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THE LINEAR SPACES ON 15 POINTS

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The linear spaces on 15 points *)

by

A.E. Brouwer

ABSTRACT

We classify all linear spaces (pairwise balanced designs with $\lambda = 1$) without lines of size two, on 15 points.

KEY WORDS & PHRASES: *linear space*

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*) This report will be submitted for publication elsewhere.

0. INTRODUCTION

In [6] KELLY and NWANKPA catalogued all linear spaces without lines of size two (which they call Sylvester-Gallai Designs) on at most 14 points. In [1] BATTEN, TOTTEN and DE WITTE characterized all linear spaces with lines of size n-1 or n on v points for $v < n^2$, $v \neq 15$. It seems that v = 15 is a rather difficult and in some ways exceptional case. In this note we analyze this case completely.

Some notations:

A *linear space* is a pair $(X,\mathcal{B})$, where elements of X are called *points*, elements of $\mathcal{B}$ are called *lines*, lines are sets of points, and any two distinct points determine a unique line. We write $v = |X|$ and $b = |\mathcal{B}|$.

A k-*line* is a line containing k-points; an r-*point* is a point on r lines. A $k_1^{e_1}k_2^{e_2}\ldots k_h^{e_h}$ -*point* is a point on $\sum_{i=1}^{h} e_i$ lines, e_i of which are k_i-lines $(1 \leq i \leq h)$. Since any point is joined to any other point we have

$$\sum_{i=1}^{h} e_i(k_i-1) = v-1.$$

In our particular case ($v = 15$, $k_i \geq 3$) it follows that a point is on at most 7 lines. But any point not on a k-line has valency at least k - so any line that does not contain all of the points contains at most seven of them, and if there is a 7-line then all other lines have size three. Therefore we have the following possibilities for the points:

A.	3^7	H.	$4^2 5^2$
B.	$3^4 4^2$	I.	$3^2 6^2$
C.	$3^1 4^4$	J.	$3^2 4^1 6^1$
D.	$3^5 5^1$	K.	$4^3 6^1$
E.	$3^3 5^2$	L.	$3^1 4^1 5^1 6^1$
F.	$3^1 5^3$	M.	$3^4 7^1$
G.	$3^2 4^2 5^1$	N.	15^1

We shall use notations like F-point instead of $3^1 5^3$-point. Also, the capitals

A-N will denote the number of points of the corresponding type. We shall see that all types occur except for H, I, K, L.

THEOREM. *A linear space without lines of size two on* 15 *points is one of the following:*

(i) {N = 15}. *A* 15-*line. Unique.*

(ii) {A = 8, M = 7}. *A* 7-*edge colouring of* K_8, *or in other words, an* STS(15) *with a Fano subsystem replaced by a* 7-*line. There are six of these.*

(iii) {B = 9, J = 6}. *A unique solution.*

(iv) {E = 15}. *Pairs, points and* 1-*factors of a* 6-*set. Unique.*

(v) {D = 15}. *A Latin square of order* 5. *There are two nonisomorphic solutions.*

(vi) {A = 2, D = 12, F = 1}. *Two solutions.*

(vii) {C = 2, F = 1, G = 12}. *Unique solution.*

(viii) {A = 3, D = 9, E = 3}. *Two solutions.*

(ix) {A = 1, B = 12, C = 2}. *Derived from an* STS(13) *with two almost parallel classes missing the same point. No solution.*

(x) {A = 2, B = 12, C = 1}. *Two solutions.*

(xi) {B = 10, C = 5}. *Two solutions derived from the two nonisomorphic linear spaces on* 16 *points with ten* $3^3 4^3$-*points and six* 4^5-*points by deleting one of latter type.*

(xii) {A = 1, C = 14}. *The unique design obtained by adding a point at infinity to each of the groups of* GD[4,1,2;14].

(xiii) {A = 3, B = 12}. *Two solutions.*

(xiv) {A = 5, B = 10}. *Sixteen solutions.*

(xv) {C = 15}. *The unique group divisible design* GD[4,1,3;15], *or in other words:* AG(2,4) *minus one point.*

(xvi) {A = 15}. *An* STS(15). *There are eighty of them.*

(120 *nonisomorphic spaces altogether*).

The rest of this note is devoted to the proof of this theorem. We shall describe all spaces explicitly, except for the 80 Steiner triple systems on 15 points, which are well known and may be found e.g. in [2].

Some of the results on Latin squares, STS(13) or GD[4,1,2;14] may be of some independent interest.

1. THERE IS A 15-LINE

If there is a 15-line then there is nothing else and we have case (i) of the theorem. The group of automorphism is S_{15} and has order 15!.

2. THERE IS A 7-LINE

As observed before, all other lines are 3-lines and intersect the 7-line. If we put an STS(7) on the 7-line we get an STS(15). (For STS(15)'s we use the numbering of BUSSEMAKER & SEIDEL [2].) On the other hand, removing the 7-line we find an edge 7-colouring of K_8 (i.e., a 1-factorization of K_8). As shown by SAFFORD [7] there are 6 nonisomorphic such 1-factorizations (see also WALLIS [9]). This is case (ii) of the theorem.

We may find invariants as follows: given two colors (1-factors) of K_8, their union is either a Hamilton circuit or consists of two 4-cycles. Construct a hypergraph on 7 points with as vertices the colors and as 2-edges the pairs of colors whose union is C_4+C_4, and as 3-edges the triples of colors whose union is K_4+K_4. If we draw a picture then a 3-edge is indicated by a circle inside the triangle formed by the three underlying edges. Now we have the following edgecolourings of K_8:

I. The directions of AG(3,2). The group of automorphisms is GA(3,2) and has order 8.7.6.4 = 1344. The corresponding hypergraph is K_7 together with seven triples forming a Fano plane $\mathcal{F}_0$. We may extend to an STS(15) by adding a Fano plane $\mathcal{F}$ on the colors - this can be done in 4 ways: if $\mathcal{F} = \mathcal{F}_0$ we get PG(3,2), design #1 on Bussemaker & Seidel's list; if $|\mathcal{F}\cap\mathcal{F}_0| = 3$ we get #2; if $|\mathcal{F}\cap\mathcal{F}_0| = 1$ we get #3; if $|\mathcal{F}\cap\mathcal{F}_0| = 0$ we get #16. Listing of the colors:

a) 12 34 56 78
b) 13 24 57 68
c) 14 23 58 67
d) 15 26 37 48

e) 16 25 38 47
f) 17 28 35 46
g) 18 27 36 45.

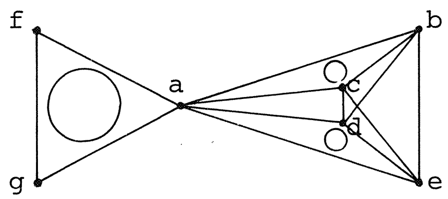
(This is Wallis' 1-factorization F_1 (p.93).)

Note that the STS(15) obtained by replacing the 7-line by the triples of F will contain exactly 1+2t hyperplanes (Fano subspaces) when t is the number of triples in the hypergraph that are also on F. In particular we find that the systems #1, #2, #3, #16 have 15,7,3,1 hyperplanes, respectively.

II. Replace in the above colouring colors f and g by

f) 17 28 36 45
g) 18 27 35 46.

