Systematic Derivation of the Weakly Non-Linear Theory of Thermoacoustic Devices

P.H.M.W. in ’t panhuis, S.W. Rienstra, J. Molenaar

Dep. of Mathematics and Computer Science
Technische Universiteit Eindhoven
P.O. Box 513, 5600 MB Eindhoven, The Netherlands

Abstract

Thermoacoustics is the field concerned with transformations between thermal and acoustic energy. This paper teaches the fundamentals of two kinds of thermoacoustic devices: the thermoacoustic prime mover and the thermoacoustic heat pump or refrigerator.

Two technologies, involving standing wave and traveling wave modes, are considered. We will investigate the case of a porous medium and two heat exchangers placed in a gas-filled resonator, in which either a standing or traveling wave is maintained. The central problem is the interaction between the porous medium and the sound field in the tube. The conventional thermoacoustic theory is reexamined and a systematic and consistent weakly non-linear theory is constructed based on dimensional analysis and small parameter asymptotics.

The difference with conventional thermoacoustic theory lies in the dimensional analysis. This is a powerful tool in understanding physical effects which are coupled to several dimensionless parameters that appear in the equations, such as the Mach number, the Prandtl number, the Laucret number and several geometrical quantities. By carefully analyzing limiting situations in which these parameters differ in orders of magnitude, we can study the behavior of the system as a function of parameters connected to geometry, heat transport and viscous effects.

1. Introduction

The most general interpretation of thermoacoustics, as described by Rott [19], includes all effects in acoustics in which heat conduction and entropy variations of the (gaseous) medium play a role. In this paper, however, we will focus specifically on thermoacoustic devices exploiting thermoacoustic concepts to produce useful refrigeration, heating, or work.

1.1 A brief history

Thermoacoustics has a long history that dates back more than two centuries. For the most part heat-driven oscillations were subject of these investigations. The reverse process, generating temperature differences using acoustic oscillations, is a relatively new phenomenon. The first qualitative explanation was given in 1887 by Lord Rayleigh in his classical work "The Theory of Sound" [14]. He explains the production of thermoacoustic oscillations as follows:
"If heat be given to the air at the moment of greatest condensation (compression) or taken from it at the moment of greatest rarefaction (expansion), the vibration is encouraged".

Rayleigh’s qualitative understanding turned out to be correct, but a quantitatively accurate theoretical description of those phenomena was not achieved until much later. The breakthrough came in 1969, when Rott and coworkers started a series of papers [15], [16], [18], [20] in which a successful linear theory of thermoacoustics was given. The first to give a comprehensive picture, was Swift [21] in 1988. He implemented Rott’s theory of thermoacoustic phenomena into creating practical thermoacoustic devices. His work included a short history, experimental results, discussions on how to build these devices, and a coherent development of the theory based on Rott’s work. Since then Swift and others have contributed a lot to the development of thermoacoustic devices. In 2002 Swift [23] also wrote a textbook, providing a complete introduction into thermoacoustics and treating several kinds of thermacoustic devices. Recently Tijani [24] wrote a Ph.D thesis based on Swift’s work in which he also discusses the linear theory of thermoacoustics. Most of the work on thermoacoustic devices has been been reviewed by Garrett [4] in 2003.

1.2 Classification

We consider thermoacoustic devices of the form shown in Fig. 1, that is, an acoustically resonant tube, containing a fluid (usually a gas) and a porous solid medium. The porous medium is modelled as a stack of parallel plates. Following Garrett [4] we classify the devices based on several criteria:

![Figure 1: Thermoacoustic device](image)

I. Prime mover vs. heat pump or refrigerator:

A thermoacoustic prime mover absorbs heat at a high temperature and exhausts heat at a lower temperature while producing work as an output. A refrigerator or heat pump absorbs heat at a low temperature and requires the input of mechanical work to exhaust more heat to a higher temperature (Fig. 2). The only difference between a heat pump and a refrigerator is whether the purpose of the device is to extract heat at the lower temperature (refrigeration) or to reject heat at the higher temperature (heating). Therefore, from now on, when we talk about a thermoacoustic refrigerator we mean either a refrigerator or a heat pump. Often the term thermoacoustic engine is used as well, either to indicate a thermoacoustic prime mover or as a general term to describe all thermoacoustic devices. To avoid confusion, we will refrain from using the term thermoacoustic engine.
II. Stack-based devices vs. regenerator-based devices:
A second classification depends on whether the porous medium used to exchange heat with the working fluid is a ”stack” or a ”regenerator”. Inside a regenerator the pore size is much smaller than inside a stack. Garrett [4] uses the so-called Laucret number $N_L$ to indicate the difference between a stack and regenerator. The Laucret number is defined as the ratio between the half pore size and the thermal penetration depth $1$. If $N_L \gg 1$ the porous medium is called a stack and if $N_L \ll 1$ it is called a regenerator. This definition of stacks and regenerators is slightly different from Garret’s $2$, but is chosen to stress that the pore size inside a regenerator is very small.

III. Standing-wave devices vs. traveling-wave devices:
Finally thermoacoustic devices can also be categorized depending on whether there is a traveling or a standing sound wave inside the thermoacoustic device. In section 3.1.7 we show that it is beneficiary to use a stack inside a standing-wave device and a regenerator inside a traveling-wave device. Therefore one could also classify thermoacoustic devices as either standing-wave stack-based devices or traveling-wave regenerator-based devices.

1.3 Basic principle of the thermoacoustic effect
The thermoacoustic effect can be understood by following a given parcel of fluid as it moves through the stack or regenerator. Fig. 3 displays the (idealized) cycles a typical fluid parcel goes through as it oscillates alongside the plate. The fluid parcel follows a four-step cycle which depends on the kind of device.

- **Stack-based devices:**
The basic thermodynamic cycle in a stack-based acoustic refrigerator or prime mover consists of two reversible adiabatic steps (step 1,3 in Fig. 3(a,b)) and two irreversible isobaric heat-transfer steps (step 2,4 in Fig. 3(a,b)).
As $N_L \gg 1$, there will be an imperfect thermal contact between the fluid and the solid. As a result a phase shift, or time delay, arises between the pressure and the temperature of the gas parcels that are at a distance of a few thermal penetration depths from the stack plate.

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$^1$The thermal penetration depth is the distance heat can diffuse through within a characteristic time
$^2$Garrett defines the porous medium to be a stack if $N_L \geq 1$, and a regenerator if $N_L < 1$
Figure 3: Typical fluid parcels executing the four steps (1-4) of the thermodynamic cycle in (a) a stack-based standing-wave refrigerator, (b) a stack-based standing wave prime mover, (c) a regenerator-based traveling wave refrigerator and (d) a regenerator-based traveling wave prime mover.
Parcels that are farther away have no thermal contact and are simply compressed and expanded adiabatically and reversibly by the sound wave. Therefore parcels that are about a thermal penetration depth away from the plate have good enough thermal contact to exchange some heat with the plate, but at the same time are in poor enough contact to produce a time delay between motion and heat transfer.

The fact that the operation of stack-based thermoacoustic devices requires pressure and displacement to be primarily in phase, explains why stack-based devices are also called standing-wave devices.

The difference between the prime-mover and the refrigerator depends on the magnitude of the temperature gradient along the stack plates. During compression (step 1) the fluid parcel is both warmed (adiabatically) and displaced along the plate. Next, if the temperature gradient along the stack is large enough, the plate temperature will be larger than the fluid parcel temperature. Hence heat will flow from the plate to the fluid (step 2). This is the case in Fig. 3(b). Then the parcel expands and moves back to the original position (step 3). There the temperature of the parcel will still be higher than the plate temperature and heat will flow from the fluid to the plate (step 4). As a result heat is transported from a high to a low temperature, thus producing a certain amount of work. In other words the device acts as a prime mover. Similarly if the temperature gradient along the plate is small enough, we find that heat is transported from a low to a high temperature, thus requiring a certain amount of work. Therefore the cycle shown in Fig. 3(a) corresponds to a refrigerator.

Thus we find that a low temperature gradient along the plate is the condition for a refrigerator and a high temperature gradient is the condition for a prime mover. The critical temperature gradient is where the temperature change along the plate just matches the adiabatic temperature change of the fluid parcel.

• Regenerator-based devices:

The basic thermodynamic cycle in a regenerator-based acoustic refrigerator or prime mover consists of two isochoric displacement steps during which heat is exchanged (step 1,3 in Fig. 3(c,d)) and two isothermal compression and expansion steps (step 2,4 in Fig. 3(c,d)).

Because the pores in a regenerator are so small compared to the thermal penetration depth, there will be an almost perfect thermal contact between the fluid and the solid. Therefore during the motion (step 2, 4) the temperature of the wall and the fluid parcel will be the same. As a result there will be a continuous exchange of heat between the gas and the solid, which takes place over a vanishingly small temperature difference and therefore only a negligibly amount of entropy is created. During the compression and expansion (step 1,3), the temperature remains constant.

The gas oscillating inside a regenerator requires the same phasing between pressure and velocity as a traveling acoustic wave. Therefore regenerator-based devices are also called traveling-wave devices.

The main advantage of regenerator-based devices with respect to stack-based devices is that there are no irreversible processes, so that the ideal efficiency is equal to the Carnot efficiency. On the other hand, because the pores are so narrow, there may be significant viscous dissipation which could lower the efficiency dramatically.

Usually the displacement of one fluid parcel is small with respect to the length of the plate. Thus there will be an entire train of adjacent fluid parcels, each confined to a short region of length $2x_1$ and
passing on heat as in a bucket brigade (Fig. 4). Although a single parcel transports heat \( \delta Q \) over a very small interval, \( \delta Q \) is shuttled along the entire plate because there are many parcels in series.

\[
\begin{align*}
\rho \left[ \frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \nabla) \bar{v} \right] &= -\nabla \bar{p} + \mu \nabla^2 \bar{v} + \left( \xi + \frac{\mu}{3} \right) \nabla (\nabla \cdot \bar{v}), \\
\frac{\partial \bar{p}}{\partial t} + \nabla \cdot (\bar{p} \bar{v}) &= 0,
\end{align*}
\]

**Figure 4:** Work and heat flow inside a thermoacoustic refrigerator. The oscillating fluid parcels work as a bucket brigade, shuttling heat along the stack plate from one parcel of gas to the next. As a result heat \( Q \) is transported from the left to the right, using work \( W \). Inside a prime mover the arrows will be reversed, i.e. heat \( Q \) is transported from the right to the left and work \( W \) is produced.

### 1.4 Scope

Motivated by the work of Swift [21], [23] and Tijani [24], this paper tries to reconstruct the linear theory of thermoacoustics in a systematic and consistent manner using dimensional analysis and small parameter asymptotics.

We will start in section 2 with a detailed description of the model and an overview of the governing equations and boundary conditions. Then in section 3 we try to solve the equations assuming there is no mean steady flow. This is done both for stack- and regenerator-based devices. This is repeated in section 4, but now with a mean steady flow. Finally section 5 shows the different energy flows and their interaction in thermoacoustic devices.

### 2. General thermoacoustic theory

We will model the thermoacoustic devices as depicted in Fig. 5 (see also [21] and [24]), where the device is modeled as an acoustically resonant tube, containing a gas and a porous solid medium. For now we will not make any assumptions on whether the tube is open or closed. The porous medium is modeled as a stack of parallel plates, each of thickness \( 2l \) and length \( L \). The space between the plates is equal to \( 2R \).

#### 2.1 Governing equations

We will focus on what happens inside the stack. The general equations describing the thermodynamic behavior are [6]

\[
\begin{align*}
\rho \left[ \frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \nabla) \bar{v} \right] &= -\nabla \bar{p} + \mu \nabla^2 \bar{v} + \left( \xi + \frac{\mu}{3} \right) \nabla (\nabla \cdot \bar{v}), \\
\frac{\partial \bar{p}}{\partial t} + \nabla \cdot (\bar{p} \bar{v}) &= 0,
\end{align*}
\]
Figure 5: A thermoacoustic device modelled as an acoustically resonant tube, containing a gas, a stack of parallel plates and heat exchangers at both sides of the stack.

\[
\rho c_p \left( \frac{\partial T}{\partial t} + \tilde{v} \cdot \nabla \tilde{T} \right) - \beta \tilde{T} \left( \frac{\partial \tilde{p}}{\partial t} + \tilde{v} \cdot \nabla \tilde{p} \right) = K \nabla \cdot (\nabla \tilde{T}) + \tilde{\Sigma} : \nabla \tilde{v}. \quad (2.1.3)
\]

Here \( \tilde{\rho} \) is the density, \( \tilde{v} \) is the velocity, \( \tilde{p} \) is the pressure, \( \tilde{T} \) is the temperature, \( \tilde{s} \) is the entropy per unit mass, \( \mu \) and \( \xi \) are the dynamic (shear) and second (bulk) viscosity, respectively; \( K \) is the gas thermal conductivity, \( c_p \) is the specific heat per unit mass, \( \beta \) is the thermal expansion coefficient and \( \tilde{\Sigma} \) is the viscous stress tensor, with components

\[
\tilde{\Sigma}_{ij} = \mu \left( \frac{\partial \tilde{v}_i}{\partial x_j} + \frac{\partial \tilde{v}_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial \tilde{v}_k}{\partial x_k} \right) + \xi \delta_{ij} \frac{\partial \tilde{v}_k}{\partial x_k}. \quad (2.1.4)
\]

Furthermore \( \tilde{\rho} \) is related to \( \tilde{p} \) and \( \tilde{T} \) according to (A.4.9)

\[
d\tilde{p} = \frac{\gamma}{\tilde{c}_p^2} d\tilde{p} - \tilde{\rho} \beta \ d\tilde{T}. \quad (2.1.5)
\]

Finally, the temperature \( \tilde{T}_s \) in the plates satisfies the diffusion equation

\[
\tilde{\rho}_s c_s \frac{\partial \tilde{T}_s}{\partial t} = K_s \nabla^2 \tilde{T}_s, \quad (2.1.6)
\]

where \( K_s \), \( c_s \) and \( \rho_s \) are the thermal conductivity, the specific heat per unit mass and the density of the stack’s material, respectively.

