A Note on Simultaneous Embedding of Planar Graphs (Abstract)*

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1 Introduction

Let G_1 and G_2 be a pair of planar graphs such that $V(G_1) = V(G_2) = V$. A *simultaneous embedding* [6] $\Psi = (\Gamma_1, \Gamma_2)$ of G_1 and G_2 is a pair of crossing-free drawings Γ_1 and Γ_2 of G_1 and G_2, respectively, such that for every vertex $v \in V$ we have $\Gamma_1(v) = \Gamma_2(v)$. If every edge $e \in E(G_1) \cap E(G_2)$ is represented with the same simple open Jordan curve both in Γ_1 and in Γ_2 we say that Ψ is a *simultaneous embedding with fixed edges*. If the edges of G_1 and G_2 are represented with straight-line segments in Γ_1 and Γ_2 we say that Ψ is a *simultaneous geometric embedding*. The existence of simultaneous geometric embeddings for pairs of paths, cycles, and caterpillars is shown in [2], where also counter-examples for pairs of general planar graphs, pairs of outerplanar graphs, and triples of paths are presented.

Concerning the the computation of (non-geometric) simultaneous embeddings, Erten and Kobourov [6] presented an $\mathcal{O}(n)$-time algorithm to simultaneously embed any pair of planar graphs on the $\mathcal{O}(n^2)\times\mathcal{O}(n^2)$ grid with at most three bends per edge, where n is the number of vertices of G_1 and G_2. If the two graphs are trees then the number of bends per edge can be reduced to one. Furthermore, in [6] an $\mathcal{O}(n)$-time algorithm to compute a simultaneous embedding with fixed edges of a tree and a path on the $\mathcal{O}(n) \times \mathcal{O}(n^2)$ grid with no bends on the path-edges and at most one bend per edge on the tree-edges is described.

In this note we revisit the elegant technique of Erten and Kobourov [6] to present some new results on simultaneous embeddings with fixed edges. We prove that the pairs outerplanar graph - path and outerplanar graph - cycle admit a simultaneous embedding with fixed edges and at most one bend per edge. For the pair outerplanar graph - path, the edges of the path are straight-line segments. We also present some extensions of the results in [6] about simultaneous embeddings of planar graphs that are immediate consequence of existing literature. For reasons of space some proof are sketched or omitted.

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2 Preliminaries

The algorithms to compute a simultaneous embedding of a pair of planar graphs and of a pair of trees presented in [6] are an elegant combination of the technique by Kaufmann and Wiese [7] for point-set embedding and of the simultaneous embedding strategy for two paths by Brass et al. [2]. In this section, we first recall the main ideas of Erten and Kobourov for simultaneous embedding of two planar graphs G_1 and G_2 and then make a couple of observations on how to combine these ideas with known literature in order to extend some of the results in [6].

Suppose first that both G_1 and G_2 are Hamiltonian. Let C_1 and C_2 be two Hamiltonian cycles of G_1 and G_2, respectively. For each cycle, one arbitrarily chosen edge (called *closing edge* in the following) is removed in order to obtain two Hamiltonian paths P_1 and P_2 for G_1 and G_2, respectively. Paths P_1 and P_2 are simultaneously embedded by the algorithm of Brass et al. [2]. Erten and Kobourov add the remaining edges of G_1 and of G_2 to the drawing by using the technique of Kaufmann and Wiese [7]. Namely, let δ be the maximum slope of a segment of the path defined by P_1. The closing edge for cycle C_1 is drawn as a polyline with two segments whose slopes are δ' and $-\delta'$, where $\delta' = \delta + \epsilon$ for an arbitrary small *epsilon* > 0. The remaining edges of G_1 are divided into edges that are inside C_1 and edges that are outside C_1. The edges that are inside (outside) C_1 are drawn inside (outside) C_1 as polylines each consisting of two segments having slopes δ' and $-\delta'$, respectively. Possible overlaps between segments corresponding to different edges can be removed by a simple rotation technique described in [7]. The drawing of G_2 is computed with an analogous procedure starting form the drawing of path P_2. Erten and Kobourov show that the overall time complexity of the procedure is $\mathcal{O}(n)$ where n is the number of vertices in G_1 and in G_2; also, if the bends may not be at integer grid points, the size of the grid is $\mathcal{O}(n^2) \times \mathcal{O}(n^2)$ ($\mathcal{O}(n^3) \times \mathcal{O}(n^3)$ else).