(This is Wallis' 1-factorization F_2.)
The group of automorphisms is the subgroup of PGL(3,2) fixing a nest (point, line, plane) and has order 64. The corresponding hypergraph is



If we extend to an STS(15) then there are 4 possibilities: if F contains the triples afg, abc, ade then we find #2 again (#2 has seven Fano subpalnes, one gives rise to a colouring of type I, and the other six (-equivalent under Aut(#2)) give rise to a colouring of type II); if F only contains afg then we find #5; if F only contains abc (or ade) then we find #4; if F contains more of afg, abc, ade then we find #8.

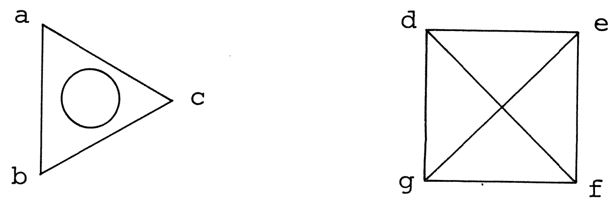
III. Replace in colouring I colours a,b,c by

a) 12 34 57 68
b) 13 24 58 67
c) 14 23 56 78.

(This is Willis' 1-factorization F_4.)
The group of automorphisms is the subgroup of PGL(3,2) fixing a plane, a line in that plane, and an orientation on that line and has order 96. It is generated by

(234)(678)(abc)(efg)
(57)(68)(df)(eg)
(58)(67)(dg)(ef)
(15)(27)(36)(48)(bc)(ef)
(12)(56)(78)(34)

and is sharply transitive on the triples (i,u,v), $i \in \{1,2,3,4,5,6,7,8\}$, $u \in \{a,b,c\}$, $v \in \{d,e,f,g\}$. The corresponding hypergraph is



Under this group the thirty ways of putting of Fanoplane on the colors are partitioned into orbits with sizes 2,3,1,4,8 and 12.

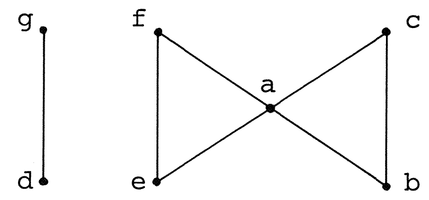
Representative:	Orbitsize	STS(15):
abc, ade, afg, bdf, beg, cdg, cef	2	#3
abc, ade, afg, bdg, bef, cdf, ceg	3	#5
abc, adg, aef, bde, bfg, cdf, ceg	1	#7
abf, ace, bcg, adg, bde, cdf, efg	4	#17
abg, acf, bce, ade, bdf, cdg, efg	8	#14
abf, acg, bce, ade, bdg, cdf, efg	12	#13

Note that we saw STS(15) #3 and #5 before: in fact #3 has three Fano subplanes, one of type I and two (equivalent under Aut(#3)) of type III; likewise #5 has three Fano subplanes, two are equivalent and of type II, the third is of type III. Also #7 has three Fano subplanes - they are equivalent and of type III.

IV. Colouring:

a)	12	34	56	78
b)	13	24	57	68
c)	14	23	58	67
d)	15	27	38	46
e)	16	25	37	48
f)	17	28	36	45
g)	18	26	35	47

The corresponding hypergraph is:



(This is Wallis' 1-factorization F_3.)
The group of automorphisms is generated by

(18)(27)(35)(46)(bc)
(15)(26)(38)(47)(bc)
(12)(34)(56)(78)(dg)
(58)(67)(dg)(ef)

and has order 16.
The 30 Fano planes on the colors fall into orbits of sizes 2,4,4,4,8,8.

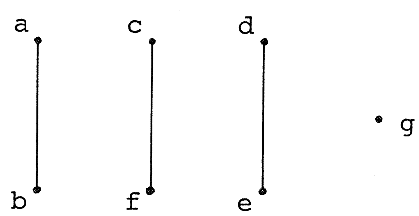
Representative:	Orbitsize:	STS(15):
abc, aef, adg, bde, bfg, cdf, ceg	2	#6
abc, ade, afg, bdg, bef, cdf, ceg	4	#4
aef, abd, acg, beg, bcf, cde, dfg	4	#15
adg, abe, acf, bdf, bcg, cde, efg	4	#18
abe, acd, afg, bcf, bdg, ceg, def	8	#10
abe, acd, afg, bcg, bdf, cef, deg	8	#9

Note that we saw #4 before: it has three Fano subplanes, one of type II and two (equivalent) of type IV. Also #6 has three Fano subplanes - they are all equivalent and of type IV.

V. Colouring:

a)	12	34	56	78
b)	13	24	57	68
c)	14	25	38	67

Corresponding hypergraph:



d) 15 28 36 47
e) 16 27 35 48
f) 17 23 46 58
g) 18 26 37 45

(This is Wallis' 1-factorization F_5.)
The group of automorphisms is generated by

(18)(26)(37)(45)(ab)(cf)(de)
(12)(34)(57)(68)(cf)(de)
(14)(67)(23)(58)(de)
(275)(346)(afd)(bce)

and has order 24.
The 30 Fano planes on the colors fall into orbits of sizes 2,12,8,8.

Representative:	Orbitsize:	STS(15):
abg, cfg, deg, acd, aef, bce, bdf	2	#19
abg, acd, aef, bcf, bde, ceg, dfg	12	#11
acg, bdg, efg, abf, ade, bce, cdf	8	#12
acg, bdg, efg, abe, adf, bcf, cde	8	#20.

VI. Colouring: Corresponding hypergraph:

a) 12 38 47 56
b) 13 24 58 67
c) 14 26 35 78
d) 15 28 37 46 <empty>.
e) 16 23 48 57
f) 17 25 34 68
g) 18 27 36 45

(This is Wallis' 1-factorization F_6.)
The group of automorphisms is generated by

(2345678)(abcdefg)
(27)(36)(45)(af)(be)(cd)
(253)(467)(adb)(cef)

and has order 42. (Note that it fixes the digit 1 - in all other cases the group is transitive on the digits.)
The 30 Fano planes on the colors fall into orbits of sizes 2,14,14.

Representative:	Orbitsize:	STS(15):
abd, bce, cdf, deg, efa, fgb, gaç	2	#61
agf, gbe, gcd, adb, ace, bcf, def	14	#21
gaf, gbc, gde, adb, ace, bef, cdf	14	#22.

Summary

type \ #	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	61
I	15	1	1	-	-	-	-	-	-	-	-	-	-	-	-	1	-	-	-	-	-	-	-
II	-	6	-	1	2	-	-	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
III	-	-	2	-	1	-	3	-	-	-	-	-	1	1	-	-	1	-	-	-	-	-	-
IV	-	-	-	2	-	3	-	-	1	1	-	-	-	-	1	-	-	1	-	-	-	-	-
V	-	-	-	-	-	-	-	-	-	-	1	1	-	-	-	-	-	-	1	1	-	-	-
VI	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1	1	1

(Although not necessary for the proof of the theorem we thus determined the structure of the 23 Steiner triple systems on 15 points containing a subsystem of size 7. At the time I wrote this report I did not have the reference to Wallis and considered this computation a useful check on the enumeration of the colourings of K_8. But the results seem of sufficient independent interest to retain them here.)
(About 1-factorizations we may remark in passing that GELLING & ODEH [5] determined the 396 different 1-factorizations of K_{10}. They also give some information about the six 1-factorizations of K_8.)

3. THERE IS A 6-LINE

LEMMA. *Let a finite linear space on* v *points contains a* k*-line and an* ℓ*-line but no lines of size two. If these lines are disjoint then*

$$v-k-\ell \geq \max(k,\ell);$$

if they intersect then

$$v-k-\ell+1 \geq \max(k-1,\ell-1).$$

When equality holds in either case, then $|k-\ell| \neq 1$.