These equations will be linearized and simplified using the following assumptions

- The theory is linear; second-order effects, such as acoustic streaming and turbulence, are neglected.
- The plates are fixed and rigid.
- The temperature variations along the stack are much smaller than the absolute temperature.
• The temperature dependence of viscosity is neglected.
• Oscillating variables have harmonic time dependence at a single angular frequency \( \tilde{\omega} \).

At the boundaries we impose the no-slip condition
\[
\tilde{v} = 0, \quad \tilde{y} = \pm R.
\] (2.1.7)

The temperatures in the plates and in the gas are coupled at the solid-gas interface where continuity of temperature and heat fluxes is imposed.
\[
\tilde{T} \bigg|_{\tilde{y} = \pm R} = \tilde{T}_s \bigg|_{\tilde{y}' = \mp l} =: \tilde{T}_b(x), \quad (2.1.8a)
\]
\[
K \left( \frac{\partial \tilde{T}}{\partial \tilde{y}} \right) \bigg|_{\tilde{y} = \pm R} = K_s \left( \frac{\partial \tilde{T}_s}{\partial \tilde{y}'} \right) \bigg|_{\tilde{y}' = \mp l}, \quad (2.1.8b)
\]

We do not impose any conditions at the stack ends, as we are mainly interested about what happens inside the stack, ignoring any entrance effects.

The next step is the rescaling of the variables in (2.1.1), (2.1.2) and (2.1.3) such that the equations are dimensionless. We assume a 2D-model and rescale as follows
\[
\tilde{x} = L x, \quad \tilde{y} = R y, \quad \tilde{y}' = ly', \quad \tilde{t} = \frac{L}{C} t, \quad (2.1.9a)
\]
\[
\tilde{u} = C u, \quad \tilde{v} = \varepsilon C v, \quad \tilde{p} = DC^2 p, \quad \tilde{\rho} = D \rho, \quad \tilde{T} = \frac{C^2}{c_p} T, \quad (2.1.9b)
\]
\[
\tilde{\rho}_s = D_s \rho_s, \quad \tilde{T}_s = \frac{C^2}{c_p} T_s, \quad (2.1.9c)
\]
\[
\tilde{c} = C c, \quad \tilde{\beta} = \frac{c_p}{C^2} \beta, \quad (2.1.9d)
\]

where \( C \) is a typical speed of sound, \( D \) and \( D_s \) are typical densities for the fluid and solid, respectively, and \( \varepsilon \) is the aspect ratio of a stack pore defined as
\[
\varepsilon = R/L \ll 1. \quad (2.1.10)
\]

Clearly, \( \varepsilon \) is a dimensionless parameter. In total there are 14 physical parameters in this problem expressible in 5 independent fundamental physical quantities. Therefore, using the Buckingham \( \pi \) theorem [1], we know that 9 independent dimensionless parameters can be constructed from the original 14 parameters. In addition to \( \varepsilon \), we will use the following dimensionless parameters:
\[
\varepsilon_1 = l/L, \quad \vartheta = \frac{RK_s}{lK}, \quad \omega = \frac{\tilde{\omega} L}{C}, \quad \gamma = \frac{c_p}{c_v}, \quad (2.1.11a)
\]
\[
A = \frac{U}{C}, \quad Wo = \sqrt{2} \frac{R}{\delta_v}, \quad N_L = \frac{R}{\delta_k}, \quad N_s = \frac{l}{\delta_s}, \quad Pr = \frac{2N_L^2}{Wo^2}. \quad (2.1.11b)
\]

where \( c_v \) is the isochoric specific heat, \( \lambda = 2\pi \tilde{\vartheta}/\tilde{\omega} \) is the wavelength, \( \tilde{\omega} \) is the frequency, \( U \) is a typical fluid speed, \( A \) is a Mach number, \( Pr \) is the Prandtl number, \( Wo \) is the Womersley number and \( N_L \) and \( N_s \) are the Lautrec numbers (as defined by Garrett [4]) related to the fluid and solid, respectively. Here
the parameters $\delta_\nu$, $\delta_k$ and $\delta_s$ are the viscous penetration depth, and the thermal penetration depths for the fluid and solid, respectively.

$$
\delta_\nu = \sqrt{\frac{2\nu}{\omega}}, \quad \delta_k = \sqrt{\frac{2\kappa}{\omega}}, \quad \delta_s = \sqrt{\frac{2\kappa_s}{\omega}}.
$$

(2.1.12)

Here $\nu = \mu/D$ is the kinematic viscosity and $\kappa = K/(Dc_p)$ is the thermal diffusivity of the fluid. It can be shown that the first 8 of the dimensionless parameters in (2.1.11) together with $\varepsilon$ form 9 independent dimensionless parameters. Obviously the Prandtl number is determined completely by the Womersley and Laucret number.

We will use the following weakly non-linear expansion for the fluid variables in powers of the amplitude $A$ of the acoustic velocity oscillations.

$$
q(x, y, t) = q_0(x, y) + \sum_{k=1}^{\infty} A^k q'_k(x, y, t), \quad A \ll 1.
$$

(2.1.13)

where we assume a small amplitude $A$ and a harmonic time-dependence for $q_1$ with dimensionless frequency $\omega$ (Helmholtz number), so that we can write

$$
q'_1(x, y, t) = \text{Re} \left[ q_1(x, y)e^{i\omega t} \right].
$$

(2.1.14)

We use the term weakly non-linear to indicate that, although we assume small amplitudes, we still include terms of higher order in $A$. In dimensionless form we obtain the following system of equations

$$
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0,
$$

(2.1.15)

$$
\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \frac{\omega}{Wo^2} \left( \varepsilon^2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)
$$

$$
+ \frac{\varepsilon \omega}{Wo^2} \left( \frac{\xi}{\mu} + \frac{1}{3} \right) \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} \right),
$$

(2.1.16)

$$
\varepsilon^2 \rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \frac{\varepsilon^2 \omega}{Wo^2} \left( \varepsilon^2 \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)
$$

$$
+ \frac{\varepsilon^2 \omega}{Wo^2} \left( \frac{\xi}{\mu} + \frac{1}{3} \right) \left( \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} \right),
$$

(2.1.17)

$$
\rho \left( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) - \beta T \left( \frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} \right) = \frac{\omega}{2N_L^2} \left( \varepsilon^2 \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)
$$

$$
+ \frac{\omega}{Wo^2} \left[ \left( \frac{\partial u}{\partial y} \right)^2 + O(\varepsilon) \right],
$$

(2.1.18)

$$
\rho_s \left( \frac{\partial T_s}{\partial t} \right) = \frac{\omega}{2N_L^2} \left( \frac{\varepsilon^2 \partial^2 T_s}{\partial x^2} - \frac{\partial^2 T_s}{\partial y^2} \right),
$$

(2.1.19)

$$
\rho_1 = -\rho_0 \beta T_1 + \frac{\gamma}{c_T^2} p_1.
$$

(2.1.20)
subject to

\[ v(y = \pm 1) = 0, \quad (2.1.21a) \]

\[ T(y = \pm 1) = T_s(y' = \mp 1) =: T_b(x), \quad (2.1.21b) \]

\[ \frac{\partial T}{\partial y} (y = \pm 1) = \vartheta \frac{\partial T_s}{\partial y'} (y' = \mp 1). \quad (2.1.21c) \]

3. Thermoacoustic devices without mean velocity

This chapter discusses thermoacoustic devices in the absence of a steady mean velocity. This will be done both for stack-based and regenerator-based devices.

To solve the equations given in the previous section we need to know the magnitude of the dimensionless parameters involved. First note that

\[ \omega = \frac{\tilde{\omega} L}{C} = \frac{\tilde{c} L}{\tilde{c} L} = \frac{c L}{L} \sim \frac{L}{\lambda}. \quad (3.0.22) \]

Here we will make the "short-stack" approximation \( L \ll \lambda (\omega \ll 1) \), where the stack is considered to be short compared to the engine with the restriction that \( \omega \) is the largest among the small parameters.

The Womersley number and the Laucret number are related by \( 2N_L^2 = PrW_o^2 \). The Prandtl number only depends on material parameters and is usually close to 1. As a result \( N_L \) and \( W_o \) should be of the same order of magnitude. Normally in standing wave machines \( R \sim \delta_k \) (stack) and in traveling wave machines \( R \ll \delta_k \) (regenerator). Also we assume that \( N_s \sim N_L \).

Furthermore we assume that the amplitudes of the acoustic oscillations can be taken arbitrarily small, with the restriction that \( \varepsilon^2 \ll A \), so that \( \partial^2 \tilde{q}_0 / \partial \tilde{z}^2 \) can be neglected with respect to \( \partial^2 \tilde{q}_1 / \partial \tilde{y}^2 \) \((\mathcal{O}(\varepsilon^2) \text{ versus } \mathcal{O}(A))\). Swift[21] and Tijani[24] also treated this case, although they did not make these assumptions explicit. Finally, we also have

\[ \vartheta = \mathcal{O}(1). \quad (3.0.23) \]

Otherwise, from (2.1.21c) there would be no relation between the heat flux of the gas and the heat flux of the stack at the stack-gas interface.

Summarizing we have

\[ \varepsilon^2 \ll \varepsilon^2 \ll A \ll \omega^2 \ll 1, \quad \vartheta = \mathcal{O}(1) \quad (3.0.24) \]

and

\[ W_o \sim 1, \quad N_L \sim 1, \quad N_s \sim 1, \quad \text{for a stack}, \quad (3.0.25) \]

\[ W_o \ll 1, \quad N_L \ll 1, \quad N_s \ll 1, \quad \text{for a regenerator}. \quad (3.0.26) \]

The presence of \( \omega \) as a small parameter suggests the following alternative expansions for the fluid variables \( q \)

\[ q = q_0(x, y) + ARe \left[ (q_{10}(x, y) + \omega q_{11}(x, y) + \omega^2 q_{12}(x, y)) e^{ixt} \right] + \cdots. \quad (3.0.27) \]

Moreover, we assume \( v_0 = 0 \). Note that the terms second order in \( A \) can be neglected since \( A \ll \omega^2 \ll 1 \).
3.1 Short Stack

In this section we will restrict ourselves to the case of a stack, i.e.

\[ Wo \sim 1, \quad N_L \sim 1, \quad N_s \sim 1. \quad (3.1.1) \]

3.1.1 The horizontal velocity \( u_1 \)

Substitute the expansions given in (3.0.27) in the \( y \)-component of the momentum equation (2.1.17). Collecting, the zeroth order terms we find \( \partial p_0 / \partial y = 0 \). Next collecting terms up to order \( A\omega^2 \) we also find

\[ A \frac{\partial p_{10}}{\partial y} + A\omega \frac{\partial p_{11}}{\partial y} + A\omega^2 \frac{\partial p_{12}}{\partial y} = 0. \quad (3.1.2) \]

This equation can only be satisfied if

\[ \frac{\partial p_{10}}{\partial y} = \frac{\partial p_{11}}{\partial y} = \frac{\partial p_{12}}{\partial y} = 0. \quad (3.1.3) \]

We do the same for the \( x \)-component (2.1.16). The zeroth order equation yields \( \partial p_0 / \partial x = 0 \). Keeping terms of order up to \( A\omega^2 \) we obtain

\[ iA\omega \rho_0 (u_{10} + \omega u_{11}) = -A \frac{dp_{10}}{dx} - A\omega \frac{dp_{11}}{dx} - A\omega^2 \frac{dp_{12}}{dx} + \frac{A\omega}{Wo^2} \left( \frac{\partial^2 u_{10}}{\partial y^2} + \omega \frac{\partial^2 u_{11}}{\partial y^2} \right), \quad (3.1.4) \]

Collecting terms of order \( A \) we find that \( \partial p_{10} / \partial x = 0 \). Hence

\[ p = p_0 + A \Re \left[ (p_{10} + \omega p_{11}(x)) e^{i\omega t} \right] + \cdots. \quad (3.1.5) \]

Next, collecting the terms of order \( A\omega \) and \( A\omega^2 \), we find that \( u_{10} \) and \( u_{11} \) satisfy

\[ i\rho_0 u_{10} = -\frac{dp_{11}}{dx} + \frac{1}{Wo^2} \frac{\partial^2 u_{10}}{\partial y^2}, \quad (3.1.6) \]

\[ i\rho_0 u_{11} = -\frac{dp_{12}}{dx} + \frac{1}{Wo^2} \frac{\partial^2 u_{11}}{\partial y^2}, \quad (3.1.7) \]

and applying the boundary conditions \( u_1(x, \pm 1) = 0 \), we obtain

\[ u_{10}(x, y) = \frac{i}{\rho_0} \frac{dp_{11}}{dx} \left[ 1 - \frac{\cosh(\alpha_{\nu} y)}{\cosh(\alpha_{\nu})} \right], \quad (3.1.8) \]

\[ u_{11}(x, y) = \frac{i}{\rho_0} \frac{dp_{12}}{dx} \left[ 1 - \frac{\cosh(\alpha_{\nu} y)}{\cosh(\alpha_{\nu})} \right], \quad (3.1.9) \]

where

\[ \alpha_{\nu} = (1 + i) \sqrt{\frac{R}{\rho_0} \delta_{\nu}} = (1 + i) \sqrt{\frac{\rho_0}{2} Wo}. \quad (3.1.10) \]

This coincides with the solution found by Swift [21] and Tijani [24] in a dimensional form.
3.1.2 The temperature $T_s$ in the plate

Using the rescaled energy equation (2.1.18) and the diffusion equation (2.1.19) we will try to find $T$ and $T_s$. First we solve (2.1.19) subject to (2.1.21b). Substitute the expansions from (3.0.27) into (2.1.19). To leading order we find that $\partial^2 T_{s0}/\partial y^2 = 0$. That means $T_{s0}$ is at most linear in $y'$. In view of symmetry we find that $T_{s0}$ is in fact constant in $y'$, i.e.

$$T_s(x, y', t) = T_{s0}(x) + A\text{Re} \left[ (T_{s10}(x, y') + \omega T_{s11}(x, y')) e^{i\omega t} \right] + \cdots,$$  \hspace{1cm} (3.1.11)

Since $\rho_{s0} = \rho_{s0}(p_0, T_{s0})$, we assume that also

$$\rho_s(x, y', t) = \rho_{s0}(x) + A\text{Re} \left[ (\rho_{s10}(x, y') + \omega \rho_{s11}(x, y')) e^{i\omega t} \right] + \cdots,$$  \hspace{1cm} (3.1.12)

Collecting terms of order $A$ we find

$$2i\rho_{s0} N_s^2 T_{s10} = \frac{\partial^2 T_{s10}}{\partial y'^2}.$$  \hspace{1cm} (3.1.13)

Imposing (2.1.21b)

$$T_{s10}(x, \pm 1) = T_{b10}(x),$$  \hspace{1cm} (3.1.14)

we find that the solution to (3.1.13) is given by

$$T_{s10}(x, y') = T_{b10}(x) \frac{\cosh(\alpha_s y')}{\cosh(\alpha_s)},$$  \hspace{1cm} (3.1.15)

where

$$\alpha_s = (1 + i)\rho_{s0} \frac{l}{\delta_s} = (1 + i)\rho_{s0} N_s.$$  \hspace{1cm} (3.1.16)

The exact expression for $T_{b10}$ should be determined from matching $T_s$ with $T$.