If G_1 and G_2 are not sub-Hamiltonian, then Erten and Kobourov use the $\mathcal{O}(n)$ algorithm by Chiba and Nishizeki [3] to augment the graphs in order to make them Hamiltonian. The augmentation is done by adding dummy edges and by splitting each edge with at most one dummy vertex. A simultaneous embedding of the augmented graphs can now be computed in linear time be the technique described above. After such an embedding is computed the dummy edges are removed and the dummy vertices are treated as bend points. As a result, every edge (u, v) that is split by a dummy vertex w ends up having at most three bends, one between u and w, one at w and one between w and v. Observe that the bend at w can be avoided if the two segments of (u, w) and (w, v) incident on w have the same slope. In [7] it is described how to rotate the segments incident on w so to avoid the third bend. However this rotation increases the area of the drawing, and Erten and Kobourov do not apply the rotation. As a result, they prove the existence of a simultaneous embedding of two planar graphs in an integer grid of size $\mathcal{O}(n^2) \times \mathcal{O}(n^2)$ and with at most three bends per edge (or $\mathcal{O}(n^3) \times \mathcal{O}(n^3)$ if bends are at integer grid points). In [5] it is showed a variant of the point-set embedding algorithm of Kaufmann and Wiese [7] that makes it possible to never have a third bend on the split edges and thus it does not require any rotation of the edges. A combination of the results in [5] and of the technique in [6] leads therefore to the following improvement of the result by Erten and Kobourov.

Theorem 1 *Let G_1 and G_2 be two planar graphs such that $V(G_1) = V(G_2) = V$. G_1 and G_2 can be simultaneously embedded in $\mathcal{O}(n)$ time, using at most two bend per edge, on an integer grid of size $\mathcal{O}(n^2) \times \mathcal{O}(n^2)$, where $n = |V|$.*

The approach of Erten and Kobourov is such that if the two graphs can be augmented without adding dummy vertices, then we have a simultaneous embedding with at most one bend per edge. In the special case of trees, they augment the graphs to become Hamiltonian without using dummy vertices. We recall that every sub-Hamiltonian graph has the property that it can be augmented with only edge addition to become Hamiltonian. Although it is $\mathcal{NP}$-hard to recognize the sub-Hamiltonian graphs, there are some families of graphs that are known to be sub-Hamiltonian and for which the edge augmentation can be found in time proportional to the number of the vertices. Among such families, we mention here outerplanar graphs and series-parallel graphs (see, e.g., [1, 4]). We can therefore extend Theorem 3 of [6] to families of graphs other than trees.

Theorem 2 *Let G_1 and G_2 be two graphs such that $V(G_1) = V(G_2) = V$ and G_i $(i = 1, 2)$ is either a series-parallel graph or an outerplanar graph. G_1 and G_2 can be simultaneously embedded in $\mathcal{O}(n)$ time, using at most one bend per edge, on an integer grid of size $\mathcal{O}(n^2) \times \mathcal{O}(n^2)$, where $n = |V|$.*

3 Simultaneous Embedding with Fixed Edges

In [6] the following result is proved.

Theorem 3 *[6] Let T be a tree and let P be a simple path such that $V(T) = V(P) = V$. T and P can be simultaneously embedded with fixed edges in $\mathcal{O}(n)$ time, using at most one bend for each edge of T and zero bends for each edge of P, on an integer grid of size $\mathcal{O}(n) \times \mathcal{O}(n^2)$, where $n = |V|$.*

The proof of Erten and Kobourov behind Theorem 3 exploits the technique described in Section 2. Namely, they present a linear-time recursive strategy for computing a Hamiltonian path P_T of the tree that contains all edges shared by P and T and then use such a Hamiltonian path and path P itself to compute a simultaneous embedding by the technique described Section 1. Since P and P_T are drawn with straight-line segments and the remaining edges of T have at most one bend, the theorem follows. It is worth remarking that the key idea in the proof of Erten and Kobourov for Theorem 3 is reducing the tree-path simultaneous embedding problem with fixed edges to the combinatorial question of finding an augmented Hamiltonian cycle in the tree with the additional constraint that the cycle must contain all edges of P. In this section we use the same approach of Erten and Kobourov to extend Theorem 3.

We need some more definitions. A k-pages book embedding $\phi(G)$ of a graph G is a crossing-free drawing of G such that the vertices of G are drawn as points along a straight line l called *spine*, and each edge is drawn as a simple Jordan curve on one among k half-planes, called *pages*, having l as common boundary. The minimum number of pages over all book embeddings of a graph G is called the *page number* of G. A graph has page number one if and only if it is outerplanar and that a graph has page number two if and only if it is sub-Hamiltonian [1].