PROOF. Look at the lines connecting a fixed point on one of the two lines with all points of the other line. □

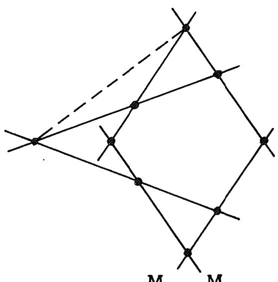
From this observation it follows that we cannot have two 6-lines or a 6-line and a 5-line. Also, that any line disjoint from our 6-line L_0 must be a 3-line. Looking at the valencies we see that each point of L_0 is in an odd number of 4-lines, and each point of $X \backslash L_0$ is an even number.

Now if there were a 3-line L_1 disjoint from L_0 then the (at least) six 4-lines (are disjoint from L_1 and) determine six triples on $X\backslash(L_0 \cup L_1)$. But $6.3 > \binom{15-6-3}{2}$, so this is impossible. Hence all lines intersect L_0.

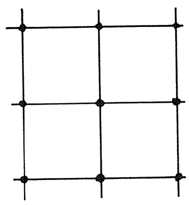
The 6 points of L_0 determine each 6 or 9 edges on $Y := X\backslash L_0$ (6 for points of type $3^3 4^1 6^1$) and 9 for points of type $4^3 6^1$). But $\binom{9}{2} = 6.6$, so all points of L_0 are of type $J = 3^3 4^1 6^1$. It follows that all points of Y are of type $B = 3^4 4^2$. Consequently, we have one 6-line, six 4-lines and eighteen 3-lines.

The points of L_0 determine parallel classes $\mathcal{P}_i$ $(1 \leq i \leq 6)$ on Y, each containing one triple M_i and three pairs.

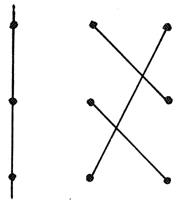
Claim: the M_i fall into two sets of three pairwise disjoint triples. For otherwise we have the picture (of the triples M_i):



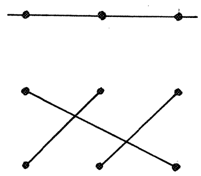
and the dotted line is a pair both in $\mathcal{P}_1$ and $\mathcal{P}_2$. Therefore the picture of the 3-lines on Y is



and the parallel classes P_i are



(mod(3,-)) and



(mod(-,3)).

The group of automorphisms is the subgroup of PGL(3,2) fixing a line and fixing or interchanging two ordered pairs on that line, and is of order 36. This is case (iii) of the theorem.

4. THERE IS A 5-LINE

Since $\binom{15}{2} \equiv \binom{4}{2} \equiv \binom{3}{2} \equiv 0 \pmod 3$ but $\binom{5}{2} \not\equiv 0 \pmod 3$ it follows that the the number of 5-lines is a multiple of three. Since the maximum number of 5-subsets of a 15-set without common pairs is $D(2,5,15) = 6$, we have at most six 5-lines.

A. When there are six 5-lines then one immediately finds a unique configuration: The points are the $\binom{6}{2}$ pairs from a 6-set Z; the 5-lines are the $\binom{6}{1}$ points of Z; the 3-lines are the 15 matchings (1-factors) on Z (i.e., the partitions in three disjoint pairs); incidence is inclusion. All points are of type $E = 3^3 5^2$. The group of automorphisms is S_6 and has order $6! = 720$. This is case (iv) of the theorem.

B. When there are three pairwise disjoint 5-lines, then we have a GD[3,1,5;15], i.e., a T[3,1;5]; in other words, a Latin square of order five. All points are of type $D = 3^5 5^1$.
This is case (v) of the theorem.

There is a unique LS(5) with a sub-LS(2) ("intercalate"):

$$\begin{array}{ccccc} \underline{1} & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \\ 3 & 5 & 1 & 2 & 4 \\ 4 & 3 & 5 & 1 & 2 \\ 5 & 4 & 2 & 3 & 1. \end{array}$$

It has four subsequences of order two and three transversals, all containing the underlined symbol $\underline{1}$. Its group of automorphism has order 12 on the square itself and permutes rows, columns and symbols in any order, hence has order 72.

Next we have the cyclic LS(5):

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{array}$$

which may viewed as the addition table of $\mathbb{Z}_5$, or as the triples determined by the lines intersecting three parallel lines in AG(2,5). Its automorphism group is the subgroup of PGL(2,5) fixing a triple of concurrent lines, and has order 600.

(The classes of Latin squares of order at most six were determined by SCHÖNHARDT [8]; cf. DÉNES & KEEDWELL [4]. The Latin squares of order five were already found by CAYLEY [3], but Cayley counted 'reduced squares' and did not consider questions of isomorphism.)

C. If the three 5-lines are concurrent in a point x_0, then x_0 is of type $F = 3^1 5^3$. Let L_0 be the 3-line incident with x_0, $L_0 = \{x_0,u,v\}$. Of the twelve other points on the 5-lines G are of type $G = 3^2 4^2 5^1$ and $D = 12-G$ are of type $D = 3^5 5^1$. Now we have $\frac{2}{3}G$ 4-lines, hence $3|G$.

I. If $G = 0$ then we have on the twelve D-points a Latin square of order four with an empty square in each row and column, where each row misses a different symbol and each column misses a different symbol; but this is nothing but a Latin square of order five with one fixed square (row, column, symbol).

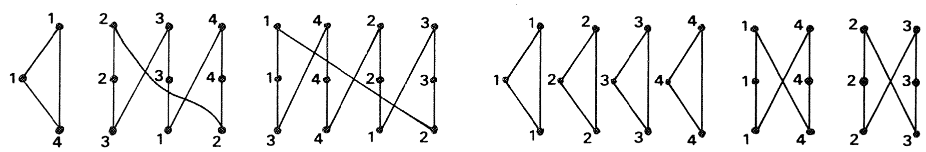
There are four of these: three derived from the LS(5) with subsquare and one from the cyclic LS(5).

a)	b)	c)	d)
-123	-431	-342	-123
1-34	2-13	4-13	4-12
24-1	43-2	24-1	34-1
324-	124-	312-	234-
No transversals,	One transversal,	Three (disjoint)	Three (disjoint)
$\|Aut\| = 18$.	One sub-LS(2),	transversals,	transversals,
	$\|Aut\| = 6$.	$\|Aut\| = 72$.	$\|Aut\| = 24$.

Graphs of missing edges:

a) b) c) d)

row:
column:
symbol:



Now in order to complete this structure to our linear space we have to find the lines through u and v. But these determine a partition of the missing - edge graph into two 1-factors. This is impossible in cases a) and c), can be done in a unique way in case b) and in two (equivalent) ways in case d).

Hence we find two linear spaces with $A = 2$, $D = 12$, $F = 1$, with automorphism groups of order 6 and 12, respectively. This is case (vi) of the theorem.