3.1.3 The temperature $T$ between the plates

We substitute the expansions shown in (2.1.13) into equation (2.1.18). To leading order we find $\partial^2 T_0/\partial y^2 = 0$. That means $T_0$ is at most linear in $y$. In view of symmetry we find that $T_0$ is in fact constant in $y$. Therefore we expand $T$ as

$$T(x, y, t) = T_0(x) + A\text{Re} \left[ (T_{10}(x, y) + \omega T_{11}(x, y)) e^{i\omega t} \right] + \cdots,$$  \hspace{1cm} (3.1.17)

Since $\rho_0 = \rho_0(p_0, T_0)$, we assume that also

$$\rho(x, y, t) = \rho_0(x) + A\text{Re} \left[ (\rho_{10}(x, y) + \omega \rho_{11}(x, y)) e^{i\omega t} \right] + \cdots,$$  \hspace{1cm} (3.1.18)

Collecting terms of order $A$, we find

$$\rho_0 u_{10} \frac{dT_0}{dx} = 0 \quad \Rightarrow \quad \left\{ \begin{array}{l} u_{10} = 0, \\
 \frac{dp_{11}}{dx} = 0. \end{array} \right.$$  \hspace{1cm} (3.1.19)
Finally, collecting terms of order $A\omega$ we obtain
\begin{equation}
\rho_0 u_{11} \frac{dT_0}{dx} + i \rho_0 T_{10} - i \beta T_0 p_{10} = \frac{1}{2N_L^2} \frac{\partial^2 T_0}{\partial y^2}.
\end{equation}

Inserting the expression for $u_{11}$ given in (3.1.9), we obtain
\begin{equation}
\left[ 1 - \frac{\cosh(\alpha_y y)}{\cosh(\alpha_y)} \right] \frac{dT_0}{dx} \frac{dp_{12}}{dx} + i \rho_0 T_{10} - i \beta T_0 p_{10} = \frac{1}{2N_L^2} \frac{\partial^2 T_0}{\partial y^2}.
\end{equation}

Applying the boundary conditions in (2.1.21), we can solve (3.1.21)
\begin{equation}
T_{10}(x, y) = \frac{\beta T_0}{\rho_0} p_{10} - \frac{1}{\rho_0} \frac{dT_0}{dx} \frac{dp_{12}}{dx} \left[ 1 - \frac{Pr}{Pr - 1} \frac{\cosh(\alpha_y y)}{\cosh(\alpha_y)} \right]
\end{equation}
\begin{equation}
- \left( \frac{\beta T_0}{\rho_0} p_{10} + \frac{1}{\rho_0} \frac{dT_0}{dx} \frac{dp_{12}}{dx} \right) \frac{\cosh(\alpha_k y)}{(1 + \varepsilon_s) \cosh(\alpha_k)}.
\end{equation}

where
\begin{equation}
\alpha_k = (1 + i) \sqrt{\frac{\rho_0}{\delta_k}} = (1 + i) \sqrt{\rho_0 N_L},
\end{equation}
\begin{equation}
\varepsilon_s = \frac{\sqrt{K_s \rho_0 \rho_{00} c_s}}{\sqrt{K_s \rho_0 \rho_{00} c_s}}, \quad f_\nu = \frac{\tanh(\alpha_\nu)}{\alpha_\nu}, \quad f_k = \frac{\tanh(\alpha_k)}{\alpha_k}.
\end{equation}

As a result we also find
\begin{equation}
T_{s_{10}}(x, y') = \frac{\varepsilon_s}{1 + \varepsilon_s} \left( \frac{\beta T_0}{\rho_0} p_{10} + \frac{1}{\rho_0} \frac{dT_0}{dx} \frac{dp_{12}}{dx} \frac{1 - \frac{f_k}{f_k}}{Pr - 1} \right) \frac{\cosh(\alpha_s y')}{\cosh(\alpha_s)}.
\end{equation}

Note that if we had taken $A \ll \varepsilon^2$ (compare with (3.0.24)), i.e. the amplitudes of the oscillations can be taken arbitrarily small, then the term $\frac{d^2 T_0}{dx^2}$ will appear in the equations. Then, for the equations to be satisfied, we must have that $T_0$ is linear in $x$ (and $T_{s_{10}}(x) = T_0(x)$).

### 3.1.4 The pressure $p$

In order to derive Rott’s wave equation, we start with the continuity equation. Restricted to terms of order up to $A\omega$, the rescaled continuity equation yields
\begin{equation}
i A \omega \rho_{10} + A \omega \frac{\partial}{\partial x}(\rho_0 u_{11}) + A \omega \rho_0 \frac{\partial v_{11}}{\partial y} + A \rho_0 \frac{\partial v_{10}}{\partial y} = 0,
\end{equation}

To leading order we find
\begin{equation}
\rho_0 \frac{\partial v_{10}}{\partial y} = 0 \quad \Rightarrow \quad v_{10} = 0,
\end{equation}

where we applied the boundary condition $v_{10}(\pm 1) = 0$. Now to leading order, we obtain
\begin{equation}
i \rho_1 + \frac{\partial}{\partial x}(\rho_0 u_{11}) + \rho_0 \frac{\partial v_{11}}{\partial y} = 0,
\end{equation}

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Using the $x$-derivative of equation (3.1.7), the equation of state (2.1.20), and inserting the expressions for $T_1$ and $u_1$, we finally find

\[
\left(\beta^2 T_0 - \frac{\gamma}{c^2}\right)p_{10} - \beta\frac{dT_0}{dx}\frac{dp_{12}}{dx} \left[ 1 - \frac{Pr}{Pr - 1}\frac{\cosh(\alpha_v y)}{\cosh(\alpha_v)}\right] \\
- \left(\beta^2 T_0 p_{10} + \beta \frac{1 + \epsilon_s f_k}{Pr - 1}\frac{dT_0}{dx}\frac{dp_{12}}{dx}\right)\frac{\cosh(\alpha_k y)}{(1 + \epsilon_s)\cosh(\alpha_k)} - \frac{d^2 p_{12}}{dx^2} + \frac{1}{Wo^2}\frac{\partial^3 u_{11}}{\partial x \partial y^2}
\]

\[+ i\rho_0 \frac{\partial v_{11}}{\partial y} = 0. \tag{3.1.29}\]

Next we integrate with respect to $y$ from 0 to 1. Note that $v(x,0) = 0$ because of symmetry. Finally we obtain

\[
i\frac{c^2}{c^5} \left[ 1 + \frac{\gamma - 1}{1 + \epsilon_s} f_k - (c^2 \beta^2 T_0 - \gamma + 1) \left(1 - \frac{f_k}{1 + \epsilon_s}\right)\right]p_{10} \\
+ \frac{f_k - f_v}{(Pr - 1)(1 + \epsilon_s)}\beta\frac{dT_0}{dx}\frac{dp_{12}}{dx} + \rho_0 \frac{d}{dx} \left(\frac{1 - f_v}{\rho_0} \frac{dp_{12}}{dx}\right) = 0, \tag{3.1.30}\]

where we used

\[
\rho_0 \frac{d}{dx} \left(\frac{1}{\rho_0}\right) = - \frac{1}{\rho_0} \frac{d\rho_0}{dx} = - \frac{1}{\rho_0} \frac{\partial \rho_0}{\partial T_0} \frac{dT_0}{dx} = \beta \frac{dT_0}{dx}. \tag{3.1.31}\]

Furthermore, if we impose the dimensionless equivalent of relation (A.4.6)

\[
c^2 \beta^2 T_0 = \gamma - 1, \tag{3.1.32}\]

then (3.1.30) transforms into Rott’s wave equation (dimensionless)

\[
\frac{1}{c^2} \left[ 1 + \frac{\gamma - 1}{1 + \epsilon_s} f_k - (c^2 \beta^2 T_0 - \gamma + 1) \left(1 - \frac{f_k}{1 + \epsilon_s}\right)\right]p_{10} \\
+ \frac{f_k - f_v}{(Pr - 1)(1 + \epsilon_s)}\beta\frac{dT_0}{dx}\frac{dp_{12}}{dx} + \rho_0 \frac{d}{dx} \left(\frac{1 - f_v}{\rho_0} \frac{dp_{12}}{dx}\right) = 0. \tag{3.1.33}\]

This result was also found by Rott [16], Swift [21] and Tijani [24]. For an ideal gas and ideal stack with $\epsilon_s = 0$ this result was first obtained in [15].

### 3.1.5 Acoustic power

The time-averaged acoustic power $\tilde{W}$ used (or produced in the case of a prime mover) in a segment of length $dx$ can be found from

\[
\frac{d\tilde{W}}{dx} = \bar{A_g} \frac{d}{dx} \left(\tilde{p}_1 \tilde{u}_1'\right), \tag{3.1.34}\]

where the overbar indicates time average (over one period), brackets $(\quad)$ indicate averaging in the $\tilde{y}$-direction and $A_g$ is the cross-sectional area of the gas within the stack. Next $\tilde{W}$ and $\bar{A_g}$ are rescaled as

\[
\tilde{W} = R^2 DC^3 \tilde{W}, \quad \bar{A_g} = R^2 A_g. \tag{3.1.35}\]
As \( p \) is independent of \( y \), we find
\[
\frac{d\dot{W}}{dx} = A^2 A_g \frac{d}{dx} \left[ p_1'(u_1') \right] \tag{3.1.36}
\]

It can be shown that the product \( p_1'(u_1') \) satisfies
\[
\overline{p_1'(u_1')} = \frac{1}{2} \text{Re} \left[ p_1 \langle u_1 \rangle \right], \tag{3.1.37}
\]
where the star denotes complex conjugation. Inserting (3.1.37) into (3.1.36) yields
\[
\frac{d\dot{W}}{dx} = \frac{1}{2} A^2 A_g \text{Re} \left[ p_1 \frac{d(u_1^*)}{dx} + \langle u_1' \rangle \frac{dp_1}{dx} \right]. \tag{3.1.38}
\]

Next, inserting (3.0.27), we find
\[
\frac{d\dot{W}}{dx} = \frac{1}{2} A^2 \omega A_g \text{Re} \left[ p_{10} \frac{d(u_{11}^*)}{dx} \right] + \cdots. \tag{3.1.39}
\]

Combining equations (3.1.9) and (3.1.33), we find
\[
dl_{12} = -i \rho_0 \langle u_{11} \rangle \frac{1}{(1 - f_\nu)}, \tag{3.1.40}
\]
and
\[
dl_{11} = i \frac{d}{dx} \left( \frac{1 - f_\nu \, dp_{12}}{\rho_0} \right) = -\frac{i}{\rho_0 c^2} \left[ 1 + \frac{\gamma - 1}{1 + \varepsilon_s} f_k \right] p_{10} - \frac{f_k - f_\nu}{(Pr - 1)(1 + \varepsilon_s)(1 - f_\nu)} \beta \frac{dT_0}{dx} \langle u_{11} \rangle, \tag{3.1.41}
\]

As a result we find
\[
\frac{d\dot{W}}{dx} = \frac{1}{2} A^2 \omega A_g \left[ \frac{1}{\rho_0 c^2} \frac{\gamma - 1}{1 + \varepsilon_s} \text{Im} (f_k) |p_{10}|^2 - \frac{\beta}{Pr - 1} \frac{dT_0}{dx} \text{Re} \left( \frac{f_k^* - f_\nu^*}{(1 + \varepsilon_s^*) (1 - f_\nu^*)} p_{10} \langle u_{11} \rangle \right) \right] + \cdots. \tag{3.1.42}
\]

This is a quantity of order \( A^2 \omega \), therefore we define \( W_{21} \) by
\[
\frac{d\dot{W}_{21}}{dx} = \frac{1}{2} A_g \left[ \frac{1}{\rho_0 c^2} \frac{\gamma - 1}{1 + \varepsilon_s} \text{Im} (f_k) |p_{10}|^2 - \frac{\beta}{Pr - 1} \frac{dT_0}{dx} \text{Re} \left( \frac{f_k^* - f_\nu^*}{(1 + \varepsilon_s^*) (1 - f_\nu^*)} p_{10} \langle u_{11} \rangle \right) \right], \tag{3.1.43}
\]
where the subscript \( 21 \) is used to indicate second order in \( A \) and first order in \( \omega \).

