A Pointed Delaunay Pseudo-Triangulation of a Simple Polygon

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Abstract

We present a definition of a pointed Delaunay pseudo-triangulation of a simple polygon. We discuss why our definition is reasonable. Our approach will be motivated from maximal locally convex functions, and extends the work of Aichholzer et al.[1]. Connections between the polytope of the pointed pseudo-triangulations and the Delaunay pseudo-triangulation will be given.

1 Introduction

Since the Delaunay triangulation is an important concept in computational geometry it is natural to ask for an equivalent in the pseudo-triangulation world. More precisely we are interested in a pointed pseudo-triangulation which is Delaunay-like. A pseudo-triangulation is pointed if every point is incident to an angle greater than π. Without the restriction to pointedness the Delaunay triangulation itself would be the most Delaunay-like pseudo-triangulation. In the following we skip the term *pointed* for the pointed Delaunay pseudo-triangulation.

As a simple case we consider the pseudo-triangulations of a simple polygon. For simple polygons the Delaunay triangulation must contain the boundary edges. Thus we are interested in generalizing the *constrained Delaunay triangulation* (CDT)[2] for simple polygons.

Delaunay-like is a quite soft expression. What we are looking for is a pseudo-triangulation which shares many of the properties of the CDT. For a survey for properties of the Delaunay triangulation see [4]. As the most important criterion we demand that for a convex polygon Delaunay triangulation and Delaunay pseudo-triangulation must coincide.

For the rest of the paper we assume that the points are given in general position.

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2 A reasonable definition of the Delaunay pseudo-triangulations of a simple polygon

Constrained regularity is a concept which can be applied on triangulations and pseudo-triangulations. A triangulation or pseudo-triangulation is said to be *regular* if it can be represented by a downward projection of a convex lifting. Since the Delaunay triangulation is the projection of the lower convex hull of the paraboloid lifting the Delaunay triangulation is regular [3]. Moreover the constrained Delaunay triangulation is constrained regular. In [1] constrained regular pseudo-triangulations were introduced. We use this approach to find a lifting map which is similar to the paraboloid lifting.

From now on we are considering a polygon P. The n vertices of P are given in a clockwise order and named $p_1, p_2, \ldots, p_n$. A vertex is called *corner* if its internal angle is smaller than π. It is called a *reflex* vertex otherwise.

Our goal is to define the Delaunay pseudo-triangulation with help of a certain maximal locally convex function. These functions were studied in [1] and we are using the *optimality theorem* of this paper for our definition. A function is locally convex on P, if it is convex on every line segment in P. Let h_i be a given height for every vertex p_i. f^* is the maximal locally convex function which satisfies $f^*(p_i) \leq h_i$. The lifting induced by f^* is piecewise linear and projects down to a pseudo-triangulation $\mathcal{PT}(h)$. f^* and $\mathcal{PT}(h)$ are unique. If we set $h_i = |p_i|^2$ the CDT of P is $\mathcal{PT}(h)$.

The pseudo-triangulation $\mathcal{PT}(h)$ is not necessaryly pointed. Since we require pointedness we have to choose the heights in such a way that f^* will induce a pointed pseudo-triangulation. The corners of P are pointed because their outer angle is greater than π. The reflex vertices must contain the big angle inside. Thus they must part of a reflex chain of some pseudo-triangle (in [1] these vertices are called *incomplete*). The *optimality theorem* says that in this case $f^*(p_i) > h_i$. Their height is defined by the 3 corners of the pseudo-triangle which contains the reflex angle of the reflex vertex.

To assure that the reflex vertices are part of a reflex chain, we let their h_i be very high. With such a lifting it is not possible to obtain local convexity when $f^*(p_i) = h_i$ for any reflex vertex. Hence they will be pointed.

Now we discuss how an appropriate lifting will look like. We define

$$h_d := \begin{cases} f_1(p_i) & \text{if } p_i \text{ is corner in } P \\ f_2(p_i) & \text{otherwise} \end{cases}$$

For f_1 we choose $f_1(p_i) = |p_i|^2$. This guarantees that for convex polygons the Delaunay triangulation and the Delaunay pseudo-triangulation coincide. f_2 can be any strictly concave function with $\min_P(f_2(p_i)) > \max_P(f_1(p_i))$.