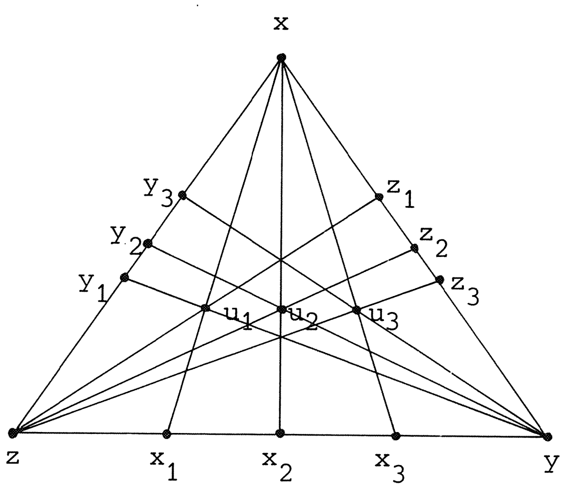
II. If $G = 3$ then there are two 4-lines both containing the three G-points. Contradiction.

III. If $G = 6$ then we have four 4-lines on the six G-points and u and v. But this is a $T[4,1;2]$, which is impossible.

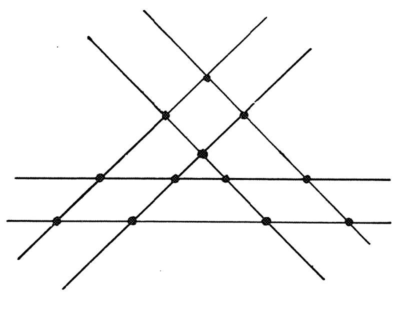
IV. If $G = 9$ then we have six 4-lines, and each of u, v is incident with at most three hence with exactly three 4-lines. But each point is incident with an even number of 4-lines. Contradiction.

V. If $G = 12$ then $X \backslash x_0 \cong AG(2,4)$-two points; in other words, we have the linear space obtained by removing from $PG(2,4)$ six points, two on a line and four on another line in such a way that their point of intersection x_0 is not removed. This linear space is unique and has a group of autorphisms of order 16. This is case (vii) of the theorem.

D. What remains is the case where the 5-lines intersect pairwise but do not all pass through the same point. The points of intersection must have valency at least five, hence are of type $E = 3^3 5^2$. This gives the picture



where u_i $(i = 1,2,3)$ are the three points not on a 5-line. If some point is on two 4-lines then the six other points on these lines must each also be in another 4-line so that there are at least five 4-lines. When there are exactly five 4-lines then 10 points are on a 4-line, and the graph of pairs not covered by a 4-line is the Petersen graph. But this graph contains at least one of the triangles $\{x_1,x_2,x_3\}$, $\{y_1,y_2,y_3\}$ or $\{z_1,z_2,z_3\}$, a contradiction. When there are six 4-lines then all 12 points are on a 4-line, each 4-line being disjoint from a unique other one. There is a unique such configuration:



and one sees immediately that its complementary graph is not a union of triangles and 3-stars in the required way.

Hence there are no 4-lines and $A = 3$, $D = 9$, $E = 3$.

Now if $\{x_i,y_i,z_i\}$ are 3-lines ($i = 1,2,3$) then the remaining 3-lines on our twelve points form together with the triples $\{x_1,x_2,x_3\}$, $\{y_1,y_2,y_3\}$ and $\{z_1,z_2,z_3\}$ a Latin square of order four (a transversal design with groups $\{u_i,x_i,y_i,z_i\}$, $i = 1,2,3$), with one distinguished set of three pairwise disjoint blocks (i.e., a partial transversal of length three).
There are two of these:

a)

$$\begin{array}{cccc} \underline{1} & 3 & 4 & 2 \\ 4 & \underline{2} & 1 & 3 \\ 2 & 4 & \underline{3} & 1 \\ 3 & 1 & 2 & 4 \end{array}$$

The partial transversal can be extended to a full transversal
$|\mathrm{Aut}| = 9.2 = 18$
($PGL(2,4)$ with three noncollinear points fixed, plus an involution).

b)

$$\begin{array}{cccc} \underline{1} & 3 & 2 & 4 \\ 4 & \underline{2} & 1 & 3 \\ 2 & 4 & \underline{3} & 1 \\ 3 & 1 & 4 & 2 \end{array}$$

No transversals,
$|\mathrm{Aut}| = 6$.

Thus we find two solutions, with automorphism groups of order 18 and 6, respectively. This is case (viii) of the theorem.

But we made the assumption that the sets $\{x_i,y_i,z_i\}$ ($i = 1,2,3$) are 3-lines. If two of them are 3-lines, then so is the third. If only one of them is a 3-line then $\{u_1,u_2,u_3\}$ is not, but if $\{u_1,u_2,u_3\}$ is not then all three sets $\{x_i,y_i,z_i\}$ must be lines (as one finds by writing down the incidence matrix - there is only one possibility, corresponding to case b) above). Remains the case where $\{u_1,u_2,u_3\}$ is a line but none of $\{x_i,y_i,z_i\}$ is. Trying for a solution with $\{u_1,x_2,y_2\}$ as a line we find none, so any line $\{u_i,x_j,y_k\}$ or $\{u_i,x_j,z_k\}$ or $\{u_i,y_j,z_k\}$ has i,j,k pairwise different. Now there is a unique solution but it contains $\{x_i,y_i,z_i\}$, that is, it is case a) again.

This finishes the case where there is a 5-line.

5. ALL LINES HAVE SIZE THREE OR FOUR

Now we have A points of type 3^7, B points of type $3^4 4^2$ and C points of type $3^1 4^4$, where $A+B+C = 15$.

A. <u>All points are of type A</u>. This is the case of a Steiner triple system on 15 points - case (xvi) of the theorem. It is well known that there are 80 of them (see [10], [2]) and I am too lazy to compute all automorphism groups. Probably someone did this already.

B. <u>All points are of type B</u>. In this case we have $15.2/4 = 7\frac{1}{2}$ 4-lines, which is impossible.

C. <u>All points are of type C</u>. This is the case of a GD[4,1,3;15], i.e., AG(2,4) minus a point. The group of automorphisms has order 180. This is case (xv) of the theorem.

D. <u>All points have type A or B (and both occur)</u>. In this case we have $\frac{1}{2}B$ 4-lines, so B is even and A is odd. Obviously $B \le \binom{\frac{1}{2}B}{2}$, i.e., (B = 0 or) $B \ge 10$.

I. <u>B = 10</u>

There is a unique configuration of five 4-lines on ten points such that each point is on two 4-lines. The graph of the non-covered edges is the Petersen graph. One immediately sees a beautiful solution:

The points are the $\binom{5}{1}+\binom{5}{2}$ points and pairs from a 5-set Z; the 4-lines are the five sets $L_i = \{(i,j) \mid j \ne i\}$ $(i \in Z)$; the 3-lines are of the form $\{(i,j), i, j\}$ $(i,j \in Z)$ or of the form $\{(i,j),(k,\ell),m\}$ $(Z = \{i,j,k,\ell,m\})$.

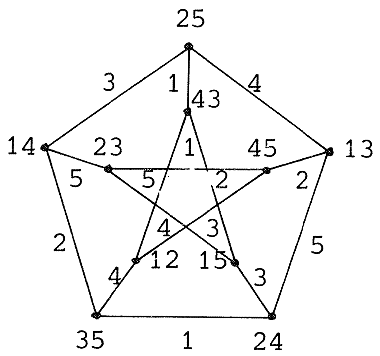
This configuration has a group of automorphisms S_5 of order $5! = 120$. It has a parallel class of triples so that it may be extended to a linear space on 16 points with 1 point of type 4^5, 5 points of type $3^6 4^1$ and 10 points of type $3^3 4^3$.

Let us first study linear spaces that have such a parallel class. Removing a 1-factor from the Petersen graph leaves two pentagons - in particular there cannot exist two disjoint parallel classes.