The first term is the thermal relaxation dissipation term. This term is present whenever a wave interacts with a solid surface, and has a dissipative effect in thermoacoustics. The second term contains the temperature gradient \( dT_0/dx \) and is called the source or sink term because it either absorbs (refrigerator) or produces acoustic power (prime mover). This term is the unique contribution to thermoacoustics.
3.1.6 The dissipation in acoustic power

One could repeat the previous analysis for a long stack ($\omega = O(1)$) as well, in which case we get

$$\frac{d\dot{W}_2}{dx} = \frac{1}{2} A_g \left\{ \frac{\omega \rho_1 \text{Im}(f_\nu)}{|1-f_\nu|^2} |\langle u_1 \rangle|^2 + \omega \frac{\gamma - 1}{\rho_0 c^2} \text{Im} \left[ \frac{f_k^*}{1 + \varepsilon_s^*} \right] |p_1|^2 \right. $$

$$- \frac{\beta}{Pr - 1} \frac{dT_0}{dx} \text{Re} \left[ \frac{f_k^* - f_\nu^*}{(1-f_\nu^*)(1 + \varepsilon_s^*)} p_1 \langle u_1 \rangle \right] \left. \right\}. \quad (3.1.44)$$

This expression contains an additional term, representing the viscous dissipation, which we do not see in (3.1.43). To see the effect of reducing the pore size, we test how (3.1.44) behaves for small $N_L$. It can be shown that for small $\omega$ and $N_L$, (3.1.43) behaves as

$$\frac{d\dot{W}_2}{dx} = \text{sink term} - \text{viscous dissipation} - \text{thermal dissipation relaxation} = O(\omega) - O\left(\frac{\omega^3}{N_L^2}\right) - O\left(\omega N_L^2\right). \quad (3.1.45)$$

Dissipation is usually undesirable, so the dissipation terms should be smaller than the sink term. This gives the inequalities

$$\omega \ll N_L \ll 1. \quad (3.1.46)$$

If $\omega$ and $N_L$ satisfy this equation, then viscous and thermal relaxation dissipation can be neglected with respect to the sink term. In other words reducing the pore size reduces the thermal relaxation dissipation and increases the viscous dissipation. However, the latter can be prevented by taking the stack short enough, as indicated in (3.1.46).

3.1.7 The sink/source term in acoustic power

We saw in (3.1.43) that the sink/source term, which we will define as $W_{21}$, was of greatest interest in thermoacoustic engines and refrigerators. To interpret $W_{21}$, we will assume $\varepsilon_s = 0$ and neglect viscosity, so that $f_\nu = 0$ and $Pr = 0$,

$$\frac{d\dot{W}_{21}}{dx} = \frac{A_g \beta}{2} \frac{dT_0}{dx} \left[ \text{Re} \left( f_k \right) \text{Re} \left( p_{10} \langle u_{11} \rangle \right) + \text{Im} \left( -f_k \right) \text{Im} \left( p_{10} \langle u_{11} \rangle \right) \right]. \quad (3.1.47)$$

In a standing wave system $\text{Im} \left( p_{10} \langle u_{11} \rangle \right)$ is large and therefore $\text{Im} \left( -f_k \right)$ is important. Fig. 6 shows that the maximal value is attained for $N_L \sim 1$, which is exactly the condition assumed for a stack in a standing wave system. In the case of a traveling wave system $\text{Re} \left( p_{10} \langle u_{11} \rangle \right)$ is large and $\text{Re} \left( f_k \right)$ is important. Fig. 6 shows that $\text{Re} \left( f_k \right)$ reaches its maximal value for $N_L \ll 1$, which was the condition for a regenerator.

3.1.8 Time-averaged energy flux $\dot{E}$

Finally we will derive an expression for the time-averaged energy flux $\dot{E}$ in the stack, correct to second order in $A$. We consider the thermoacoustic refrigerator as shown in Fig. 5(a), driven by a loudspeaker. We assume that the refrigerator is thermally insulated from the surroundings, except at
Figure 6: Real and imaginary part of $f_k$, plotted as a function of the Laucret number $N_L$.  

the two heat exchangers, so that heat can be exchanged with the outside world only via the two heat exchangers. Work can only be exchanged at the loudspeaker piston.

By conservation of energy we have (A.5.6)

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \tilde{\rho} \tilde{v}^2 + \tilde{\rho} \tilde{e} \right) = -\tilde{\nabla} \cdot \left[ \tilde{v} \left( \frac{1}{2} \tilde{\rho} \tilde{v}^2 + \tilde{\rho} \tilde{h} \right) - K \tilde{\nabla} \tilde{T} - \tilde{v} \cdot \tilde{\Sigma} \right],$$  

(3.1.48)

where $\tilde{e}$ and $\tilde{h}$ are the internal energy and enthalpy per unit mass, respectively and $\tilde{\Sigma}$ is the viscous stress tensor as defined in (2.1.4). The expression on the left represents the rate-of-change of the energy in a unit volume of the fluid, while that on the right is the divergence of the energy flux density which consists of transfer of mass by the motion of the fluid, transfer of heat and energy flux due to internal friction, respectively. In steady state, for a cyclic refrigerator (prime mover) without heat flows to the surroundings, the time-averaged energy flux $\dot{\tilde{E}}_2$ (correct up to second order) along $\tilde{x}$ must be independent of $\tilde{y}$. Assuming steady state, integrating (3.1.48) with respect to $\tilde{y}$ (and $\tilde{y}'$) from $\tilde{y} = 0$ to $\tilde{y}' = 0$ and time-averaging yields

$$\frac{d}{d\tilde{x}} \left[ \int_0^R \rho \tilde{u} \tilde{v}^2 \, d\tilde{y} + \int_0^R \rho \tilde{u} \tilde{h} \, d\tilde{y} - \int_0^R K \frac{\partial \tilde{T}}{\partial \tilde{x}} \, d\tilde{y} - \int_0^l K_s \frac{\partial \tilde{T}_s}{\partial \tilde{x}} \, d\tilde{y}' - \int_0^R \tilde{u} \tilde{\Sigma}_{11} + \tilde{v} \tilde{\Sigma}_{21} \, d\tilde{y} \right] = 0.$$  

(3.1.49)

The quantity within the square brackets is $\dot{\tilde{E}}/\tilde{\Pi}$, the time-averaged energy flux per unit perimeter along $x$. Rescaling yields

$$\frac{\dot{E}}{\Pi} = \varepsilon^2 \int_0^1 \rho \omega^2 \, dy + \int_0^1 \rho \omega \, dy - \varepsilon^2 \omega \int_0^1 \frac{\partial \tilde{T}}{\partial \tilde{x}} \, dy - \varepsilon^2 \omega \int_0^1 \frac{\partial \tilde{T}_s}{\partial \tilde{x}} \, dy' - \frac{\varepsilon \omega}{W_0^2} \int_0^1 u \frac{\partial \mu}{\partial y} \left[ 1 + \mathcal{O}(\varepsilon^2) \right] \, dy,$$  

(3.1.50)
where \( \hat{E}, \bar{h} \) and \( \bar{\Pi} \) were rescaled as
\[
\hat{E} = R^2 D C^3 \bar{E}, \quad \bar{h} = C^2 \bar{s}, \quad \bar{\Pi} = R \Pi.
\] (3.1.51)

The second integral on the right hand side, can be written up two second order as follows.
\[
\rho u h = A \rho_0 h_0 u'_1 + A^2 \rho_0 h_0 u'_2 + A^2 h_0 \rho_1 u'_1 + A^2 \rho_0 h_1 u'_1 + \cdots.
\] (3.1.52)

The first term disappears because \( u'_1 = Re [u_1 e^{it}] = 0 \). The second term is zero because the second order time-averaged mass flux is zero
\[
\int_0^1 (\rho_0 \bar{u}_2 + \bar{p}_1 \bar{u}_1) \, dy = 0.
\] (3.1.53)

Therefore, inserting the relation
\[
dh = T ds + \frac{1}{\rho} dp = dT + \frac{1}{\rho} (1 - \beta T) dp,
\] (3.1.54)

we obtain
\[
\frac{\dot{E}}{\Pi} = A^3 \varepsilon^2 \int_0^1 \rho_0 u_1 s_1^3 \, dy + A^2 \int_0^1 (\rho_0 T_0 \bar{u}_1 s_1 + \bar{p}_1 \bar{u}_1) \, dy
\]
\[
- (\varepsilon^2 + \vartheta \varepsilon_1^2) \frac{\omega}{2 N_L^2} \frac{dT_0}{dx} - A^2 \varepsilon \frac{\omega}{W_0^2} \int_0^1 \frac{\partial u_1}{\partial y} \, dy.
\] (3.1.55)

Clearly the first and last term on the right hand side can be neglected with respect to the second term. As a result we are left with
\[
\frac{\dot{E}}{\Pi} = A^2 \int_0^1 \rho_0 u_1 T_1^2 \, dy + A^2 \int_0^1 (1 - \beta T_0) p_1 u_1 \, dy
\]
\[
- (\varepsilon^2 + \vartheta \varepsilon_1^2) \frac{\omega}{2 N_L^2} \frac{dT_0}{dx}.
\] (3.1.56)

Substituting the expansions given in (3.0.27) we find
\[
\dot{E} = \frac{1}{2} A^2 \omega A_g \int_0^1 \rho_0 \text{Re} [u_1^* T_{10}] \, dy + \frac{1}{2} A^2 \omega A_g \int_0^1 (1 - \beta T_0) \text{Re} [p_{10} u_{11}] \, dy
\]
\[
- (\varepsilon^2 + \vartheta \varepsilon_1^2) \frac{\omega}{2 N_L^2} \frac{dT_0}{dx} + \cdots
\]
\[
=: (\varepsilon^2 + \vartheta \varepsilon_1^2) \omega \dot{E}_{012} + A^2 \omega \dot{E}_{210} + \cdots.
\] (3.1.58)

with \( \dot{E}_{012} \) and \( \dot{E}_{210} \) defined as
\[
\dot{E}_{012} = - \frac{A_g}{2} \frac{dT_0}{2 N_L^2} \frac{dx},
\] (3.1.59)

\[
\dot{E}_{210} = \frac{A_g}{2} \int_0^1 \rho_0 \text{Re} [u_1^* T_{10}] \, dy + \frac{A_g}{2} \int_0^1 (1 - \beta T_0) \text{Re} [p_{10} u_{11}] \, dy.
\] (3.1.60)
where the index 012 indicates second order in $\varepsilon^2$ and $\varepsilon^2_1$ and first order in $\omega$, and the index 210 indicates second order in $A$ and first order in $\omega$. Inserting the expression obtained for $T_{10}$ into (3.1.60), we find

$$\dot{E}_{210} = \frac{Ag}{2} \text{Re} \left[ p_{10}(u_{11}) \left( 1 - \frac{\beta T_0(f_k - f_{n1})}{(1 + \varepsilon_s)(1 - f_{n1}^*)(1 + Pr)} \right) \right]$$

$$+ \frac{Ag |u_{11}|^2}{2(1 - Pr)|1 - f_k|^2} \frac{dT_0}{dx} \text{Im} \left[ f_{n1}^* + \frac{(1 + \varepsilon_s f_{n1}^*)(f_k - f_{n1}^*)}{(1 + \varepsilon_s)(Pr + 1)} \right].$$

(3.1.61)

$\dot{E}_{012}$ represents the contribution of the conduction of heat through gas and stack material to the total energy flux and $\dot{E}_{210}$ represents the contribution of the thermoacoustic heat flow. Therefore we find in the case of a short stack with the assumptions given in (3.0.24), that the energy flux is determined mainly by $\dot{E}_{210}$.

### 3.2 Short Regenerator

In the previous section we considered the case of a stack, where the Laucret and Womersley number were of order 1. Here we will discuss the regenerator, for which $Wo$ and $N_L$ are very small. Therefore we have to relate $Wo$ and $N_L$ to the other small parameters in (3.0.24). In section 3.1.6 we saw that to reduce viscous dissipation $N_L$ should satisfy $\omega \ll N_L \ll 1$. The remaining assumptions are slightly altered, so that

$$\frac{\varepsilon^2_1}{\omega} \ll \frac{\varepsilon^2}{\omega} \ll A \ll \omega^2 \ll N_L^2 \ll 1, \quad \vartheta, Pr = O(1),$$

(3.2.1)

We expand the fluid variables in the same way as we did in (3.0.27) for the short stack

$$q = q_0(x, y) + A \text{Re} \left[ (q_{10}(x, y) + \omega q_{11}(x, y)) e^{i\omega t} \right] + \cdots.$$  

(3.2.2)

#### 3.2.1 The horizontal velocity $u_1$

Substitute the expansions given in (3.0.27) in the $y$-component of the momentum equation (2.1.17). To leading order we find $\partial p_0/\partial y = 0$. Hence

$$p = p_0 + A \text{Re} \left[ (p_{10}(x) + \omega p_{11}(x)) e^{i\omega t} \right] + \cdots.$$  

(3.2.3)

Next, neglecting terms with order higher than $A\omega$, we obtain

$$A \frac{\partial p_{10}}{\partial y} + A\omega \frac{\partial p_{11}}{\partial y} = 0.$$  

(3.2.4)

This equation can only be satisfied if $\partial p_{10}/\partial y = \partial p_{11}/\partial y = 0$. We do the same for the $x$-component (2.1.16). To leading order we find $\partial p_0/\partial x = 0$. Next, we only keep terms of order up to $A\omega$ and $A\omega^2/Wo^2$ and neglect higher order terms. This leads to

$$iA\omega \rho_0 u_{10} = -A \frac{dp_{10}}{dx} - A\omega \frac{dp_{11}}{dx} + \frac{A\omega}{Wo^2} \left( \frac{\partial^2 u_{10}}{\partial y^2} + \omega \frac{\partial^2 u_{11}}{\partial y^2} \right).$$  

(3.2.5)

To solve this equation we have to couple $Wo$ to $\omega$ in some way. The most natural choice seems to be

$$Wo = \sqrt{\omega} \ll 1 \quad \Rightarrow \quad \omega \ll N_L = \sqrt{\frac{2}{Pr}} Wo \ll 1.$$  

