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AFDELING ZUIVERE WISKUNDE
(DEPARTMENT OF PURE MATHEMATICS)

ZW 91/77 FEBRUARI

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EXTENSION OF COLOURINGS OF THE EDGES OF A COMPLETE (UNIFORM HYPER)GRAPH

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Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O).

AMS(MOS) subject classification scheme (1970): 05A17, 05B30

Extension of colourings of the edges of a complete (uniform hyper)graph

by

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ABSTRACT

Let $1 \leq m < n$ and consider the complete graph on 2m points K_{2m} as a subgraph of K_{2n}. We prove that if an edge-colouring of K_{2m} (with 2m-1 colours) is given, this colouring can be extended to a colouring of K_{2n} (with 2n-1 colours) iff $2m \leq n$. The corresponding problem for complete h-uniform hypergraphs is discussed, the case h = 3 is solved completely and asymptotic results are given for arbitrary h.

KEYWORDS & PHRASES: *parallelism*

0. INTRODUCTION

Let X be a finite set and let $P_h(X)$ be the collection of all h-element subsets of X. A *parallelism* on $P_h(X)$ is an equivalence relation of $P_h(X)$ such that the members of each equivalence class form a partition of X. Obviously for the existence of a parallelism $h \mid \#X$ is necessary, and in Baranyai [1] it is shown that this condition suffices. A subset Y of X (provided with a given parallelism) is called a *subspace* when the restriction of the equivalence relation on $P_h(X)$ to $P_h(Y)$ yields a parallelism on Y [-in other words, when it never happens that $H_1 \mathbin{//} H_2$ and $H_1 \subset Y$ but H_2 intersects both Y and X\Y]. Cameron [5] remarked that if Y is a proper subspace of X then $2\,\#Y \le \#X$, and in Brouwer [3] it is shown that if $2h \mid \#X$ then there exists a //ism on X with a subspace Y such that $\#Y = \frac{1}{2}\,\#X$. More generally it can be shown in the same way that if $th \mid \#X$ then there exists a //ism on X with a subspace Y such that $\#Y = \frac{1}{t} \cdot \#X$ (see [2], [4]). We conjecture that the requirements $2\,\#Y \le \#X$ and $\#Y \equiv \#X \equiv 0 \pmod h$ suffice in all cases for the existence of a //ism on X with subspace Y. In this note we prove this conjecture for h = 2 or 3 and for h arbitrary, n sufficiently large.

0A. Graph theoretic terminology and upper bound.

These results can be phrased in the language of (hyper)graphs as follows:

A parallelism on $P_h(X)$, where $\#X = n$, is a colouring of the complete h-uniform hypergraph on n vertices with $\frac{h}{n}\binom{n}{h} = \binom{n-1}{h-1}$ colours, where edges with the same colour are disjoint. If Y is a subspace of X, where $\#Y = m$, then any such colouring of Y (with $\binom{m-1}{h-1}$ colours) can be extended to a colouring of X with $\binom{n-1}{h-1}$ colours. A necessary condition for this to be possible is that $m \le \frac{1}{2}n$ [for: the $\binom{m-1}{h-1}$ colours used to colour the h-subsets of Y colour $\frac{n-m}{h}\binom{m-1}{h-1}$ h-subsets of X\Y, so that $\frac{n-m}{h}\binom{m-1}{h-1} \le \binom{n-m}{h}$ hence $\binom{m-1}{h-1} \le \binom{n-m-1}{h-1}$, and consequently $m \le n-m$].

0B. A general existence theorem.

Define for fixed X and Y (where $Y \subset X$, $\#X = n$, $\#Y = m$) the *weight* of an h-subset H of X as $\#(H \cap Y)$. In order to prove the existence of a parallelism on X with subspace Y it suffices to indicate a suitable weight distribution of the parallel classes (by the theorem quoted below). If the parallel classes are $\mathcal{F}_z (z=1,\ldots,\binom{n-1}{h-1})$ and $\mathcal{F}_z$ contains X_{gz} elements of weight g $(0 \le g \le h)$ then obviously the X_{gz} satisfy

(1) $$\sum_g X_{gz} = \frac{n}{h}$$

(2) $$\sum_g gX_{gz} = m$$

(3) $$\sum_z X_{gz} = \binom{m}{g}\binom{n-m}{h-g}.$$

Conversely, given a matrix (X_{gz}) satisfying these equations (where the X_{gz} are nonnegative integers), there exists a parallelism on X with this weight distribution. In particular if for $\binom{m-1}{h-1}$ values of z we have $X_{0z} = \frac{n-m}{h}$ and $X_{hz} = \frac{m}{h}$ and $X_{gz} = 0$ $(1 \le g \le h-1)$, then Y will be a subspace of this parallelism.