I found five solutions, given below in the representation of a Petersen graph with edges labeled with digits and points labeled with pairs of digits, such that at each vertex all five digits 1-4 occur, and in the whole graph all ten pairs occur. (Clearly the intention is that the pointset of the linear space contains the points of the Petersen graph together with points 1,2,3,4,5. If the edge ab has label 3 then {a,b,3} will be a 3-line; if the point a has label 2,5 then {a,2,5} will be a 3-line.)

In order to distinguish nonisomorphic solutions I draw an edge ab with a heavy line whenever the labels of a and b are intersecting pairs. This suffices to show the nonisomorphism of the following five solutions. The graphs are drawn in such a way that the parallel class is formed by the edges connecting the outer and the inner pentagon.

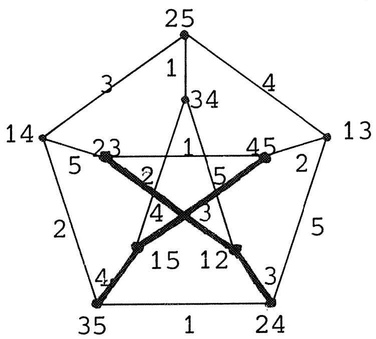
a)



This is the nice solution described above, the unique one without heavy edges - also without the assumption of a parallel class. |Aut| = 120.

There are 6 (equivalent) parallel classes.

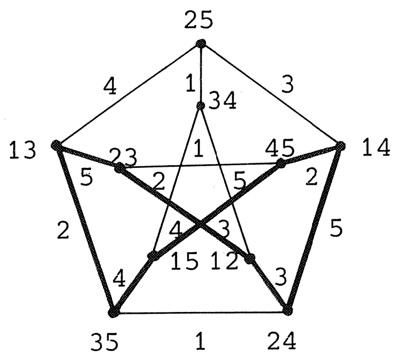
b)



Aut = <(25),(34)> of order 4.

There are 2 (equivalent) parallel classes.

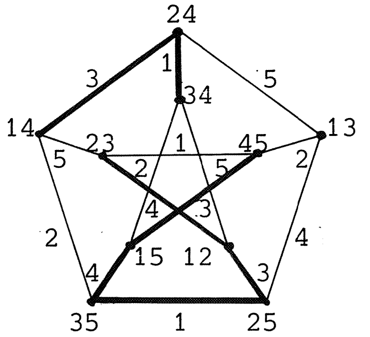
c)



Aut = <(25),(34),(23)(45)> of order 8.

There are 2 (equivalent) parallel classes.

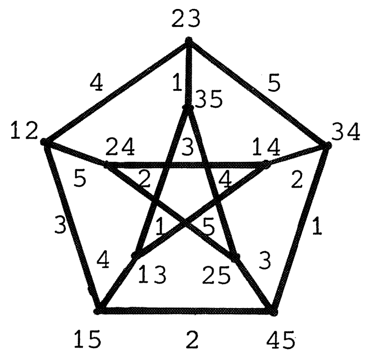
d)



Aut = {1} of order 1.

There is one parallel class.

e)

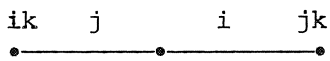


Aut = <(12345)> of order 5.

There is one parallel class.

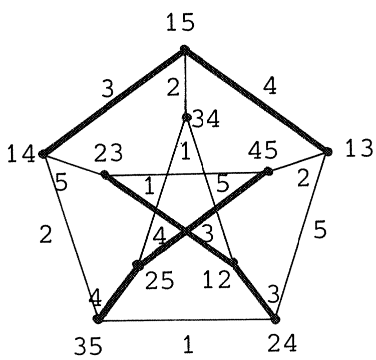
Now we drop the assumption that there exists a parallel class. If we have a solution and it contains the subconfiguration

then by replacing this by



we obtain a new solution. In this way b) is obtained from a) and c), d) from b). Also from b) comes

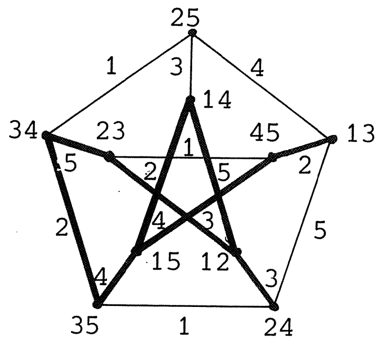
f)



Aut = <(125),(34)(25)> of order 6.

and

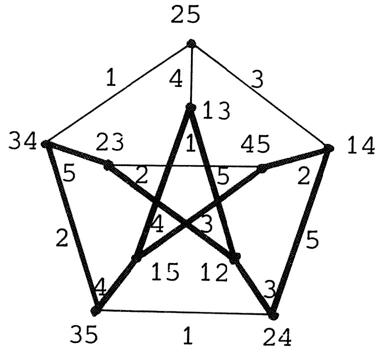
g)



- the heavy edges form a 6-gon with two pending edges. Aut = <(25)> of order 2.

From c) we get

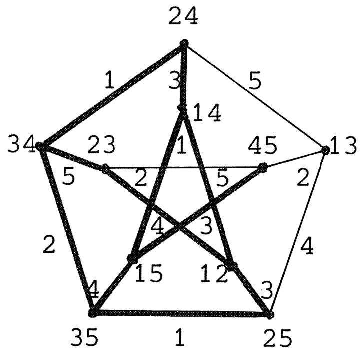
h)



- the heavy edges form an 8-gon with a diagonal of length 2. Aut = <(25)> of order 2.

From d) we get

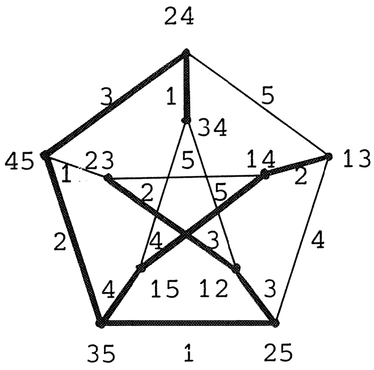
i)



|Aut| = 1, 11 heavy edges,

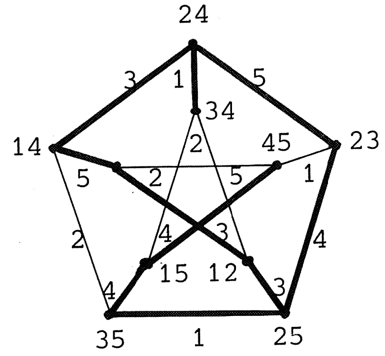
and

j)



Aut = <(124)> of order 3, 9 heavy edges, and

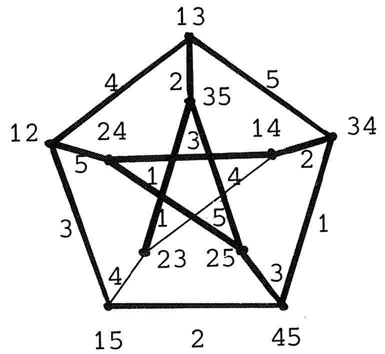
k)



Aut = < > of order 1.
10 heavy edges forming a 6-gon with a pending edge and a pending 3-path.
The 5 thin edges form two 2-paths and a loose edge.

From e) we get

ℓ)

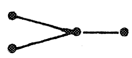


|Aut| = 1, 13 heavy edges. The thin - edge graph is a path of length two.
(Note that we may obtain k from ℓ, i.e., all solutions obtained thus far are connected by the operation of switching along a 2-path.)