(3.2.6)
Different choices are possible, but this choice simplifies the equation significantly and still satisfies \( \omega \ll N_L \). To simplify the equations even further, we also assume \( N_L = N_s \). Collecting the terms of order \( A \), we find that \( u_{10} \) satisfies

\[
\frac{\partial^2 u_{10}}{\partial y^2} = \frac{dp_{10}}{dx}
\]  

(3.2.7)

and applying the boundary conditions \( u_{10}(x, \pm 1) = 0 \), we obtain

\[
u_{10}(x, y) = -\frac{1}{2} \frac{dp_{10}}{dx} (1 - y^2).
\]  

(3.2.8)

Finally, collecting the terms of order \( A\omega \), we find

\[
-\frac{i}{2} \frac{dp_{10}}{dx} \rho_0 (1 - y^2) = -\frac{dp_{11}}{dx} + \frac{\partial^2 u_{11}}{\partial y^2},
\]  

(3.2.9)

where we substituted (3.2.8) for \( u_{10} \). Applying the boundary conditions \( u_{11}(x, \pm 1) = 0 \), we obtain

\[
u_{11}(x, y) = \frac{i}{24} \frac{dp_{10}}{dx} \rho_0 (1 - y^2) (5 - y^2) - \frac{1}{2} \frac{dp_{11}}{dx} (1 - y^2).
\]  

(3.2.10)

### 3.2.2 The temperature \( T_s \) in the plate

Using the rescaled energy equation (2.1.18) and the diffusion equation (2.1.19) we will try to find \( T \) and \( T_s \). First we solve (2.1.19) subject to (2.1.21b). Substitute the expansions from (3.0.27) into (2.1.19). To leading order we obtain \( \partial^2 T_s / \partial y^2 = 0 \). That means \( T_s \) is at most linear in \( y \). In view of symmetry we find that \( T_{s_0} \) is in fact constant in \( y \), i.e.

\[
T_s(x, y', t) = T_{s_0}(x) + A \text{Re} \left[ (T_{s_{10}}(x, y') + \omega T_{s_{11}}(x, y') \ e^{i\omega t}) \right] + \cdots,
\]  

(3.2.11)

Collecting terms up to order \( A\omega \) we find

\[
iA \omega \text{Pr} \rho_s T_{s_{10}} = A \frac{\partial^2 T_{s_{10}}}{\partial y^2} + A \omega \frac{\partial^2 T_{s_{11}}}{\partial y^2} = 0.
\]  

(3.2.12)

Then the terms of order \( A \) yield

\[
\frac{\partial^2 T_{s_{10}}}{\partial y^2} = 0.
\]  

(3.2.13)

Imposing (2.1.21b) we find

\[
T_{s_{10}}(x, y') = T_{b_{10}}(x),
\]  

(3.2.14)

Finally, collecting the terms of order \( A\omega \), we find

\[
i \text{Pr} \rho_s T_{b_{10}} = \frac{\partial^2 T_{s_{11}}}{\partial y^2}.
\]  

(3.2.15)

Applying the boundary conditions in (2.1.21b) we find

\[
T_{s_{11}}(x, y') = -\frac{i}{2} \text{Pr} \rho_s T_{b_{10}}(x) (1 - y^2) + T_{b_{11}}(x).
\]  

(3.2.16)

The exact expression for \( T_{b_{10}} \) should be determined from matching \( T_s \) with \( T \).
3.2.3 The temperature $T$ between the plates

We substitute the expansions shown in (2.1.13) into equation (2.1.18). To leading order we find $\partial^2 T_0 / \partial y^2 = 0$. That means $T_0$ is at most linear in $y$. In view of symmetry we find that $T_0$ is in fact constant in $y$. Therefore we expand $T$ as

$$T(x, y, t) = T_0(x) + A \text{Re} \left[ \left( T_{10}(x, y) + \omega T_{11}(x, y) \right) e^{i\omega t} \right] + \cdots,$$

(3.2.17)

Since $\rho_0 = \rho_0(p_0, T_0)$, we assume that also

$$\rho(x, y, t) = \rho_0(x) + A \text{Re} \left[ \left( \rho_{10}(x, y) + \omega \rho_{11}(x, y) \right) e^{i\omega t} \right] + \cdots,$$

(3.2.18)

Collecting terms up to order $A\omega$ we find

$$A\rho_0 u_{10} \frac{dT_0}{dx} + iA\omega \rho_0 T_{10} - iA\omega \beta T_{11} p_{10} = \frac{1}{Pr} \left( A \frac{\partial^2 T_{10}}{\partial y^2} + A\omega \frac{\partial^2 T_{11}}{\partial y^2} \right).$$

(3.2.19)

Now to leading order we find

$$\rho_0 \frac{Pr}{2} \frac{dp_{10}}{dx} \frac{dT_0}{dx} (y^2 - 1) = \frac{\partial^2 T_{10}}{\partial y^2},$$

(3.2.20)

where we substituted the expression for $u_{10}$ given in (3.1.8). Applying the boundary conditions in (2.1.21b), we find

$$T_{10}(x, y) = \rho_0 \frac{Pr}{24} \frac{dp_{10}}{dx} \frac{dT_0}{dx} (y^4 - 6y^2 + 5) + T_{b_{10}}.$$

(3.2.21)

Now $T_{b_{10}}$ still needs to be determined. The remaining boundary conditions in (2.1.21c), however, cannot be satisfied, unless

$$\frac{dp_{10}}{dx} = 0 \quad \Rightarrow \quad \begin{cases} u_{10}(x, y) = 0 \\ u_{11}(x, y) = -\frac{1}{2} \frac{dp_{11}}{dx} \left( 1 - y^2 \right). \end{cases}$$

(3.2.22)

With (3.2.22) we find

$$T_{10}(x) = T_{s_{10}}(x) = T_{b_{10}}(x),$$

(3.2.23)

where $T_{b_{10}}$ is yet to be determined. Finally we turn to the $A\omega$-equation. Collecting the $A\omega$ terms, we find

$$i\rho_0 T_{b_{10}} - i\beta T_0 p_{10} = \frac{1}{Pr} \frac{\partial^2 T_{11}}{\partial y^2}.$$

(3.2.24)

Applying the boundary conditions in (2.1.21b), we find

$$T_{11}(x, y) = -\frac{i}{2} Pr \left( \rho_0 T_{b_{10}} - \beta T_0 p_{10} \right) \left( 1 - y^2 \right) + T_{b_{11}}.$$

(3.2.25)

The boundary conditions in (2.1.21c) lead to

$$T_{b_{10}} = \frac{\beta T_0 p_{10}}{\rho_0 + \beta \rho_{s_0}}.$$

(3.2.26)
As a result we find

\[ T_{11}(x, y) = \frac{Pr}{2} \frac{\partial p_{0}}{\rho_{0} + \varrho \rho_{0}} \beta T_{0} p_{10} (1 - y^2) + T_{b11}, \]  
(3.2.27)

\[ T_{s11}(x, y') = -\frac{Pr}{2} \frac{\beta T_{0} p_{10}}{\rho_{0} + \varrho \rho_{0}} (1 - y'^2) + T_{b11}, \]  
(3.2.28)

\[ \frac{\partial v_{10}}{\partial y} = 0 \quad \Rightarrow \quad v_{10} = 0, \]  
(3.2.30)

where we applied the boundary condition \( v_{10}(\pm 1) = 0 \). Now to leading order, we obtain

\[ i \rho_{10} + \frac{\partial}{\partial x} (\rho_{0} u_{11}) + \rho_{0} \frac{\partial v_{11}}{\partial y} = 0, \]  
(3.2.31)

Using equation (3.2.22), the equation of state (2.1.20), and inserting the expressions for \( T_1 \) and \( u_{11} \) we finally find

\[ -i \rho_{0} \beta T_{b10} + i \frac{\gamma}{\epsilon^2} p_{10} - \frac{1}{2} \frac{d^2 p_{11}}{dx^2} (1 - y^2) + \rho_{0} \frac{\partial v_{11}}{\partial y} = 0. \]  
(3.2.32)

Next we integrate with respect to \( y \) from 0 to 1. Note that \( v(x, 0) = 0 \) because of symmetry and \( v(x, 1) = 0 \) because of the boundary conditions. We obtain

\[ \frac{d}{dx} \left( \rho_{0} \frac{d p_{11}}{dx} \right) = 3i \frac{\gamma}{\epsilon^2} p_{10} - 3i \rho_{0} \beta T_{b10} = 3i \beta^2 T_{0} p_{10} \left( \frac{\gamma}{\gamma - 1} - \frac{\rho_{0}}{\rho_{0} + \varrho \rho_{0}} \right). \]  
(3.2.33)

In the last equality we inserted the expression for \( T_{b10} \) and applied thermodynamic expression (A.4.6). Integrating twice with respect to \( y \), we can express \( p_{11} \) as follows

\[ p_{11}(x) = 3i \beta^2 p_{10} \int_{0}^{x} \int_{0}^{\xi} T_{0}(\xi) \left( \frac{\gamma}{\gamma - 1} - \frac{\rho_{0}(\xi)}{\rho_{0}(\xi) + \varrho \rho_{0}(\xi)} \right) d\xi d\xi' + A_1 x + A_2. \]  
(3.2.34)

The constants \( A_1 \) and \( A_2 \) can be determined if boundary conditions at \( x = 0 \) and \( x = 1 \) are imposed. During the derivation leading to this result, the following assumptions were used

\[ \frac{\varepsilon^2}{\omega} \ll \frac{\varepsilon^2}{\omega} \ll \varrho A \ll \omega^2 \ll 1, \quad \varrho = O(1), \quad \omega = \omega_{0}^2 = \frac{2N_{L}^2}{Pr} = \frac{2N_{s}^2}{Pr}, \]  
(3.2.35)

### 3.2.4 The pressure \( p \)

In order to derive Rott’s wave equation, we start with the continuity equation. To leading order the rescaled continuity equation yields

\[ \rho_{0} \frac{\partial v_{10}}{\partial y} = 0 \quad \Rightarrow \quad v_{10} = 0, \]  
(3.2.30)

\[ i \rho_{10} + \frac{\partial}{\partial x} (\rho_{0} u_{11}) + \rho_{0} \frac{\partial v_{11}}{\partial y} = 0, \]  
(3.2.31)

Using equation (3.2.22), the equation of state (2.1.20), and inserting the expressions for \( T_1 \) and \( u_{11} \) we finally find

\[ -i \rho_{0} \beta T_{b10} + i \frac{\gamma}{\epsilon^2} p_{10} - \frac{1}{2} \frac{d^2 p_{11}}{dx^2} (1 - y^2) + \rho_{0} \frac{\partial v_{11}}{\partial y} = 0. \]  
(3.2.32)

Next we integrate with respect to \( y \) from 0 to 1. Note that \( v(x, 0) = 0 \) because of symmetry and \( v(x, 1) = 0 \) because of the boundary conditions. We obtain

\[ \frac{d}{dx} \left( \rho_{0} \frac{d p_{11}}{dx} \right) = 3i \frac{\gamma}{\epsilon^2} p_{10} - 3i \rho_{0} \beta T_{b10} = 3i \beta^2 T_{0} p_{10} \left( \frac{\gamma}{\gamma - 1} - \frac{\rho_{0}}{\rho_{0} + \varrho \rho_{0}} \right). \]  
(3.2.33)

In the last equality we inserted the expression for \( T_{b10} \) and applied thermodynamic expression (A.4.6). Integrating twice with respect to \( y \), we can express \( p_{11} \) as follows

\[ p_{11}(x) = 3i \beta^2 p_{10} \int_{0}^{x} \int_{0}^{\xi} T_{0}(\xi) \left( \frac{\gamma}{\gamma - 1} - \frac{\rho_{0}(\xi)}{\rho_{0}(\xi) + \varrho \rho_{0}(\xi)} \right) d\xi d\xi' + A_1 x + A_2. \]  
(3.2.34)

The constants \( A_1 \) and \( A_2 \) can be determined if boundary conditions at \( x = 0 \) and \( x = 1 \) are imposed. During the derivation leading to this result, the following assumptions were used

\[ \frac{\varepsilon^2}{\omega} \ll \frac{\varepsilon^2}{\omega} \ll A \ll \omega^2 \ll 1, \quad \varrho = O(1), \quad \omega = \omega_{0}^2 = \frac{2N_{L}^2}{Pr} = \frac{2N_{s}^2}{Pr}, \]  
(3.2.35)

### 3.2.5 Acoustic power

Remember from the previous section that the time-averaged acoustic power \( \dot{W} \) is given by

\[ \frac{d\dot{W}}{dx} = \frac{1}{2} A^2 A_{g} \frac{d}{dx} \left( \text{Re} \left[ p_{1}(u_{1}) \right] \right) \]  
(3.2.36)
Inserting (3.2.2) into (3.2.36) yields
\[
\frac{d\dot{W}}{dx} = \frac{1}{2} A^2 \omega A_g \Re \left[ p_{10} \frac{d(u_{11}^*)}{dx} \right] + \cdots .
\]  \hspace{1cm} (3.2.37)

Combining equations (3.2.22) and (3.2.33), we find
\[
\frac{d\langle u_{11} \rangle}{dx} = -\frac{1}{3} \frac{d^2 p_{10}}{dx^2} = -\frac{1}{3 \rho_0} \frac{d}{dx} \left( \frac{\rho_0}{\rho_0} \frac{dp_{10}}{dx} \right) + \frac{1}{3 \rho_0} \frac{dp_{10}}{dx} \frac{dp_{10}}{dx}
\]
\[
= -i \beta T_0 p_{10} \left( \frac{\gamma}{\gamma - 1} - \frac{\rho_0}{\rho_0 + \vartheta \rho_s} \right) + \beta \frac{dT_0}{dx} \langle u_{11} \rangle
\]  \hspace{1cm} (3.2.38)

Inserting this result into (3.2.37), we find
\[
\frac{d\dot{W}}{dx} = \frac{1}{2} A^2 \omega A_g \Re \left[ \beta \frac{dT_0}{dx} p_{10} \langle u_{11} \rangle^* \right] + \beta \frac{dT_0}{dx} \langle u_{11} \rangle^* + \cdots
\]  \hspace{1cm} (3.2.39)

This is a quantity of order $A^2 \omega$, therefore we define $W_{21}$ by
\[
\frac{d\dot{W}_{21}}{dx} = \frac{1}{2} A^2 \omega A_g \beta \frac{dT_0}{dx} \Re \left[ p_{10} \langle u_{11} \rangle^* \right].
\]  \hspace{1cm} (3.2.40)

where the subscript 21 is used to indicate second order in $A$ and first order in $\omega$.