That the above is true can be proved in the same way as it was proved in the case $n = 2m$ in [3]; on the other hand, it is a special case of a very general theorem in [2].

1. THE CASE h = 2

By what was stated in section 0B we have to find nonnegative integers X_{gz} such that for $z = 1,\ldots,n-1$ we have

$$\sum_{g=0}^{2} X_{gz} = \tfrac{1}{2}n$$

$$\sum_{g=0}^{2} gX_{gz} = m$$

$$\sum_z X_{gz} = \binom{m}{g}\binom{n-m}{2-g} \qquad (g=0,1,2)$$

and for m-1 values of z we have

$$X_{0z} = \tfrac{1}{2}(n-m), \quad X_{2z} = \tfrac{1}{2}m \text{ and } X_{1z} = 0.$$

The unique solution is

$$X_{0z} = \frac{n}{2} - m,\ X_{1z} = m,\ X_{2z} = 0 \text{ for n-m values of z}$$

and

$$X_{0z} = \tfrac{1}{2}(n-m),\ X_{1z} = 0,\ X_{2z} = \tfrac{1}{2}m \text{ for m-1 values of z.}$$

In particular there is a solution.

2. THE CASE $m \mid n$

Suppose $n = mt$. Then (as already remarked in [2] and [4]) a solution exists. For any ordered t-tuple $(h_1,\ldots,h_t)$ with $\sum h_j = h$ take $\frac{h}{n} \Pi_j \binom{m}{h_j}$ columns z with ($X_{gz} = 0$ if g does not occur among the h_j and)

$$X_{gz} = \textstyle\sum_{g=h_j} \frac{m}{h}.$$

Obviously

$$\textstyle\sum_g X_{gz} = t\cdot \frac{m}{h} = \frac{n}{h},$$

$$\textstyle\sum_g gX_{gz} = \sum_j h_j \frac{m}{h} = m; \text{ also}$$

$$\sum_z X_{gz} = \sum_{\substack{(h_1\ldots h_t)\\ \Sigma h_j=h}} \left(\sum_{g=h_j} \frac{m}{h}\right)\cdot \frac{h}{n} \Pi_j \binom{m}{h_j} =$$

$$= \frac{1}{t} \sum_{\substack{(h_1,\ldots,h_t)\\ \Sigma h_j=h}} \left(\sum_{g=h_j} 1\right)\cdot \Pi_j \binom{m}{h_j} =$$

$$= \sum_{\substack{(h_1,\ldots,h_t)\\ \Sigma h_j=h,\, g=h_1}} \Pi_j \binom{m}{h_j} = \binom{m}{g}\binom{n-m}{h-g},$$

Hence this yields a solution of (1) - (3).

Perhaps you remark that

$$\frac{h}{n} \Pi_j \binom{m}{h_j}$$

need not be an integer; but, since the t-tuples $(h_1,\ldots,h_t)$, $(h_t,h_1,\ldots,h_{t-1})$, $,\ldots,(h_2,\ldots,h_t,h_1)$ yield the same columns all we need is that

$$\frac{h\sigma(\sigma)}{n} \Pi_j \binom{m}{h_j}$$

is an integer, where $\sigma(\sigma)$ is the order of the cyclic permutation $(h_1,\ldots,h_t)$ $\to (h_t,h_1,\ldots,h_{t-1})$. Now

$$\frac{h\sigma(\sigma)}{n} \Pi_j \binom{m}{h_j} = \sum_{i=1}^{t} \frac{\sigma(\sigma)}{t} (\Pi_{j\neq i} \binom{m}{h_j}) \cdot \binom{m-1}{h_i-1} =$$

$$\sum_{i=j}^{\sigma(\sigma)} (\Pi_{j\neq i} \binom{m}{h_j}) \cdot \binom{m-1}{h_i-1},$$ which is an integer.

It remains to prove that for $\binom{m-1}{h-1}$ values of z we have $X_{hz} = \frac{m}{h}$ and $X_{gz} = 0$ $(1 \le g \le h-1)$. We obtain such solutions from the t-tuples $(0\ldots h\ldots 0)$. The number of solutions of this type is $\frac{h\sigma(\sigma)}{n} \Pi_j \binom{m}{h_j} = \binom{m-1}{h-1}$ as required.