These were all solutions that can be switched into a solution with a parallel class.

Now that we have some idea of what the solutions look like we start a systematic search. We classify the solutions according to the structure of the subgraphs G_s induced by the Petersen graph on the set of points $\{st \mid t \neq s\}$. We have the following possibilities:

α) a 3-path

β) $K_{1,3}$ 

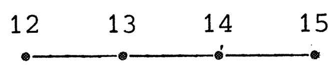
γ) a 2-path plus isolated vertex

δ) two disjoint edges

ε) an edge and two isolated points

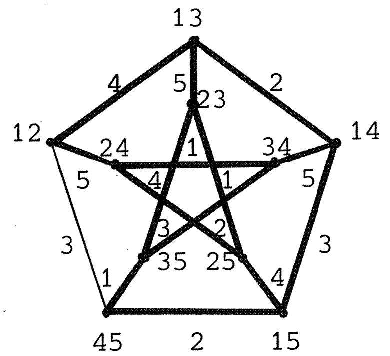
ζ) four isolated points.

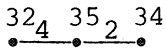
(α) Searching for solutions containing a configurations isomorphic to



we find (apart from k,ℓ,e,i,d) also

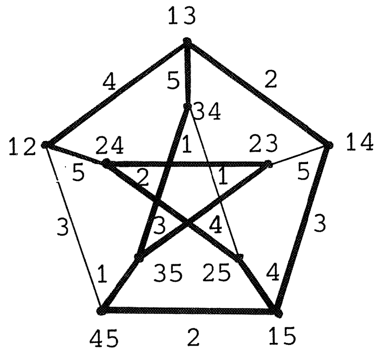
m) |Aut| = 1,14 heavy edges:



and (switching along 

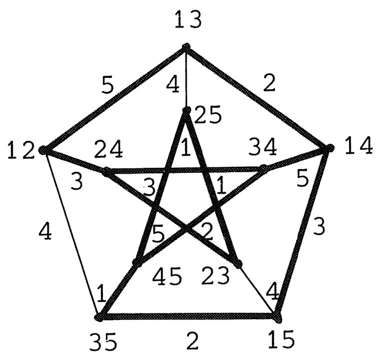
)

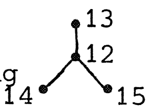
n) Aut = <(135)> of order 3,
12 heavy edges, the thin edges have distinct labels;



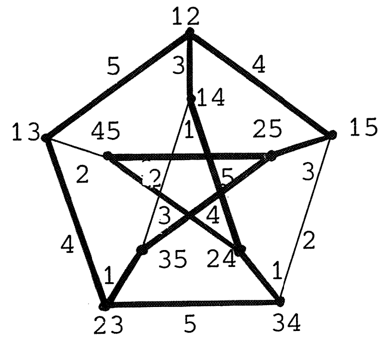
and, looking alike but nonisomorphic:

o) Aut = <(125)> of order 3,
12 heavy edges. The light edges have the same label s(=4). The subgraph G_s is isomorphic to $K_{1,3}$.



(β) Next we search for solutions containing 

. We find n,o,j and one more solution resembling o) but nonisomorphic with it:

p) 

Aut = <(135,134)> of order 12, transitive on the 12 heavy edges. The light edges bear the same label s(=2); the subgraph G_s is empty.

(γ) Next we search for solutions containing $\overset{12}{\bullet}\text{—}\overset{13}{\bullet}\text{—}\overset{14}{\bullet}\quad\overset{15}{\bullet}$ (but no configurations α or β). We find b,g,h,f,c but nothing new.

(δ) Next we search for solutions containing $\overset{12}{\bullet}\text{—}\overset{13}{\bullet}\quad\overset{14}{\bullet}\text{—}\overset{15}{\bullet}$. One immediately sees that such configurations cannot exist.

(ε) Next we search for solutions containing $\overset{12}{\bullet}\text{—}\overset{13}{\bullet}\quad\overset{14}{\bullet}\quad\overset{15}{\bullet}$. Again we see immediately that there are none.

(ζ) The last case is that without heavy edges. But we already remarked that a) is the unique solution with this property.

Thus, there are exactly 16 solutions with A = 5, B = 10. This is case (xiv) of the theorem.

Summary.

# of heavy edges	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
solutions	a	–	–	–	b	–	f	d	c,g	j	h	i	n,o,p	ℓ	m	e

II. B = 12.

Now we have A = 3, B = 12, six 4-lines and twenty-three 3-lines. There is a unique configuration of six 4-lines on twelve points such that each point is on two lines (see picture in section 4D). It has |Aut| = 48. Let Y be the set of B-points and Z the set of A-points, $X = Y \cup Z$. There are 18 3-lines intersecting both Y and Z and five contained entirely within either Y or Z.

a) If Z is a 3-line itself, then the remaining four 3-lines not intersecting Z form a parallel class on Y. This can be chosen in an essentially unique way. If we number the 4-lines

1 2 3 4, 1 5 9 11, 3 7 9 10, 5 6 7 8, 2 6 10 12, 4 8 11 12

then we have two equivalent choices for the parallel class of triples; one is:

1 7 12, 2 8 9, 3 6 11, 4 5 10.

The graph of remaining edges is uniquely 3-colourable, so we have furthermore

1 6 a, 2 5 a, 3 12 a, 4 9 a, 7 11 a, 8 10 a,
1 8 b, 2 7 b, 3 5 b, 4 6 b, 9 12 b, 10 11 b,
1 10 c, 2 11 c, 3 8 c, 4 7 c, 5 12 c, 6 9 c

and finally the triple $Z = \{a,b,c\}$.
Thus there is a unique solution with Z a 3-line, and $|\mathrm{Aut}| = 24$.

b) If Z is not a 3-line then we have (the above 4-lines and) the 3-lines

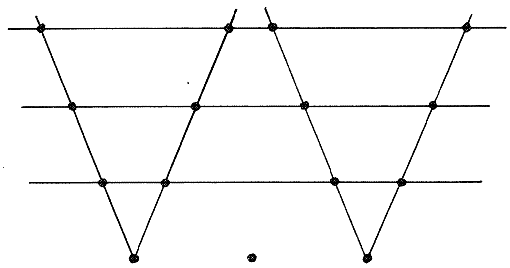
abp, acq, bcr where $Z = \{a,b,c\}$ and $p,q,r \in Y$.

If P_1 and P_2 are the two parallel classes of triples on Y compatible with the given 4-lines (as we found above) then each triple from P_1 intersects three triples from P_2. Consequently, the five triples that are 3-lines inside Y cannot be divided as 2+3 among P_1 and P_2, so we must take all triples from P_1, say, and one triple from P_2. Let P_1 be the parallel class given above under a), and let the fifth triple be 1 8 10. This determines p,q,r: we have 1bc, 8ab, 10ac. Again the reamining graph is uniquely 3-colourable with the given restrictions, and we find the same lines as before, except for 8 10 a, 18b, and 1 10 c. (Note that indeed the four lines abc, 8 10 a, 18b, 1 10 c cover the same edges as the four lines 1 8 10, 1bc, 8ab, 10ac.) The latter space has $|\mathrm{Aut}| = 6$. This was case (viii) of the theorem.