We see that the acoustic power is determined completely by the sink/source term. It can either absorb (refrigerator) or produce acoustic power (prime mover) depending on the magnitude of the temperature gradient along the stack. Thus, as expected, not only the viscous dissipation, but also the thermal relaxation dissipation can be neglected. This is due to the perfect thermal contact between the gas and the plate ($N_L \ll 1$), which was not the case in (3.1.43) for the short stack. Viscous dissipation can be neglected because the regenerator is short enough ($\omega \ll N_L$).

Moreover, if we look at (3.2.40), then we can see that this exactly coincides with the sink term in (3.1.47), for $N_L \ll 1$ ($\Im(-f_k) \to 0$ and $\Re(f_k) \to 1$).

### 3.2.6 Time-averaged energy flux $\dot{E}$

In section 3.1.8 we saw that the time-averaged energy flux $\dot{E}$ in the stack is given by
\[
\dot{E} = A^2 A_g \int_0^1 \rho_0 u_1^* T_1^* \, dy + A^2 \int_0^1 (1 - \beta T_0) p_{10}^* u_1^* \, dy - \frac{\varepsilon^2 + \vartheta \varepsilon_1^2}{Pr} \frac{dT_0}{dx}.
\]  \hspace{1cm} (3.2.41)

Substituting the expansions given in (3.2.2) we find

\[
\dot{E} = \frac{1}{2} A^2 \omega A_g \int_0^1 \rho_0 \Re \left[ u_1^* T_{10} \right] \, dy + \frac{1}{2} A^2 \omega A_g \int_0^1 (1 - \beta T_0) \Re \left[ p_{10} u_{11}^* \right] \, dy
\]
\[
- \frac{\varepsilon^2 + \vartheta \varepsilon_1^2}{Pr} \frac{A_g}{dx} \frac{dT_0}{dx} + \cdots
\]
\[
= (\varepsilon^2 + \vartheta \varepsilon_1^2) \dot{E}_{002} + A^2 \omega \dot{E}_{210} + \cdots
\]  \hspace{1cm} (3.2.42)
with $\dot{E}_{002}$ and $\dot{E}_{210}$ defined as

$$\dot{E}_{002} = -\frac{A_g}{Pr} \frac{dT_0}{dx},$$  
(3.2.43)

$$\dot{E}_{210} = \frac{A_g}{2} \int_0^1 \rho_0 \text{Re} [u_{11}^{*} T_{10}] \ dy + \frac{A_g}{2} \int_0^1 (1 - \beta T_0) \text{Re} [p_{10} u_{11}^{*}] \ dy,$$
(3.2.44)

where the index 002 indicates second order in $\varepsilon_2$ and $\varepsilon_1$ and the index 210 indicates second order in $A$ and first order in $\omega$. Inserting the expression obtained for $T_{10}$ into (3.2.44), we find

$$\dot{E}_{210} = \frac{A_g}{2} \left[ 1 + \vartheta (1 - \beta T_0) \right] \text{Re} [p_{10} \langle u_{11}^{*} \rangle].$$
(3.2.45)

$\dot{E}_{002}$ represents the contribution of the conduction of heat through gas and stack material to the total energy flux and $\dot{E}_{210}$ represents the contribution of the thermoacoustic heat flow. The term $\dot{E}_{002}$ is an undesirable nuisance, but can normally be neglected provided $\varepsilon$ and $\varepsilon_1$ are small enough as assumed in (3.2.1) ($\varepsilon^2 \ll A\omega$).

### 4. Thermoacoustic devices with mean velocity

This chapter discusses thermoacoustic devices in the presence of a steady mean velocity. This will be done both for stack-based and regenerator-based devices. With the addition of a steady non-zero mean velocity along $x$, the gas moves through the tube in a repetitive “102 steps forward, 98 steps backward” manner. As a result we will have to adapt our expansions in (2.1.13) to include a steady mean flow $\bar{u}_m$. However, as we still assume that $u_m = \bar{u}_m/C \ll 1$. We include this in our expansion by demanding

$$u_m = A^\alpha u_{m,\alpha} + o(A^\alpha),$$
(4.0.46)

for some constant $\alpha > 0$ and with $u_{m,\alpha} \neq 0$. This suggests the following alternate expansion for $v$ and the remaining fluid variables $q$

$$v(x, y, t) = \sum_{j=1}^{\infty} A^{\alpha_j} v_{m,\alpha_j}(x, y) + \sum_{k=1}^{\infty} A^k v'_k(x, y, t),$$
(4.0.47a)

$$q(x, y, t) = q_{m,0}(x, y) + \sum_{j=1}^{\infty} A^{\alpha_j} q_{m,\alpha_j}(x, y) + \sum_{k=1}^{\infty} A^k q'_k(x, y, t),$$
(4.0.47b)

where $\alpha_j = j \alpha$ and $q'_k$ is assumed to be of the form

$$q'_k(x, y, t) = \text{Re} \left[ q_1(x, y) e^{i\omega t} \right].$$
(4.0.48)

We will insert these expansions in the equations and boundary conditions given in (2.1.15)-(2.1.21).

#### 4.1 Short Stack

We will assume the same ordering for the dimensionless parameters as in (3.0.24) for the case without a mean flow

$$Wo = O(1), \quad N_L = O(1), \quad N_s = O(1), \quad \vartheta = O(1),$$
(4.1.1a)
\[ \varepsilon_1^2 \ll \varepsilon^2 \ll A \ll \omega \ll 1, \] (4.1.1b)

and expand the fluid perturbation variables \( q_1 \) and \( q_{m,\alpha} \) as

\[ q_1(x, y) = q_{10}(x, y) + \omega q_{11}(x, y) + \omega^2 q_{12}(x, y) + \cdots, \] (4.1.2a)

\[ q_{m,\alpha}(x, y) = q_{m,\alpha 0}(x, y) + \omega q_{m,\alpha 1}(x, y) + \omega^2 q_{m,\alpha 2}(x, y) + \cdots. \] (4.1.2b)

### 4.1.1 The steady horizontal velocity component \( u_{m,\alpha} \)

We substitute the expansions (4.0.47) into the time-averaged momentum equation. For the \( y \)-component we find, neglecting higher order terms,

\[ \frac{\partial p_{m,0}}{\partial y} + A^\alpha \frac{\partial p_{m,\alpha}}{\partial y} = 0. \] (4.1.3)

Consequently

\[ \frac{\partial p_{m,0}}{\partial y} = \frac{\partial p_{m,\alpha}}{\partial y} = 0. \] (4.1.4)

For the \( x \)-component we find

\[ A^2 \omega \text{Re} [i \rho_1 u_1^*] + A^2 \rho_0 \text{Re} \left[ u_1^* \frac{\partial u_1}{\partial x} + v_1^* \frac{\partial u_1}{\partial y} \right] = -\frac{dp_{m,0}}{dx} - A^\alpha \frac{dp_{m,\alpha}}{dx} + A^\alpha \frac{\omega}{W_0^2} \frac{\partial^2 u_{m,\alpha}}{\partial y^2}. \] (4.1.5)

To leading order we find \( p_{m,0} \) is constant. Now we can distinguish three cases \( \alpha < 2, \alpha = 2 \) and \( \alpha > 2 \). We will only consider the first two cases. These are the most interesting cases, since the acoustic power and total energy will hardly be affected by the mean flow if \( \alpha > 2 \). Furthermore, if \( \alpha > 2 \) then the left hand side of this equation would have to be zero, which in general is not the case. Therefore it seems that \( \alpha \) cannot be larger than 2.

**\( \alpha < 2 \):**

\( A^\alpha \) is the leading term. Collecting the \( A^\alpha \) terms and expanding in \( \omega \), we find \( p_{m,\alpha 0} \) is constant and

\[ \frac{1}{W_0^2} \frac{\partial^2 u_{m,\alpha k}}{\partial y^2} = \frac{dp_{m,\alpha(k+1)}}{dx}, \quad \text{for all } k \text{ with } A^\tau \ll \omega^{k+1}, \] (4.1.6)

where \( \tau = \min\{\alpha, 2 - \alpha\} \). Next, integrating twice with respect to \( y \) and imposing the boundary conditions \( u_{m,\alpha}(x, \pm 1) = 0 \), we find

\[ u_{m,\alpha k}(x, y) = -\frac{W_0^2}{2} \frac{dp_{m,\alpha(k+1)}}{dx} (1 - y^2), \quad \text{for all } k \text{ with } A^\tau \ll \omega^{k+1}. \] (4.1.7)

**\( \alpha = 2 \):**

Now both \( A^\alpha \) and \( A^2 \) are the leading terms. Furthermore, as \( \alpha > 1 \) the calculation of the first order
perturbation variables $T_1, T_s, u_1$ and $p_1$ in the previous chapter remains unaffected. Collecting the $A^2$ terms and expanding in $\omega$, we find

$$A^2 \omega^2 \text{Re} \left[ i \left( \rho_0 T_{10} + \frac{\gamma}{c^2} p_{10} \right) u_{11}^* + \rho_0 u_{11}^* \frac{\partial u_{11}}{\partial x} + \rho_0 v_{11}^* \frac{\partial u_{11}}{\partial y} \right] = -A^2 \frac{dp_{m,20}}{dx}$$

$$+ A^2 \omega \left\{ \frac{1}{Wo^2} \frac{\partial^2 u_{m,20}}{\partial y^2} - \frac{dp_{m,21}}{dx} \right\} + A^2 \omega^2 \left\{ \frac{1}{Wo^2} \frac{\partial^2 u_{m,21}}{\partial y^2} - \frac{dp_{m,22}}{dx} \right\}.$$  \hspace{1cm} (4.1.8)

Here we used $u_{10} = v_{10} = 0$ as calculated in (3.1.19) and (3.1.27). Now successively we find $p_{m,20}$ is constant,

$$u_{m,20}(x, y) = -\frac{Wo^2}{2} \frac{dp_{m,21}}{dx} (1 - y^2), \hspace{1cm} (4.1.9)$$

and $u_{m,21}$ satisfies

$$\frac{1}{Wo^2} \frac{\partial^2 u_{m,21}}{\partial y^2} = \frac{dp_{m,22}}{dx} - \text{Re} \left[ i \left( \rho_0 T_{10} + \frac{\gamma}{c^2} p_{10} \right) u_{11}^* + \rho_0 u_{11}^* \frac{\partial u_{11}}{\partial x} + \rho_0 v_{11}^* \frac{\partial u_{11}}{\partial y} \right]. \hspace{1cm} (4.1.10)$$

### 4.1.2 The steady temperature $T_{s m, \alpha}$ in the plate

The time-averaged part of the diffusion equation for $T_s$ reads

$$\frac{1}{2} A^2 \omega \text{Re} \left[ i \rho_{s10} T_{s10}^* \right] + \frac{\omega}{2 N_s^2} \left( \frac{\partial^2 T_{s m,0}}{\partial y^2} + A^\alpha \frac{\partial^2 T_{s m,0}}{\partial y^2} \right) = 0. \hspace{1cm} (4.1.11)$$

Here we only included terms up to order $A^\alpha$ and $A^2$. As a result $T_{s m,0}$ is a function of $x$ only and can be determined as

$$T_{s m,0}(x) = T_{m,0}(x, 1), \hspace{1cm} (4.1.12)$$

As a result we have three boundary conditions left for $T_{m,0}$:

$$T_{m,0}(x, -1) = T_{m,0}(x, 1), \hspace{1cm} (4.1.13a)$$

$$\frac{\partial T_{m,0}}{\partial y}(x, \pm 1) = 0. \hspace{1cm} (4.1.13b)$$

$\alpha < 2$:

For all $k$ with $A^\tau \ll \omega^k$, we find $T_{s m, \alpha k}$ is a function of $x$ only and can be determined as

$$T_{s m, \alpha k}(x) = T_{m, \alpha k}(x, 1), \hspace{1cm} (4.1.14)$$

As a result we again have three boundary conditions left for $T_{m, \alpha k}$:

$$T_{m, \alpha k}(x, -1) = T_{m, \alpha k}(x, 1), \hspace{1cm} (4.1.15a)$$

$$\frac{\partial T_{m, \alpha k}}{\partial y}(x, \pm 1) = 0. \hspace{1cm} (4.1.15b)$$

Note that in general it will not not be possible for $T_{m, \alpha k}$ to satisfy three boundary conditions.