Let $v_0, v_1, \ldots, v_{n-1}$ be the vertices of G in the order they are encountered along the spine from left to right. We say that $v_0, v_1, \ldots, v_{n-1}$ is the *linear ordering induced by* $\phi(G)$. An edge $e_1 = (v_i, v_j) \in E(G)$ is said to be *nested inside* another edge $e_2 = (u_h, v_l) \in E(G)$ in $\phi(G)$ if e_1 and e_2 are drawn on the same page and $h < i < j < l$. A *r-rainbow* is a set of r edges $e_0, e_1, \ldots, e_{r-1} \in E(G)$ such that e_{i+1} is nested inside e_i $(0 \leq i < r - 1)$. Edges e_0 and e_1 are called the *top edge* and the *second edge* of the r-rainbow, respectively.

In what follows we are interested in book embeddings on at most two pages. In this case the book embedding is a planar drawing and the two pages are the two half-planes defined by the straight line representing the spine. We assume that the spine is drawn as an horizontal straight line and we refer to the two pages as the *top page* and as the *bottom page*.

Also, when we consider a 1-page book embedding we shall assume that the only page is the top page. The next two lemmas use 1-page book embeddings to show some combinatorial properties of outerplanar graphs.

Lemma 4 *Let G be an outerplanar graph, let $\phi(G)$ be a 1-page book embedding of G, and let $v_0, v_1, \dots, v_{n-1}$ the linear ordering induced by $\phi(G)$. Let $E' \subseteq E(G)$ be a set of disjoint edges and let $e^* = (v_i, v_j)$, $0 < i < j < n-1$ be an edge of E'. It is possible to add edges to G in such a way that the augmented graph G' is planar and contains a path π with the following properties: (i) π starts at v_i and ends at v_{j+1}; (ii) π contains e^*, all the edges of E' nested inside e^*, and all vertices v_l with $i < l < j+1$.*

Sketch of Proof. Let ϕ' be the 1-page book embedding obtained by deleting from $\phi(G)$ all edges of $E(G)$ that do not belong to E'. For each edge $e \in E'$, the *weight* of e is the maximum r such that there exists a r-rainbow in ϕ' having e as its top edge; if e is not the top edge of any rainbow, the weight of e is 0.

We prove the statement by induction on the weight w of e^*. If $w = 0$, i.e. no edge of E' is nested inside e^*, we proceed as illustrated in Figure 1: if there is no vertex between v_i and v_j along the spine, we choose $\pi = v_i, v_j, v_{j+1}$; otherwise, π is obtained by concatenating the path v_i, v_j with the path $v_j, v_{j-1}, \dots, v_{i+1}$, and with edge (v_{i+1}, v_{j+1}). The edges of π that are not in $\phi(G)$ are augmenting edges. We draw all the augmenting edges on the bottom page. Let $\phi(G')$ be the augmented drawing and let G' be the corresponding augmented graph. We prove that $\phi(G')$ is a 2-page book embedding and therefore G' is planar. The edges in the top page do not cross each other because they are all edges of G and do not cross in $\phi(G)$. All edges in the bottom page connect pairs of consecutive vertices except, possibly, edge (v_{i+1}, v_{j+1}). It follows that there cannot be any crossing in the bottom page and hence G' is planar.