Lemma 1 *The maximal locally convex function in the domain of P defined by h_d projects down to a pointed pseudo-triangulation.*

Proof. Let p_c be a reflex vertex of P and let l be a line in P that crosses the neighborhood of p_c. The line l ends at the boundary of P. We name its endpoints e_1 and e_2. The point e_1 lies on the boundary segment (b_1, b_2) of P; the point e_2 on the segment (b_3, b_4). Figure 1 illustartes the situation.

We know that for $i \in \{1, 2, 3, 4\} f^*(b_i) \leq f_2(b_i)$ by the definition of f^*. Since f_2 is concave and f^* convex we know that $f^*(e_1) \leq f_2(e_1)$ and $f^*(e_2) \leq f_2(e_2)$. Thus we have $f^*(e_1) < f_2(p_c)$ and $f^*(e_2) < f_2(p_c)$ (f_2 is strictly concave). But since f^* must be convex on l we deduce that $f^*(p_c) < f_2(p_c)$. Thus $f^*(p_c) < h_c$ for all reflex vertices and as discussed above $\mathcal{PT}(h_d)$ is pointed. ☐

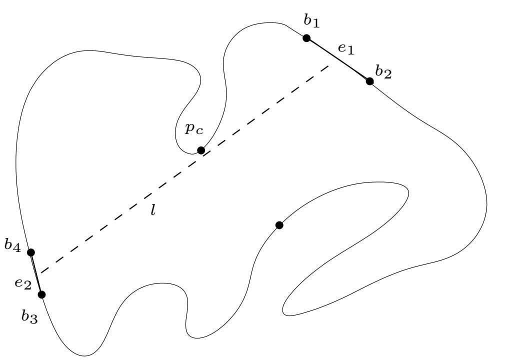
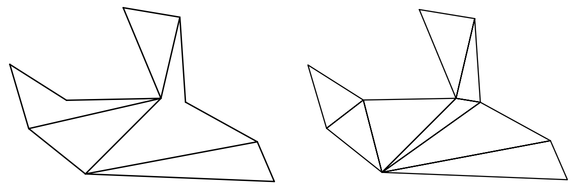


Figure 1: Illustration of the proof of Lemma 1

Since we have defined a unique pointed pseudo-triangulation by a maximal locally convex function, we are able to define the Delaunay pseudo-triangulation with help of h_d. It turns out that the height of the reflex vertices are not relevant.

Definition 1 *Let P be a simple polygon and h_d the lifting defined above. We name $\mathcal{PT}(h_d)$ the Delaunay pseudo-triangulation of P.*

Figure 2.a shows an easy example. We have a polygon with 10 vertices, 3 of them are reflex. The locally convex surface induced by the paraboloid lifting of the complete vertices is shown in Figure 3.



(a) (b)

Figure 2: A Delaunay pseudo-triangulation of a polygon (a) and the CDT of the same polygon (b)

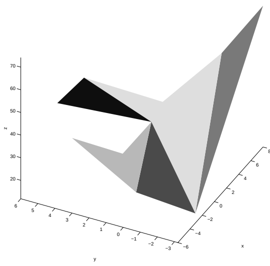


Figure 3: The lifted polygon from Figure 2

The $\mathcal{PT}(h_d)$ can be constructed by flipping to optimality [1]: We start with any triangulation of P and then flip away all reflex edges by convexifying and edge removing flips. After at most $O(n^2)$ flips we will reach the optimum. For every flip a linear system of equations must be solved. This can be done in $O(n + i^3)$ time, where i is the number of incomplete vertices. By flipping to optimality the $\mathcal{PT}(h_d)$ can be computed in $O(n^5)$.

3 Relations to the PPT Polytope

This section presents some evidence why our choice of definition is reasonable. The Delaunay triangulation can be expressed as an optimal vertex of a high-dimensional polytope. We will show a similar interpretation for the Delaunay pseudo-triangulation.

The secondary polytope of a point set S represents every regular triangulation of S as a vertex in $\mathbf{R}^{|S|}$. Moreover flips corresponds to edges in this polytope. The scalar product of a vertex of the secondary polytope and a height vector gives the volume under the lifted triangulation. If the height vector lifts the points to the paraboloid the minimization of the volume will give us the Delaunay triangulation. Recently a polytope for regular pseudo-triangulations of simple polygons was introduced [1, Lemma 9.2]. Again we can represent the volume under the lifted surface as a scalar product with some height vector. Thus $\mathcal{PT}(h_d)$ can be defined as an optimal vertex in this polytope. For practical computation this approach is not useful, because the polytope is not given by a set of inequalities.

