 \text{ paren}$$

$$\text{dus gegeven STS $\gamma$ : } \frac{7 \cdot 6 \cdot 5 \cdot 12}{30} = 7 \cdot 12 \text{ paren}$$

$$\text{dus gegeven paar centers: } \frac{7 \cdot 12}{\binom{7}{2}} = 4 \text{ paren}$$

dus van  $i \neq j$