3. THE CASE h = 3

For arbitrary h we can somewhat simplify our equations: If we let $X_{0z} = \frac{n-m}{h}$, $X_{hz} = \frac{m}{h}$ and $X_{gz} = 0$ $(1 \le g \le h-1)$ for $z = \binom{n-1}{h-1} - \binom{m-1}{h-1} + 1$, $,\ldots, \binom{n-1}{h-1}$ then we have to solve

(1') $$\sum_{g=1}^{h-1} X_{gz} \le \frac{n}{h}$$

(2') $$\sum_{g=1}^{h-1} gX_{gz} = m$$

(3') $$\sum_z X_{gz} = \binom{m}{g} \binom{n-m}{h-g} \qquad (1 \le g \le h-1)$$

where now z runs from 1 up to $\binom{n-1}{h-1} - \binom{m-1}{h-1}$ [since for these z we can define X_{0z} and X_{hz} by $X_{0z} = \frac{n}{h} - \sum_{g=1}^{h-1} X_{gz}$ and $X_{hz} = 0$ and from the other equations it follows that (3) holds, that is, $\sum_{\text{all } z} X_{0z} = \binom{m}{0} \binom{n-m}{h-0}$].

Note that

(4')
$$\sum_z 1 = \binom{n-1}{h-1} - \binom{m-1}{h-1}$$

follows from (1') - (3').
Our solutions (X_{gz}) will contain many identical columns say columns (Y_{gi}) with multiplicity N_i. Rewriting (1') - (3') we get:

(1")
$$\sum_{g=1}^{h-1} Y_{gi} \le \frac{n}{h}$$

(2")
$$\sum_{g=1}^{h-1} gY_{gi} = m$$

(3")
$$\sum_i N_i Y_{gi} = \binom{m}{g}\binom{n-m}{h-g}$$

(4")
$$\sum_i N_i = \binom{n-1}{h-1} - \binom{m-1}{h-1}.$$

In the special case h = 3 we need two different columns; in the table below we give N_1 and Y_{2i} (i=1,2) - then $Y_{1i} = m - 2Y_{2i}$, and $N_2 = \binom{n-1}{2} - \binom{m-1}{2} - N_1$.

Case	N_1	Y_{21}	Y_{22}
$m \le \frac{n}{3}$, m even	$\frac{1}{2}(n-m)(n-m-1)$	0	$\frac{m}{2}$
$m \le \frac{n}{3}$, m odd	$\frac{1}{2}(n-m)(n-2m)$	0	$\frac{m}{3}$
$m \ge \frac{n}{3}$:			
$n \equiv m \equiv 0 (\mathrm{mod}\ 2)$	$\frac{3}{2}\ m(n-m-1)$	$\frac{1}{6}(4m-n)$	$\frac{m}{2}$
$n \equiv 1,\ m \equiv 0 (\mathrm{mod}\ 2)$	$\frac{3}{2}\ m(n-m)$	$\frac{1}{6}(4m-n+1)$	$\frac{m}{2}$
$n \equiv m \equiv 1 (\mathrm{mod}\ 2)$	$\frac{3}{2}\ (m-1)(n-m-3)$	$\frac{1}{6}(4m-n-3)$	$\frac{m-1}{2}$
$n \equiv 0,\ m \equiv 1 (\mathrm{mod}\ 2)$	$\frac{3}{2}\ (m-1)(n-m)$	$\frac{1}{6}(4m-n)$	$\frac{m-1}{2}$

That this is indeed a solution can be readily verified.

3. THE CASE h = 4

In this case we have an easy solution for $n \geq 4m$; we did not bother to look for solutions if $2m < n < 4m$.
Here the matrices $(Y_{gi})_{1 \leq g \leq 3,\ 1 \leq i \leq 3}$ can be taken as

$$\begin{bmatrix} m & 0 & 0 \\ 0 & \frac{1}{2}m & 0 \\ 0 & 0 & \frac{1}{3}m \end{bmatrix} \quad \text{if } 3|m \quad \text{and}$$

$$\begin{bmatrix} m & 0 & \frac{1}{4}m \\ 0 & \frac{1}{2}m & 0 \\ 0 & 0 & \frac{1}{4}m \end{bmatrix} \quad \text{otherwise.}$$

The multiplicities N_i are uniquely determined from the (Y_{gi}) and (3"), (4").

5. ASYMPTOTIC RESULTS

For n large (for instance $n \geq mh^{3/2}$) we can give an explicit solution as follows:
We define the matrix (Y_{gi}) and multiplicities N_i $1 \leq g \leq h-1$, $1 \leq i \leq h-1$ with help of the numbers Y_g $2 \leq g \leq h-1$ which are to be chosen later.
The matrix (Y_{gi}) will contain 0'-s except in the first row and the main diagonal - this explains why the indices g and i will be a little bit mixed.
Let $Y_{gi} = \delta_{gi} Y_g$ for $2 \leq g \leq h-1$ and $1 \leq i \leq h-1$

$$Y_{1i} = m - \sum_{g=2}^{h-1} gY_{gi} = m - iY_i \quad (\text{supposing } Y_1 = 0)$$

$$N_g = \frac{1}{Yg} \binom{m}{g} \binom{n-m}{h-g}$$

$$N_1 = \binom{n-1}{h-1} - \binom{m-1}{h-1} - \sum_{i=2}^{h-1} N_i$$

For this to be a solution first of all the Y_{gi} and the N_i must be nonnegative integers, that is,

(5) $$0 \le Y_i \le \frac{m}{i},$$

(6) $$Y_g \mid \binom{m}{g}\binom{n-m}{h-g},$$

(7) $$\binom{n-1}{h-1} - \binom{m-1}{h-1} - \sum_{g=2}^{h-1} \frac{1}{Y_g} \binom{m}{g}\binom{n-m}{h-g} \ge 0$$

and in order to satisfy (1") we need $n \ge mh$, while (2") - (4") are satisfied automatically.