III. <u>B = 14</u>

Finally we have A = 1, B = 14, twenty-one 3-lines and seven 4-lines. Each 4-line is disjoint from two others, i.e., if we make a graph with the 4-lines as vertices and connect them when they are disjoint, then we have a union of polygons, i.e., a 7-gon or the union of a 3-gon and a 4-gon.

a) If we have a 3-gon plus a 4-gon then the 4-lines lie as follows:



The line connecting p and q must be pqx. If we remove p and q from the 4-lines containing them and replace 3-lines pab or qab by yab, where y is a new symbol, add a new point z to the three remaining 'horizontal' 4-lines, and finally replace pqx by xyz, then we obtain a linear space with A = 2 (the points x,y), F = 1 (the point z) and D = 12 (all other points). But this is a case we studied before (there were two solutions). Trying to go back (i.e., divide the edges incident with y among p and q) we see that this is impossible.

b) If we have a 7-gon then we may label the seven lines with elements of $\mathbb{Z}_7$ and the points on them with pairs (ij) where $i,j \in \mathbb{Z}_7$, $i-j \neq 0,\pm 1$, in such a way that the lines become $\{(13),(14),(15),(16)\}$ (mod 7). Trying to complete this structure, I found no solution.

E. <u>All points have type A or C (and both types occur)</u>. Each A-point lies on the unique 3-line through a given C-point, hence there is only one A-point. Removing it leaves us with the unique group-divisible design GD[4,1,2;14]. Conversely, one obtains the given linear space by adding a point at infinity to each of the groups of this design. (The structure of GD[4,1,2;14] may be determined as follows: first by counting it follows that each block B is disjoint from a unique other block B'; then, that B and B' intersect the same groups. This means that interchanging the points in each of the groups is an involution of the design. Next, identify both points in each of the groups. This yields a 2-(7,4,2) design. Taking block complements yields a 2-(7,3,1) design, i.e. a Fano plane, PG(2,2). Going back can be done in an essentially unique way. An explicit example is given by {0,1,4,6} (mod 14), with

$$\begin{aligned}\mathrm{Aut} = \langle x \mapsto x+1 \pmod{14},\ &(146)(2\ 3\ 12)(5\ 9\ 10)(8\ 11\ 13),\\ &(8\ 9\ 12)(125)(3\ 6\ 10),\ (25)(3\ 10)(46)(9\ 12)(11\ 13)\rangle\end{aligned}$$

of order 336.) The linear space has the same group of automorphisms, i.e. $|\mathrm{Aut}| = 336$. This is case (xii) of the theorem.

F. <u>All points have type B or C (and both occur)</u>. Now there are B points of type $3^4 4^2$ and $C = 15-B$ points of type $3^1 4^4$. There are $15-\frac{1}{2}B$ 4-lines and $B+5$ 3-lines. In particular, B is even.

Let there be a_i 4-lines with i C-points. Then we have the equations (inequalities)

$$\begin{aligned}&\sum a_i = 15-\tfrac{1}{2}B = \tfrac{1}{2}(C+15)\\ &\sum i a_i = 4C\\ &\sum \tbinom{i}{2} a_i \le \tbinom{C}{2}\end{aligned}$$

which yields

$$0 \le a_0+a_3+3a_4 \le \tfrac{1}{2}(C-3)(C-5).$$

If $C = 3$ or $C = 5$ it follows that equality holds everywhere, so that any two C-points are joined by a 4-line. We shall use this observation below.

Let there be b_i 3-lines with i B-points. Then we have the equations

$$\begin{aligned}&\sum b_i = B+5\\ &\sum i b_i = 4B\\ &\sum \tbinom{i}{2} b_i \le \tbinom{B}{2}.\end{aligned}$$

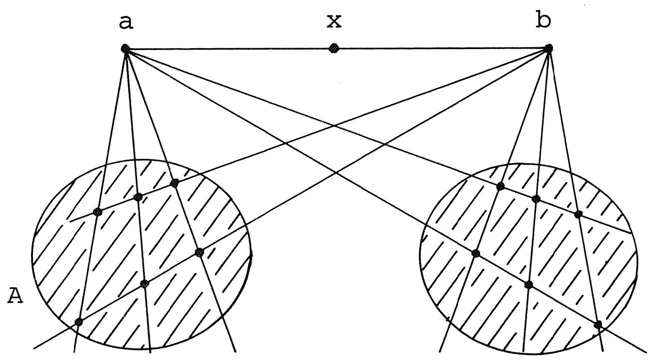
Hence

$$0 \le b_0+b_3 \le \tfrac{1}{2}(B-2)(B-5).$$

In particular $B \neq 4$. Also, if $B = 2$ then the two B-points are joined by a 3-line.

I. $\underline{B = 2}$

There are seven 3-lines, and they can intersect only in the B-points, i.e., we have the picture



Apart from the four 4-lines drawn there are ten others, six of which do not contain the point x. Each of these six lines has two points in A and two points in B, so together with the lines drawn already they cover all edges within A and B. A 4-line through x can therefore have at most one point in A or in B, i.e. it contains at most three points. Contradiction.

II. $\underline{B = 4}$

This case was already excluded.

III. $\underline{B = 6}$

Looking at the above equations for b_i we see (adding them with coefficients 3,-2,1) that

$$0 \le 3b_0+b_1 \le \tfrac{1}{2}(B-5)(B-6).$$

Now that $B = 6$, equality holds everywhere, i.e., the six B-points are pairwise connected by 3-lines; nine ($=b_2$) 3-lines contain two B-points, and two ($=b_3$) contain three B-points. The twelve 4-lines each contains a B-point and hence induce an STS(9) on the set of C-points. But this requires that these lines can be split up into six pairs of disjoint lines, which is not the case.

IV. B = 8

There are 2B incidences (B-point, 4-line), and only $15-\frac{1}{2}B$ 4-lines, so at least $2B-(15-\frac{1}{2}B) = \frac{5}{2}B-15$ pairs of B-points are covered by 4-lines. So we may improve our inequality to

$$\sum \binom{i}{2} b_i \le \binom{B}{2} - (\frac{5}{2}B-15)$$

which yields

$$0 \le 3b_0+b_1 \le \frac{1}{2}(B-6)(B-10).$$

In particular B = 8 is impossible.

V. B = 10

There are ten 4-lines and fifteen 3-lines. From the inequalities on c_i we see that any two C-points are joined by a 4-line, and that a 4-line contains at most two C-points. From the inequalities on b_i derived under IV we see that the ten 4-lines cover exactly ten pairs of B-points. Consequently each of the 4-lines contains two B-points and two C-points; five of the 3-lines contain a C-point, the other ten 3-lines contain only B-points.

The five 3-lines containing both B- and C-points from a parallel class, i.e., we can adjoin a point at infinity and obtain a new linear space with 16 points, 10 of type $3^3 4^3$ and 6 of type 4^5.

Let Y be the set of $3^3 4^3$-points and $Z = X\backslash Y$ the set of 4^5-points. The ten points in Y each determine a 1-factor in Z; each edge is in two such 1-factors. If we first look for this configuration: 10 1-factors, covering all pairs twice (but without repeated 1-factors) then we see that this is essentially unique: altogether there are 15 1-factors on K_6, covering all edges three times, so the complement of the configuration we are looking for is a colouring of K_6 with 5 colours, and up to isomorphism this is unique.

Now make a graph on the ten 1-factors (i.e., on Y), connecting two 1-factors when they have an edge in common (i.e. when the two $3^3 4^3$-points are joined by a 4-line). This graph turns out to be the Petersen graph, so constructing our linear space is equivalent with finding a decomposition of the complement of the Petersen graph into ten triangles.