Even more interesting is the connection between $\mathcal{PT}(h_d)$ and the polytope of pointed pseudo-triangulations (PPT Polytope) defined in [6]. For a class of simple polygons we present a formulation of $\mathcal{PT}(h_d)$ as an objective function of the PPT polytope. This gives us a completely independent way to define $\mathcal{PT}(h_d)$ as a generalization of a Delaunay triangulation for this class.

Our starting point is the objective function for the Delaunay triangulation in the PPT polytope. Of course this can only be done for points in convex position. Fortunately, for points in convex position, there is a connections between the secondary polytope and the PPT polytope (in this case the two polytopes are combinatorially equivalent)[6].

We abbreviate the the signed area of the triangle p_i, p_j, p_k with $[p_i, p_j, p_k]$ and the point $(0, 0)$ with 0. A possible formulation of the PPT polytope is the following:

$$\begin{array}{lrcl} \forall(p_i, p_j) & \langle p_i - p_j, v_i - v_j\rangle + d_{ij} & = & [p_i, p_j, 0]^2 \\ & d_{ij} & \leq & 0 \end{array}$$

The PPT polytope is originally expressed by the variables v_i and lives in $\mathbf{R}^{2n}$. We added the slack variables d_{ij} to simplify further calculations.

A vertex is represented by a maximal set of inequalities which hold with equality, i.e. $d_{ij} = 0$. The edges of a pointed pseudo-triangulation of a vertex can be expressed as the set $\{(i, j) \mid d_{ij} = 0\}$.

Let $a \in \mathbf{R}^n$ a point of the secondary polytope. As discussed above, the minimization of $\sum_i a_i |p_i|^2$ leads to the Delaunay triangulation. We can express a_i in terms of d_{ij}, i.e.

$$a_i = -\frac{d_{i-1,i+1}}{[p_{i-1}, p_i, p_{i+1}]} + [p_{i-1}, p_i, p_{i+1}]$$

Let us abbreviate the signed area $[p_{i-1}, p_i, p_{i+1}]$ with E_i and let

$$c_k := -\frac{|p_k|^2}{E_k}$$

This gives us for a minimization problem the objective function

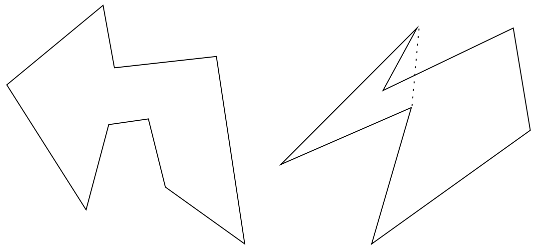
$$minimize \qquad \sum_{i \in \{1,\ldots,n\}} c_i d_{i-1,i+1} \qquad (1)$$

over the PPT polytope.

We will now perturb the convex polygon. Convexity is not necessary anymore, but we forbid heavy deformations. More precisely we demand the following:

Definition 2 *We call a polygon P neighborly visible, if the two neighbors of any corner of P can see each other.*

Figure 4.a shows a neighborly visible polygon, Figure 4.b a polygon which is not neighborly visible.



(a) (b)

Figure 4: Examples of neighborly and not neighborly visible polygons

Theorem 2 *Let P be a neighborly visible polygon. Furthermore let all corners of P lie on its convex hull. Then minimizing the objective function (1) over the PPT polytope will induce $\mathcal{PT}(h_d)$ as solution of the linear program.*

Proof. Due to the limited space we will only sketch the proof of the theorem. We construct a pointed pseudo-triangulation of the point set P as the extension of the polygon P. For this we set all d_{ij} belonging to the boundary of P to 0. Furthermore we triangulate the concave chains of P and set the used d_{ij} to 0 if necessary. If we would allow corners which are not part of the convex hull of P, we have to add pseudo-triangles. This would immediately lead to dependencies which result in a different solution of the LP. To give an idea of the proof we first observe the case where P is a convex polygon. The dual of the LP leads to the an stressed framework which is not in equilibrium condition. The stress for each edge is given by ω^*_{ij}. It holds