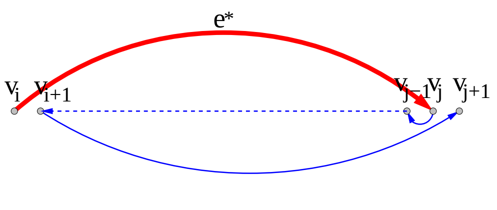

Figure 1: The base case of Lemma 4

Suppose now that $w = k \geq 1$ and that the statement is true for $w < k$. Let ϕ' be the 1-page book embedding obtained by deleting from $\phi(G)$ all edges of $E(G)$ that do not belong to E'. Let $e_0 = (v_{i_0}, v_{j_0}), e_1 = (v_{i_1}, v_{j_1}), \dots e_h = (v_{i_{h-1}}, v_{j_{h-1}})$ $(i_0 < i_1 < \dots < i_{h-1})$ be the second edges of the rainbows that have e as their top edge. The weight of each edge e_m $(0 \leq m \leq h-1)$ is at most $k-1$ and by inductive hypothesis there exists a path π_m from v_{i_m} to v_{j_m+1} that contains e_m, all edges of E' that are nested inside e_m, and all vertices between v_{i_m} and v_{j_m+1}. Also, since the edges of E' are disjoint we have that $i_0 < j_0 < i_1 < j_1 < \dots < i_{h-1} < j_{h-1}$. Denote as $\overline{\pi_m}$ the path with the same edges as π_m that starts at v_{j_m+1} and ends at v_{i_m} (that is, $\overline{\pi_m}$ is the "reverse" of π_m). We choose π as depicted in Figure 2, i.e. $\pi = v_i, v_j, v_{j-1}, \dots, \overline{\pi_{h-1}}, \dots, \overline{\pi_{h-2}}, \dots, \overline{\pi_0}, \dots, v_{i+1}, v_{j+1}$, where the edges of π that are not in G are augmenting edges. Path π starts at v_i, ends at v_{j+1}, it contains e^*, and by induction it contains all vertices between v_i and v_{j+1} and all edges of E' nested inside e.

Let $\phi(G')$ be the augmented drawing and let G' be the corresponding augmented graph. We prove that $\phi(G')$ is a 2-page book embedding and therefore G' is planar. The edges in the top page do not cross each other because they are all edges of G and do not cross in $\phi(G)$. Let d_1 and d_2 be two edges in the bottom page both connecting two vertices that are non-consecutive along the spine. If d_1 and d_2 are both edges of a path π_m $(0 \leq m \leq h-1)$, then they do not cross by induction. If d_1 and d_2 are edges of two different paths π_{m_1} and π_{m_2} $(0 \leq m_1 < m_2 \leq h-1)$, then the vertices of π_{m_1} are before those of π_{m_2} and d_1 and d_2 do not cross. The only edge in the bottom page that connects vertices that are non-consecutive along the spine and that does not belong to $\cup_{m=0}^{h-1} \pi_m$ is (v_{i+1}, v_{j+1}). Let d_1 be the edge (v_{i+1}, v_{j+1}) and let d_2 be an edge of a path π_m $(0 \leq m \leq h-1)$; we have $v_{i+1} < v_{i_m} < v_{j_m} < v_{j+1}$, which implies that the two edges do not cross and hence G' is planar. □

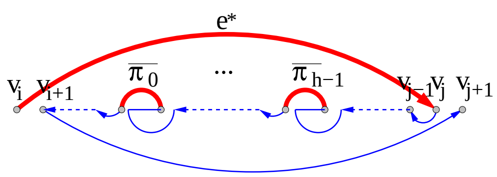

Figure 2: The inductive case of Lemma 4

Lemma 5 *Let G be an outerplanar graph and let $E' \subseteq E(G)$ be a set of disjoint edges. It is possible to add edges to G in such a way that the augmented graph G' is planar and has a Hamiltonian cycle containing all edges of E'.*

Sketch of Proof. Since G is outerplanar it admits a 1-page book embedding $\phi(G)$. Let $v_0, v_1, \dots, v_{n-1}$ the linear ordering induced by $\phi(G)$. It can be proved [1] that one can always compute $\phi(G)$ such that v_0 and v_{n-1} are not connected by an edge of $E(G) \cap E(P)$. Let $e_0 = (v_{i_0}, v_{j_0}), e_1 = (v_{i_1}, v_{j_1}), \dots e_{h-1} = (v_{i_{h-1}}, v_{j_{h-1}})$ $(0 < i_0 < i_1 < \dots < i_{h-1} < n-1)$

be the edges of E' not nested inside any other edge of E'. Since the edges of E' are disjoint then $i_0 < j_0 < i_1 < j_1 < \cdots < i_{h-1} < j_{h-1}$. Also, by Lemma 4 for each edge e_m $(0 \leq m \leq h-1)$ there exists a path π_m from v_{i_m} to v_{j_m+1} that contains edge e_m, all edges of E' nested inside e_m, and all vertices between v_{i_m} and v_{j_m+1}. We choose a cycle $C = v_0, v_1, \dots, \pi_0, \dots, \pi_1, \dots, \pi_{h-1}, v_{n-2}, v_{n-1}, v_0$, where the edges of C that are not in G are augmenting edges. By Lemma 4, C contains all vertices of G and all edges of E'. □

Lemma 6 *Let G be an outerplanar graph and let P be a simple path such that $V(G) = V(P) = V$. It is possible to add edges to G in such a way that the augmented graph G' is planar and has a Hamiltonian cycle containing all edges of $E(G) \cap E(P)$.*