If we label the vertices of the graph with pairs (ij) from $I_5 = \{1,2,3,4,5\}$ then two points are joined in the complement of the Petersen graph when the corresponding pairs have a point in common. There are two kinds of triangles: $\{(12),(13),(23)\}$ and $\{(12),(13),(14)\}$ (non-centered and centered).

a) If we take all triangles of the first kind this works.

b) The only other solution (up to isomorphism) is found by taking all triangles of the first kind containing the symbol 5, and the 4 triangles of the second kind on the set $\{1,2,3,4\}$.

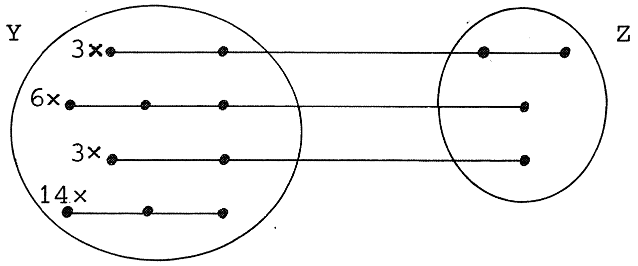
[The two kinds of triangles are really distinct, and the difference does not depend on the labeling: a triangle of the first kind is a maximal independent set while a triangle of the second kind may be extended.]

The automorphism group of the (edge)colouring of K_6 has order $6.5.2 = 60$, and any such automorphism induces an automorphism of the Petersen graph. In the first case the linear space is invariant under this whole group, i.e., we have $|\mathrm{Aut}\ (a)| = 60$; in the second case we lose a factor 5, so $|\mathrm{Aut}\ (b)| = 12$.

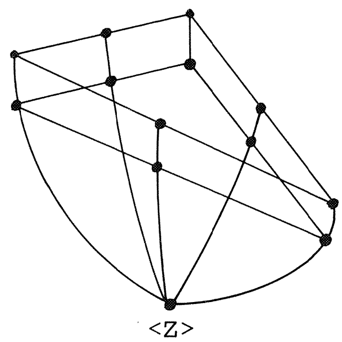
Returning to $v = 15$ amounts to choosing a fixed matching (1-factor) in the Petersen graph. But all are equivalent, so we again have two solutions, with automorphism groups of order 10 and 2, respectively. This is case (xi) of the theorem.

VI. <u>B = 12</u>

Here we have nine 4-lines and seventeen 3-lines. Let Y be the set of B-points and $Z = X\backslash Y$ be the set of C-points. As observed alove any two C-points are joined by a 4-line and each 4-line contains a C-point. Schematically we have



(i.e., $b_0 = b_1 = 0$, $b_2 = 3$, $b_3 = 14$; $a_0 = a_3 = a_4 = 0$, $a_1 = 6$, $a_2 = 3$). On Y we see twenty triples; the edges not covered by them form a 1-factor, i.e. identifying Z to a point we obtain a STS(13). Each point of Z defines two disjoint triples, and these two intersect the same groups of the GD[3,1,2;12] on Y. This gives us the subconfiguration

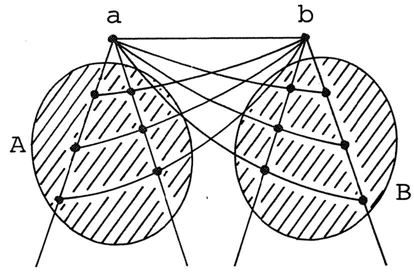


in the STS(13), where <Z> is the point to which Z was identified.

Trying to complete this structure one may write down w.l.o.g. the missing edges incident with the six 'corners', but then there is no way to continue. Hence B = 12 is impossible.

VII. B = 14

Here we have one C-point, 14 B-points, nineteen 3-lines and eight 4-lines. When we remove the C-point we get a linear space on 14 points with one 2-line, twenty-two 3-lines and four 4-lines containing a parallel class of four 3-lines and one 2-line. Twelve of the points have type $3^5 4^1$, and two have type $2^1 3^3 4^2$. This space must look like



There are sixteen triples intersecting both A and B, taking care of 32 cross-edges. But there are 36 cross-edges, so not all can be covered. Contradiction.

G. All points have type A,B or C (and all occur). For any point of type C, all A-points are on the unique 3-line incident with it. Hence (and since B is even, i.e. A+C is odd) we have either C = 2, A = 1 or C = 1, A = 2.

I. A = 1, C = 2

We have nineteen 3-lines and eight 4-lines. Removing the two C-points we get a linear space on 13 points with 3-lines only, i.e. an STS(13). (For: the two C-points must have been joined by a 3-line.) This STS(13) contains two almost parallel classes (sets of four pairwise disjoint blocks) missing the same point (the former A-point). Now the cyclic STS(13) has for each point a unique almost parallel class missing it, and the other STS(13) contains a unique pair of almost parallel classes missing the same point.

Let the cyclic STS(13) be given by {1,3,9} and {2,6,5} (mod 13). The unique almost parallel class missing the point 0 is

{8,10,3}, {9,11,4}, {12,1,7}, {2,6,5}.

Switching the arrow on 2,4,5,6,10,12 (i.e., replacing {2,4,10}, {10,12,5}, {2,6,5}, {4,6,12} by {10,2,5}, {10,4,12}, {2,4,6}, {5,6,12}) we get the other STS(13). The cyclic automorphism $x \mapsto x+1$ (mod 13) has been lost, but $x \mapsto 3x$ (mod 13) still acts.

A new almost parallel class must contain one of these four new blocks. One type is

$$\{10,2,5\},\ \{11,0,6\},\ \{1,3,9\},\ \{4,8,7\}$$

missing 12; there are three such ones, one for each of the blocks $\{10,2,5\}$, $\{2,4,6\}$, $\{5,6,12\}$ permuted by $x \mapsto 3x \pmod{13}$. There is no disjoint almost parallel class with the same missing point.

The other point is

$$\{10,12,4\},\ \{3,5,11\},\ \{6,8,1\},\ \{7,9,2\} \quad \text{missing } 0$$

and

$$\{10,12,4\},\ \{11,2,1\},\ \{3,7,6\},\ \{5,9,8\} \quad \text{missing } 0.$$

But these are not disjoint, and the second STS(13) doesn't contain the block $\{2,6,5\}$ anymore, so again we don't find the required two almost parallel classes. Thus, there is no solution in this case. [In my notes it said that there was a unique solution, but I really don't see it anymore.] This is case (ix) of the theorem.

II. <u>A = 2, C = 1</u>

There are seven 4-lines; four through the unique C-point and three pairwise disjoint in the set Y of B-points. If we take these three lines as groups (rows, columns, symbols) then Y becomes a Latin square of order 4 with one missing symbol in each row or column; two rows missing different symbols and two columns missing different symbols, i.e., a Latin square of order five with a fixed block.

We studied this configuration above in section 4CI and found four solutions. Now our Latin square has a transversal (the four 4-lines through the C-point), so case a) is impossible. Also, the missing edges must be assigned to the two A-points, and just as before this is impossible in cases a) and c). Case b) can be extended in a unique way and yields a solution with $|\mathrm{Aut}| = 6$. Case d) can be extended in an essentially unique way and yields a solution with $|\mathrm{Aut}| = 4$. This is case (x) of the theorem.

This ends the proof of the theorem. Clearly the number of cases considered is very large - often one sentence in the paper corresponds to half an hour of calculation. Consequently, it is unlikely that there is no mistake somewhere. I hope to do this analysis again by computer.

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