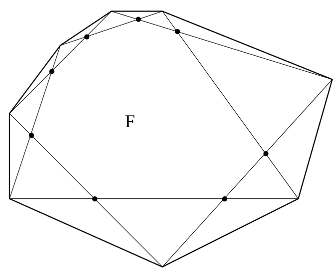


Figure 5: The construction of an equilibrium stress used in the proof

$$\forall i \ \sum_j \omega^*_{ij}(p_i - p_j) = -c_{i+1}(p_i - p_{i+2}) - c_{i-1}(p_i - p_{i-2})$$

We can adjust this stress by introducing additional arcs between every pair (p_i, p_{i+2}) (see Figure 5). Then

$$\omega_{ij} := \begin{cases} \omega^*_{ij} + c_{i+1} & \text{if } (i+2) \bmod n = j \\ \omega^*_{ij} & \text{otherwise} \end{cases}$$

forms an equilibrium stress. With help of the technique used in [5] we can turn the constructed graph into a planar one. Now we are able to apply the reverse of the Maxwell-Cremona theorem [8]. This leads to a lifting of the stressed graph. The heights of all points induced by the lifting can be calculated with help of the newly added edges and *without* knowing the Delaunay triangulation of P. First we fix the location of the central face F; then we tilt the points upwards. The stress $\omega_{i,i+2}$ gives us the information how high the point p_i will be tilted. Figure 6 gives an idea how the tilting process looks like. After some calculation it turns out that the height is exactly $|p_i|^2$. Furthermore the sign of the stress guarantees that the edges of the Delaunay triangulation span a convex surface.

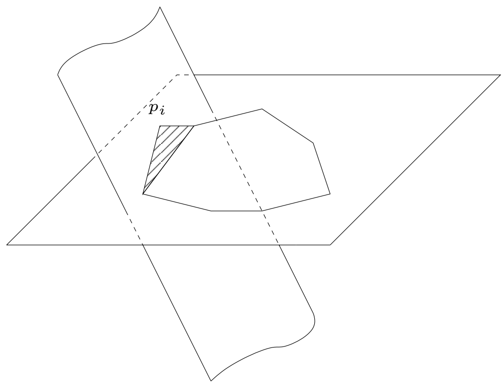


Figure 6: The tilting process

For a general neighborly visible polygon P we can now deduce the following. Since P is neighborly visible, there exists a single central face F. All corners of P will lifted to the paraboloid, because the arguments for the convex case still hold. The reflex vertices corresponds to 2 vertices in the lifting. The first one has height 0 and the second one is shifted vertically. The second vertex is the result of a tilting process outside P and belongs to the lifted pseudo-trinagulation surface. Again the pseudo-triangulation surface is convex due to the signs of the stress. Therefore the solution of the LP coincides with $\mathcal{PT}(h_d)$. □

4 Future work

Defining a Delaunay pseudo-triangulation for a simple polygon is just a first step towards a pointed Delaunay pseudo-triangulation of a general augmented polygon. Since we give a number of arguments why the definition of the Delaunay pseudo-triangulation is reasonable we are convinced that our approach can be generalized in a natural way.

One can think of using the concept of complete and incomplete vertices introduced in [1] for non-pointed Delaunay pseudo-triangulation of point sets. If we define which vertices of a point set are complete and incomplete in advance, we can define a maximal locally convex function which preserves the state of the vertices. Lifting all pointed vertices to the paraboloid would give us a Delaunay-like pseudo-triangulation. This approach needs further investigation and seems to be promising.

Since pointed pseudo-triangulations have no lifting it is completely unclear how to define a pointed Delaunay-like pseudo-triangulations. In [6] a “canonical” pseudo-triangulation has been defined as optimal vertex of the PPT polytope with the help a certain canonical objective function. This canonical pseudo-triangulation has several nice properties like an interesting lifting function for the convex case (see[7]). But it turned out that it is not a good candidate for the pointed Delaunay pseudo-triangulation. It has several properties which are not characteristic for the Delaunay Triangulation. One possible way to overcome the “non regularity” of pointed pseudo-triangulations could be to find a different objective function over the PPT polytope.

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