Sketch of Proof. The subgraph of G induced by the edges of $E(G) \cap E(P)$ is a forest of paths; we denote by $\pi_0, \pi_1, \dots \pi_{h-1}$ the paths of this forest. Let $\phi(G)$ be a 1-page book embedding of G and let $v_0, v_1, \dots, v_{n-1}$ the linear ordering induced by $\phi(G)$. As in Lemma 5, we can assume that v_0 and v_{n-1} are not connected by an edge of $E(G) \cap E(P)$. Let H be the graph defined as follows. The vertices of H are all vertices of G except those having two edges of $E(G) \cap E(P)$ incident on them; for each path π_m with endvertices v_i and v_j, H has the edge $e_m = (v_i, v_j)$ $(0 \leq m \leq h-1)$.

Graph H is a set of disjoint edges and we can apply Lemma 5 to H with $E' = E(H)$. We obtain an augmented graph H' that is planar and has a Hamiltonian cycle D containing all edges of H (actually, cycle D coincides with H' since E' coincides with $E(H)$). The technique in the proof of Lemma 5 defines H' by computing a 2-page book embedding $\phi(H')$ of H'. $\phi(H')$ is computed by adding edges to a given 1-page book embedding $\phi(H)$ of H; we assume here that $\phi(H)$ is obtained from $\phi(G)$ by deleting the vertices and edges of G that are not in H and by adding in the top page the edges of H that replace each π_m $(0 \leq m \leq h-1)$. Let G' be the graph obtained by adding to G the augmenting edges of H' and let C be the cycle obtained by replacing each edge e_m of H in H' with the corresponding path π_m $(0 \leq m \leq h-1)$. Cycle C is a simple cycle of G' and by construction it contains all vertices of G and all edges of $E(G) \cap E(P)$.

Consider the drawing $\phi(G')$ of G' defined as follows. The vertices of G' are drawn as in $\phi(G)$; the edges of $E(G') \cap E(G)$ are drawn as in $\phi(G)$, i.e. they are drawn on the top page; the edges of $E(G') \cap E(H')$ are drawn as in $\phi(H')$, i.e. they are drawn on the bottom page. We have that the edges above the spine do not cross since they do not cross in $\phi(G)$ and the edges below the spine do not cross since they do not cross in $\phi(H')$. It follows that $\phi(G')$ is a 2-page book embedding and therefore G' is planar. □

Theorem 7 *Let G be an outerplanar graph and let P be a simple path such that $V(G) = V(P) = V$. G and P can be simultaneously embedded with fixed edges in $\mathcal{O}(n)$ time, using at most one bend for each edge of G and zero bends for each edge of P, on an integer grid of size $\mathcal{O}(n) \times \mathcal{O}(n^2)$, where $n = |V|$.*

Theorem 7 can be extended to the pair outerplanar graph - cycle.

Theorem 8 *Let G be an outerplanar graph and let C be a simple cycle such that $V(G) = V(C) = V$. G and C can be simultaneously embedded with fixed edges in $\mathcal{O}(n)$ time, using at most one bend per edge, on an integer grid of size $\mathcal{O}(n^2) \times \mathcal{O}(n^2)$, where $n = |V|$.*

Theorems 7 and 8 can be extended to the more general case in which the two graphs have only a subset of their vertices in common. Let G_1 and G_2 be a pair of planar graphs such that $V(G_1) \cap V(G_2) = V$. A *simultaneous embedding with fixed edges* of G_1 and of G_2 is a a pair of crossing-free drawings Γ_1 and Γ_2 of G_1 and G_2, respectively, such that for every vertex $v \in V$ we have $\Gamma_1(v) = \Gamma_2(v)$ and for every edge $e \in E(G_1) \cap E(G_2)$ we have $\Gamma_1(e) = \Gamma_2(e)$.

Theorem 9 *Let G_1 be an outerplanar graph and let G_2 be either a simple path or a simple cycle such that $V(G_1) \cap V(G_2) = V$. G_1 and G_2 can be simultaneously embedded with fixed edges in $\mathcal{O}(n)$ time, using at most one bend per edge, on an integer grid of size $\mathcal{O}(n^2) \times \mathcal{O}(n^2)$, where $n = |V(G_1) \cup V(G_2)|$. If G_2 is a simple path its edges are drawn as straight-line segments and the grid size is $\mathcal{O}(n) \times \mathcal{O}(n^2)$.*

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