$\alpha = 2$:

Collecting the $A^2$ terms and expanding in $\omega$, we find

$$\frac{1}{2} \text{Re} \left[ i \rho_{s10} T_{s10}^* \right] + \frac{1}{2 N_s^2} \frac{\partial^2 T_{s m,20}}{\partial y^2} = 0. \hspace{1cm} (4.1.16)$$
4.1.3 The steady temperature $T_{m,\alpha}$

Substituting the expansions (4.0.47) in the time-averaged temperature equation, we obtain

$$
\frac{1}{2} A^2 \omega \text{Re} \left[ i \rho_1 T_1^* - i \beta \rho_1 T_1^* \right] + \frac{1}{2} A^2 \text{Re} \left[ \rho_0 u_1^* \frac{\partial T_1}{\partial x} + \rho_0 v_1^* \frac{\partial T_1}{\partial y} - \beta T_0 u_1^* \frac{dp_1}{dx} \right] + A^\alpha \rho_{m,0} \left( u_{m,\alpha} \frac{\partial T_{m,0}}{\partial x} + v_{m,\alpha} \frac{\partial T_{m,0}}{\partial y} \right) = \frac{\omega}{2N_L^2} \left( \frac{\partial^2 T_{m,0}}{\partial y^2} + A^\alpha \frac{\partial^2 T_{m,\alpha}}{\partial y^2} \right) + \frac{1}{2} \frac{A^2 \omega}{\omega_0^2} \left( \frac{\partial u_1}{\partial y} \right)^2. \tag{4.1.17}
$$

Again we only included terms up to order $A^2$ and $A^\alpha$. To leading order we find $\partial^2 T_{m,0}/\partial y^2 = 0$ and thus $T_{m,0}$ is a function of $x$ only. The boundary conditions in (4.1.12) and (4.1.13) are satisfied if

$$
T_{m,0}(x) = T_{s,m,0}(x). \tag{4.1.18}
$$

If $\alpha < 2$:

$A^\alpha$ is the leading term. Collecting $A^\alpha$-terms and expanding in $\omega$ we find

$$
- \frac{W_0^2}{2} (1 - y^2) \rho_{m,0} \frac{dT_{m,0}}{dx} \left( \frac{dp_{m,\alpha1}}{dx} + \omega \frac{dp_{m,\alpha2}}{dx} \right) = \frac{\omega}{2N_L^2} \frac{\partial^2 T_{m,\alpha0}}{\partial y^2}, \tag{4.1.19}
$$

where we substituted the expression found for $u_{m,\alpha0}$ and $u_{m,\alpha1}$. To leading order we find $dp_{m,\alpha1}/dx = 0$ and therefore also $u_{m,\alpha0} = 0$. Next (4.1.19) reduces into

$$
- \frac{W_0^2}{2} (1 - y^2) \rho_{m,0} \frac{dT_{m,0}}{dx} \frac{dp_{m,\alpha2}}{dx} = \frac{1}{2N_L^2} \frac{\partial^2 T_{m,\alpha0}}{\partial y^2}. \tag{4.1.20}
$$

If we try to solve this equation, then the boundary conditions in (4.1.15) can only be satisfied if $dp_{m,\alpha2}/dx = 0$ and therefore also $u_{m,\alpha1} = 0$. Continuing this way we can show that $u_{m,\alpha k} = 0$ for all $k$ with $A^\alpha \ll \omega^{k+1}$ and $A^{2-\alpha} \ll \omega^{k+1}$. Consequently we find that either $A^\alpha u_{m,\alpha} = O \left( A^{2\alpha} \right)$, or $A^\alpha u_{m,\alpha} = O \left( A^2 \right)$, both contradicting our initial assumptions and therefore $\alpha$ must be 2. Thus the physics causes the steady mean flow term to be essentially second order, which was also suggested by Swift [23].

If $\alpha = 2$:

Collecting the $A^2$ terms and expanding in $\omega$, we find

$$
\frac{1}{2} \omega \text{Re} \left[ i \left( \rho_0 \beta T_0 + \frac{\gamma}{c^2} p_1 \right) T_0^* - i \beta p_1 T_1^* + \rho_0 u_{11}^* \frac{\partial T_1}{\partial x} + \rho_0 v_{11}^* \frac{\partial T_1}{\partial y} - \beta T_0 u_{11}^* \frac{dp_1}{dx} \right] = \frac{\omega}{2N_L^2} \frac{\partial^2 T_{m,20}}{\partial y^2} - \rho_{m,0} \frac{dT_{m,0}}{dx} \left( u_{m,20} + \omega u_{m,21} \right). \tag{4.1.21}
$$

Remember that the variables $p_1$, $T_1$, $T_{s1}$ and $u_1$ are given by the expressions in the previous chapter, as $\alpha > 1$. To leading order, we find $u_{m,20} = 0$. Next $T_{m,20}$ satisfies

$$
\frac{1}{2} \text{Re} \left[ i \left( \rho_0 \beta T_0 + \left( \frac{\gamma}{c^2} - \beta \right) p_1 \right) T_0^* + \rho_0 u_{11}^* \frac{\partial T_1}{\partial x} + \rho_0 v_{11}^* \frac{\partial T_1}{\partial y} \right] = \frac{1}{2N_L^2} \frac{\partial^2 T_{m,20}}{\partial y^2} - \rho_{m,0} \frac{dT_{m,0}}{dx} u_{m,21}. \tag{4.1.22}
$$

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Solving (4.1.16) and (4.1.22) subject to the boundary conditions

\[ T_{m,20}(x, \pm 1) = T_{s_{m,20}}(x, \pm 1), \quad (4.1.23a) \]
\[ \frac{\partial T_{m,20}}{\partial y}(x, \pm 1) = \vartheta \frac{\partial T_{s_{m,20}}}{\partial y}(x, \mp 1). \quad (4.1.23b) \]

we will find \( T_{m,20} \) and \( T_{s_{m,20}} \).

### 4.1.4 The continuity equation

Integrating the continuity equation with respect to \( y \) from 0 to 1 and time-averaging yields

\[ \frac{d\dot{M}_{21}}{dx} = 0, \quad (4.1.24) \]

where

\[ \dot{M}_{21} = \frac{1}{2} \text{Re} \left[ \int_{0}^{1} \rho_{10} u_{11}^{*} \, dy \right] + \rho_{0} \int_{0}^{1} u_{m,21} \, dy. \quad (4.1.25) \]

### 4.1.5 The unsteady terms \( p_{1}, u_{1} \) and \( T_{1} \)

As \( \alpha > 1 \) the calculation of the perturbation variables \( T_{1}, u_{1}, p_{1} \) in the previous chapter remains unaffected. Thus for \( p_{1} \) we have

\[ p_{10} = \text{constant}, \quad (4.1.26a) \]
\[ p_{11} = \text{constant}, \quad (4.1.26b) \]

\[ \frac{1}{c_{2}} \left[ 1 + \frac{\gamma - 1}{1 + \varepsilon_{s}} \frac{f_{k}}{f_{0}} \right] p_{10} + \frac{f_{k} - f_{0}}{(P_{r} - 1)(1 + \varepsilon_{s})} \beta \frac{dT_{m,0}}{dx} \frac{dp_{12}}{dx} + \rho_{m,0} \frac{d}{dx} \left( \frac{1 - f_{0} \frac{dp_{12}}{dx}}{\rho_{m,0}} \right) = 0 \quad (4.1.26c) \]

For \( u_{1} \) we have

\[ u_{10}(x, y) = 0, \quad (4.1.27a) \]
\[ u_{11}(x, y) = \frac{i}{\rho_{m,0}} \frac{dp_{12}}{dx} \left[ 1 - \frac{\cosh(\alpha_{y}y)}{\cosh(\alpha_{y})} \right]. \quad (4.1.27b) \]

Finally, for \( T_{1} \) and \( T_{s_{1}} \) we have

\[ T_{10}(x, y) = \frac{\beta T_{m,0}}{\rho_{m,0}} p_{10} - \frac{1}{\rho_{m,0}} \frac{dT_{m,0}}{dx} \frac{dp_{12}}{dx} \left[ 1 - \frac{P_{r}}{P_{r} - 1} \frac{\cosh(\alpha_{y}y)}{\cosh(\alpha_{y})} \right] \]
\[ - \left( \frac{\beta T_{m,0}}{\rho_{m,0}} p_{10} + \frac{1}{\rho_{m,0}} \frac{1 + \varepsilon_{s} \frac{f_{k}}{f_{0}}}{P_{r} - 1} \frac{dT_{m,0}}{dx} \frac{dp_{12}}{dx} \right) \frac{\cosh(\alpha_{k}y)}{(1 + \varepsilon_{s}) \cosh(\alpha_{k})}, \quad (4.1.28) \]
\[ T_{s_{10}}(x, y') = \frac{\varepsilon_{s}}{1 + \varepsilon_{s}} \left( \frac{\beta T_{0}}{\rho_{0}} p_{10} + \frac{1}{\rho_{0}} \frac{dT_{0}}{dx} \frac{dp_{12}}{dx} \frac{1 - \frac{f_{k}}{f_{0}}}{P_{r} - 1} \right) \frac{\cosh(\alpha_{s}y')}{\cosh(\alpha_{s})}. \quad (4.1.29) \]
4.1.6 Time-averaged energy flux $\dot{E}$

In the same way as we did in equation (3.1.58) in the absence of a mean flow, we can derive an equation for the time-averaged energy flux $\dot{E}$. In the case of a mean flow, we get an additional term in equation (3.1.52) involving $u_m^2$, that is

$$\rho u h = A_0 \rho_0 h_0 u_1 + A^2 \rho_0 h_0 u_1 + A^2 \rho_0 h_1 u_1 + A^2 \rho_0 h_0 u_m^2 + \cdots$$

With this change, we also get an additional steady flow term $\dot{E}_{m,210}$ in equation (3.1.58)

$$\dot{E} = A^2 \omega \dot{E}_{m,210} + A^2 \omega \dot{E}_{210} + (\varepsilon^2 + \vartheta \varepsilon^2) \omega \dot{E}_{012} + \cdots,$$

(4.1.31)

where

$$\dot{E}_{m,210} = \rho_0 h_0 \int_0^1 u_{m,21} dy.$$

(4.1.32)

4.2 Short Regenerator

We will assume the same ordering for the dimensionless parameters as in (3.2.35) for the case without a mean flow

$$\varepsilon_1 \approx \varepsilon_2 \approx A \ll 1, \quad \vartheta = O(1), \quad \omega = W_0^2 = \frac{2N_2^2}{Pr} = \frac{2N_2^2}{Pr}.$$

(4.2.1)

and again expand the fluid perturbation variables $q_1$ and $q_{m,\alpha}$ as

$$q_1(x, y) = q_{01}(x, y) + \omega q_{11}(x, y) + \omega^2 q_{12}(x, y) + \cdots,$$

(4.2.2a)

$$q_{m,\alpha}(x, y) = q_{m,\alpha 0}(x, y) + \omega q_{m,\alpha 1}(x, y) + \omega^2 q_{m,\alpha 2}(x, y) + \cdots.$$

(4.2.2b)

In the same way as we did for the stack, it can be shown that $\alpha = 2$. We will not repeat this analysis and only mention the final outcome.

4.2.1 The steady flow terms

For $u_{m,2}$ we find

$$u_{m,20}(x, y) = -\frac{1}{2} \frac{d p_{m,20}}{d x} (1 - y^2),$$

(4.2.3)

$$u_{m,21}(x, y) = -\frac{1}{2} \frac{d p_{m,21}}{d x} (1 - y^2).$$

(4.2.4)

and $u_{m,21}$ satisfies

$$\frac{\partial^2 u_{m,22}}{\partial y^2} = \frac{d p_{m,22}}{d x} - \text{Re} \left[ i \left( \rho \beta T_{10} + \gamma \frac{\gamma}{e^2} p_{10} \right) u_{11}^* + \rho_0 u_{11} \frac{\partial u_{11}}{\partial x} + \rho_0 v_{11} \frac{\partial u_{11}}{\partial y} \right].$$

(4.2.5)

subject to $u_{m,21}(x, \pm 1) = 0$. For the steady temperature terms we find

$$T_{m,20}(x) = T_{m,20}(x).$$

(4.2.6)
and \( T_{m,21} \) and \( T_{s,m,21} \) satisfy
\[
\frac{Pr}{2} \text{Re} \left[ \imath \sigma_{10} T_{s,10}^* \right] + \frac{\partial^2 T_{s,m,21}}{\partial y'^2} = 0. \tag{4.2.7}
\]
\[
\frac{1}{2} \text{Re} \left[ \imath \left\{ \rho_0 \beta T_{10} + \left( \frac{\gamma}{C_s^2} - \beta \right) p_{10} \right\} T_{10}^* + \frac{\partial T_{10}}{\partial x} \frac{d T_{10}}{d x} \right] = \frac{1}{Pr} \frac{\partial^2 T_{m,21}}{\partial y'^2} - \rho_{m,0} \frac{d T_{m,0}}{d x} u_{m,21}, \tag{4.2.8}
\]
subject to
\[
T_{m,21}(x, \pm 1) = T_{s,m,21}(x, \pm 1), \tag{4.2.9a}
\]
\[
\frac{\partial T_{m,21}}{\partial y}(x, \pm 1) = \vartheta \frac{\partial T_{s,m,21}}{\partial y}(x, \mp 1). \tag{4.2.9b}
\]

For the pressure we find that \( p_{m,20} \) is constant and \( p_{m,21} \) can be found from
\[
\frac{d M_{21}}{d x} = 0, \tag{4.2.10}
\]
where
\[
M_{21} = \frac{1}{2} \text{Re} \left[ \int_0^1 \left( \rho_0 \beta T_{10} + \frac{\gamma}{C_s^2} p_{10} \right) T_{10}^* \right] + \rho_0 \int_0^1 u_{m,21} dy. \tag{4.2.11}
\]

### 4.2.2 The unsteady flow terms

As \( \alpha > 1 \) the calculation of the perturbation variables \( T_1, u_1, p_1 \) in the previous chapter remains unaffected. Thus for \( p_1 \) we have
\[
p_{10} = \text{constant}, \tag{4.2.12a}
\]
\[
p_{11} = 3i \beta^2 p_{10} \int_0^x \int_0^\xi T_0(\xi) \left( \frac{\gamma}{\gamma - 1} - \frac{\rho_0(\xi)}{\rho_0(\xi) + \vartheta \rho_{s0}(\xi)} \right) d \xi d \zeta + A_1 x + A_2. \tag{4.2.12b}
\]
For \( u_1 \) we have
\[
u_{10}(x, y) = 0, \tag{4.2.13a}
\]
\[
u_{11}(x, y) = \frac{i}{24} \frac{d p_{10}}{d x} \rho_0(1 - y^2)(5 - y^2) - \frac{1}{2} \frac{d p_{11}}{d x}(1 - y^2). \tag{4.2.13b}
\]
Finally, for \( T_1 \) and \( T_{s1} \) we have
\[
T_{10}(x) = T_{s10}(x) = \frac{\beta T_0 p_{10}}{\rho_0 + \vartheta \rho_{s0}}, \tag{4.2.14a}
\]
\[
T_{11}(x, y) = i \frac{Pr}{2} \frac{\vartheta \rho_{s0} \beta T_0 p_{10}}{\rho_0 + \vartheta \rho_{s0}} (1 - y^2) + T_{b11}, \tag{4.2.14b}
\]
\[
T_{s11}(x, y') = -i \frac{Pr}{2} \frac{\beta T_0 p_{10}}{\rho_0 + \vartheta \rho_{s0}} (1 - y'^2) + T_{b11}. \tag{4.2.14c}
\]
4.2.3 Time-averaged energy flux $\dot{E}$

In the same way as we did for the stack we find an additional steady flow term $\dot{E}_{m,210}$, so that

$$\dot{E} = A^2 \omega \dot{E}_{m,210} + A^2 \omega \dot{E}_{210} + (\varepsilon^2 + \vartheta \varepsilon_1^2) \omega \dot{E}_{012} + \cdots,$$

(4.2.15)

where

$$\dot{E}_{m,210} = \rho_0 h_0 \int_0^1 u_{m,21} \, dy.$$  

(4.2.16)

Figure 7: (a) A standing wave refrigerator, insulated everywhere except at the heat exchangers. (b) Illustration of $\dot{W}_2$, $\dot{Q}_2$ and $\dot{E}_2$ in the refrigerator. The discontinuities in $\dot{E}_2$ are due to heat transfers at the heat exchangers.