One possible choice would be to take $Y_g = 1$ for all g. This satisfies (5) and (6), and since (7) is a polynomial in n of degree h-1 with leading coefficient $\frac{1}{(h-1)!} > 0$ this surely yields a solution when n is large enough. To get a bound that is linear in m we have to do some work:
Choose $Y_2 = [\frac{m}{2}]$; note that this satisfies (5) and (6) (since $[\frac{m}{2}] \mid \binom{m}{2}$).
If $g \mid m$ then choose $Y_g = \frac{m}{g}$; again this is OK.
In the general case choose

$$Y_g = \frac{m(h,g)}{h(m,g)} .$$

This choice satisfies (5) since $h \mid m$ so that

$$(h,g) \le (m,g) \text{ and } Y_g \le \frac{m}{h} < \frac{m}{g}.$$

also (6) is satisfied, for if $(m,g) = am + bg$ then

$$\frac{h}{m}\frac{(m,g)}{(h,g)} \binom{m}{g} = a\frac{h}{(h,g)}\binom{m}{g} + b\frac{h}{(h,g)}\binom{m-1}{g-1}$$

is integral.
Note that

$$Y_g \ge \frac{m}{h} \cdot \frac{1}{\frac{1}{2}g} = \frac{2m}{gh}$$

in this case (since if $g \nmid m$ then $(g,m) \le \frac{1}{2}g$), while also if $g \mid m$ then $Y_g = \frac{m}{g} \ge \frac{2m}{gh}$.

Now concerning (7) we find

$$\binom{n-1}{h-1} - \binom{m-1}{h-1} - \sum_{g=2}^{h-1} \frac{1}{Y_g} \binom{m}{g}\binom{n-m}{h-g} \geq$$

$$\binom{n-1}{h-1} - \binom{m-1}{h-1} - \frac{1}{Y_2}\binom{m}{2}\binom{n-m}{h-2} - \frac{h}{2}\sum_{g=3}^{h-1} \binom{m-1}{g-1}\binom{n-m}{h-g} \geq$$

$$\binom{n-1}{h-1} - \binom{m-1}{h-1} - m\binom{n-m}{h-2} - \frac{h}{2}\binom{m-1}{2}\binom{n-m}{h-3} - \frac{h}{2}\binom{m-1}{3}\binom{n-m}{h-4}$$

$$- \frac{h}{2}(h-5) \binom{m-1}{4}\binom{n-m}{h-5} \geq$$

$$\binom{n-1}{h-1} \Big\{1 - \frac{1}{2^{h-1}} - \frac{m(h-1)}{(n-1)} - \frac{h(h-1)(h-2)}{2}\binom{m-1}{2}\frac{1}{(n-1)(n-2)} -$$

$$\frac{h(h-1)(h-2)(h-3)}{2}\binom{m-1}{3}\frac{1}{(n-1)(n-2)(n-3)} -$$

$$- \frac{h(h-1)(h-2)(h-3)(h-4)(h-5)}{2(n-1)(n-2)(n-3)(n-4)}\binom{m-1}{4}\Big\}$$

$$\geq \binom{n-1}{h-1}\Big\{1 - \frac{1}{2^{h-1}} - \frac{mh}{n} - \frac{m^2h^3}{4n^2} - \frac{m^3h^4}{12n^3} - \frac{m^4h^6}{48n^4}\Big\}$$

$$\geq \binom{n-1}{h-1}\Big\{1 - \frac{1}{8} - \frac{1}{2} - \frac{1}{4} - \frac{1}{24} - \frac{1}{48}\Big\} = \binom{n-1}{h-1} \cdot \frac{3}{48} > 0$$

where we used $n \geq mh^{3/2}$ and $h \geq 4$ (and the facts that $\binom{m-1}{4}\binom{n-m}{h-5}$ is larger than $\binom{m-1}{g-1}\binom{n-m}{h-g}$ for $g \geq 6$, and that $\frac{a-1}{b-1} < \frac{a}{b}$ if $a < b$).
This proves that a solution exists when $n > mh^{3/2}$.

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