5. Energy fluxes in thermoacoustic devices

The preceding sections show the derivation of expressions for the total energy flow $\dot{E}_2$ and the acoustic power $\dot{W}_2$ absorbed in a thermoacoustic refrigerator or heat pump or produced in a prime mover. As noted earlier in equation (3.1.58), the total energy is the sum of the acoustic power, the hydrodynamic
heat flux and the conduction heat flux. This section will give an idealized illustration of the different flows and their interaction in a thermoacoustic refrigerator.

Fig. 7(a) shows a standing wave refrigerator thermally insulated from the surroundings except at the heat exchangers left and right of the stack, where heat can be exchanged with the surroundings. The arrows show the different energy flows into or out of the system except for the conductive heat flow which is neglected for ease of discussion. A loudspeaker (driver) sustains a standing acoustic wave in the resonator by supplying acoustic power $\dot{W}_2$ in the form of a traveling acoustic wave. Equation (3.1.42) shows that part of this wave will be used to sustain the standing wave against the thermal and viscous dissipations, and the rest of the power will be used by the thermoacoustic effect to transport heat from the cold heat exchanger to the hot heat exchanger.

Fig. 7(b) illustrates the behavior of the different energy flows as a function of the position in the tube. A part of the acoustic power delivered by the speaker is dissipated at the resonator wall (first and second term in (3.1.42), to the left and right of the stack. The part dissipated at the right of the stack shows up as heat at the cold heat exchanger, decreasing the effective cooling power of the refrigerator. The acoustic power in the stack decreases monotonously, as it is used to transport heat from the cold heat exchanger to the hot heat exchanger, and to overcome viscous forces inside the stack. This power used shows up as heat at the hot heat exchanger. The power dissipated at the tube wall to the left of the stack also shows up as heat at the hot heat exchanger.

Inside the stack, the heat flux grows from $\dot{Q}_C$ at the right end of the stack to $\dot{Q}_H$ at the left end of the stack. The two discontinuities in the heat flux arise as a result of the heat exchangers at $T_C$ and $T_H$, supplying heat $\dot{Q}_C$ and removing heat $\dot{Q}_H$. The total energy flow is everywhere the sum of acoustic work and heat fluxes. Within the stack the total energy remains constant.

Conservation of energy holds everywhere. Applying the principle of energy conservation to the control volume shown in Fig. 7(a), in steady state over a cycle, the energy inside the control volume cannot change. Hence the rate $\dot{E}_2$ at which energy flows out is equal to the rate $\dot{Q}_C$ at which energy flows in. Similarly we also find that $\dot{W}_2$ is equal to $\dot{Q}_H - \dot{Q}_C$.

6. Conclusions

A linear theory of thermoacoustics has been developed based on a dimensional analysis and using small parameter asymptotics. The validity domain of this linear theory can be expressed in the dimensionless parameters. For the stack this is

$$\varepsilon^2 \ll \varepsilon^2 \ll A \ll \omega^2 \ll 1,$$

$$W_0 = O(1), \quad N_L = O(1), \quad N_s = O(1), \quad \vartheta = O(1),$$

and for the regenerator we have

$$\frac{\varepsilon^2_1}{\omega} \ll \frac{\varepsilon^2}{\omega} \ll A \ll \omega^2 \ll N_L^2 \ll 1,$$

$$W_0 \sim N_L, \quad N_s \sim N_L, \quad \vartheta = O(1).$$

Using this linear theory all the relevant variables could be determined. Similar results to those of Swift [21], [23] were obtained. The effect of the dimensionless parameters is most clear in the acoustic power. Viscous dissipation is reduced by taking the stack short ($\omega \ll 1$). On the other hand,
reducing the pore size \((N_L \ll 1)\) increases the viscous dissipation and reduces the thermal relaxation dissipation. However, the former can be prevented by taking the stack short enough, such that \(\omega \ll N_L \ll 1\).

The linearization was first performed assuming no steady mean velocity. Repeating the analysis with a (small) steady mean velocity showed that the mean velocity \(u_m\) had to be second order in the (dimensionless) amplitude \(A\). In fact it is always present and arises due to the first-order acoustics, i.e. it is caused by the physics.

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A  Nomenclature

Note that the tildes are used to indicate dimensional variables

A.1  General constants and variables

\( C \)  typical speed of sound
\( \tilde{c} \)  speed of sound
\( c_p \)  isobaric specific heat
\( c_v \)  isochoric specific heat
\( D \)  typical density
\( \tilde{E} \)  total energy
\( \tilde{h} \)  specific enthalpy
\( K \)  thermal conductivity
\( l \)  plate half-thickness
\( L \)  length of the stack plates
\( \tilde{p} \)  pressure
\( \tilde{Q} \)  heat flux
\( \tilde{s} \)  specific entropy
\( \tilde{t} \)  time
\( \tilde{T} \)  temperature
\( U \)  typical fluid speed
\( \tilde{u} \)  velocity component in direction of sound propagation
\( \tilde{v} \)  velocity component perpendicular to the direction of sound propagation
\( \tilde{W} \)  acoustic power
\( \tilde{x} \)  space coordinate along sound propagation
\( \tilde{y} \)  space coordinate perpendicular to sound propagation
\( R \)  plate half-separation
\( \tilde{\beta} \)  isobaric volumetric expansion coefficient
\( \delta \)  penetration depth
\( \tilde{\epsilon} \)  specific internal energy
\( \kappa \)  thermal diffusivity
\( \lambda \)  wave length
\( \mu \)  dynamic (shear) viscosity
\( \nu \)  kinematic viscosity
\( \xi \)  second (bulk) viscosity
\( \tilde{\rho} \)  density
\( \tilde{\Sigma} \)  viscous stress tensor
\( \tilde{\omega} \) angular frequency of the acoustic oscillations

### A.2 Dimensionless Numbers

\[ A = \frac{U}{C} \] amplitude of the acoustic oscillations (Mach number) \hspace{1cm} (A.2.1)

\[ N_L = \frac{R}{\delta_k} \] Laucret number for the fluid \hspace{1cm} (A.2.2)

\[ N_s = \frac{l}{\delta_s} \] Laucret number for the solid \hspace{1cm} (A.2.3)

\[ Pr = \frac{2N_L^2}{Wo^2} \] Prandtl number \hspace{1cm} (A.2.4)

\[ Wo = \sqrt{\frac{2}{\delta_v}} R \] Womersley Number \hspace{1cm} (A.2.5)

\[ \varepsilon = \frac{R}{L} \] aspect ratio of the space between two stack plates \hspace{1cm} (A.2.6)

\[ \varepsilon_1 = \frac{l}{L} \] aspect ratio of the stack plate \hspace{1cm} (A.2.7)

\[ \gamma = \frac{c_p}{c_v} \] ratio isobaric and isochoric specific heat \hspace{1cm} (A.2.8)

\[ \vartheta = \frac{RK_s}{lK} \] ratio of the heat fluxes leaving the gas and the plate \hspace{1cm} (A.2.9)

\[ \omega = \frac{\tilde{\omega}L}{C} \] rescaled frequency of the acoustic oscillations (Helmholtz number) \hspace{1cm} (A.2.10)

### A.3 Thermoacoustic auxiliary functions and variables

\[ f_k = \frac{\tanh(\alpha_k)}{\alpha_k} \] \hspace{1cm} (A.3.1)

\[ f_\nu = \frac{\tanh(\alpha_\nu)}{\alpha_\nu}, \] Rott’s function \hspace{1cm} (A.3.2)

\[ \alpha_\nu = (1 + i) \frac{R}{\delta_\nu} \] \hspace{1cm} (A.3.3)

\[ \alpha_k = (1 + i) \frac{R}{\delta_k} \] \hspace{1cm} (A.3.4)

\[ \alpha_s = (1 + i) \frac{l}{\delta_s} \] \hspace{1cm} (A.3.5)
\[ \varepsilon_s = \frac{\sqrt{K \rho_0 c_p \tanh(\alpha_k)}}{\sqrt{K_s \rho_0 c_s \tanh(\alpha_s)}} \text{ stack heat capacity ratio} \quad (A.3.6) \]

### A.4 Thermodynamic constants and relations

These relations were taken from [3].

\[ \tilde{c}^2 = \left( \frac{\partial \tilde{h}}{\partial \tilde{p}} \right)_{\tilde{s}}, \quad (A.4.1) \]

\[ c_p = \tilde{T} \left( \frac{\partial \tilde{s}}{\partial \tilde{T}} \right)_{\tilde{p}} = \left( \frac{\partial \tilde{h}}{\partial \tilde{T}} \right)_{\tilde{p}}, \quad (A.4.2) \]

\[ c_v = \tilde{T} \left( \frac{\partial \tilde{s}}{\partial \tilde{T}} \right)_{\tilde{\rho}} = \left( \frac{\partial \tilde{\varepsilon}}{\partial \tilde{T}} \right)_{\tilde{\rho}}, \quad (A.4.3) \]

\[ \beta = -\frac{1}{\tilde{\rho}} \left( \frac{\partial \tilde{\rho}}{\partial \tilde{T}} \right)_{\tilde{p}}, \quad (A.4.4) \]

\[ \gamma = \frac{c_p}{c_v}, \quad (A.4.5) \]

\[ \tilde{c}^2 \beta^2 \tilde{T} = c_p (\gamma - 1), \quad (A.4.6) \]

\[ \tilde{p} = \tilde{\rho} \tilde{h} - \tilde{\rho} \tilde{\varepsilon}, \quad (A.4.7) \]

\[ d\tilde{s} = \frac{c_p}{\tilde{T}} d\tilde{T} - \frac{\tilde{\beta}}{\tilde{\rho}} d\tilde{\rho}, \quad \Rightarrow \quad \tilde{s}_1 = \frac{c_p}{\tilde{T}_0} \tilde{T}_1 - \frac{\tilde{\beta}}{\tilde{\rho}_0} \tilde{p}_1, \quad (A.4.8) \]

\[ d\tilde{\rho} = \frac{\gamma}{\tilde{c}_s} d\tilde{\rho} - \tilde{\rho} \tilde{\beta} d\tilde{\rho}, \quad \Rightarrow \quad \tilde{\rho}_1 = \frac{\gamma}{\tilde{c}_s} \tilde{p}_1 - \tilde{\rho}_0 \tilde{T}_1, \quad (A.4.9) \]

\[ d\tilde{h} = \tilde{T} d\tilde{s} + \frac{1}{\tilde{\rho}} d\tilde{\rho}, \quad \Rightarrow \quad \tilde{h}_1 = \tilde{T}_0 \tilde{s}_1 + \frac{\tilde{p}_1}{\tilde{\rho}_0}, \quad (A.4.10) \]

\[ d\tilde{\varepsilon} = \tilde{T} d\tilde{s} + \frac{\tilde{\rho}}{\tilde{c}_s} d\tilde{\rho}, \quad \Rightarrow \quad \tilde{\varepsilon}_1 = \tilde{T}_0 \tilde{s}_1 - \frac{\tilde{p}_0}{\tilde{c}_s} \tilde{\rho}_1, \quad (A.4.11) \]

For an ideal gas some of these equations simplify. An ideal gas is a gas for which \( \tilde{p} / (\tilde{\rho} \tilde{T}) = \text{constant} \). The value of this constant is denoted by \( R \).

\[ \tilde{p} = \tilde{\rho} R \tilde{T}, \quad (A.4.12) \]

\[ R = c_p - c_v, \quad (A.4.13) \]

\[ \tilde{\beta} \tilde{T} = 1, \quad (A.4.14) \]
Some ideal gases have the special property that their specific heats $c_p$ and $c_v$ are constant. These gases are called perfect gases\(^4\).

### A.5 Fundamental equations

These equations were taken from [6].

\[
\frac{\partial \tilde{\rho}}{\partial \tilde{t}} + \nabla \cdot (\tilde{\rho} \tilde{v}) = 0, 
\]

\[
\tilde{\rho} \left[ \frac{\partial \tilde{s}}{\partial \tilde{t}} + (\tilde{v} \cdot \nabla) \tilde{v} \right] = -\tilde{\nabla} \tilde{p} + \mu \tilde{v}^2 \tilde{v} + \left( \xi + \frac{\mu}{3} \right) \tilde{\nabla} (\tilde{\nabla} \cdot \tilde{v}), 
\]

\[
\tilde{\rho} \tilde{T} \left( \frac{\partial \tilde{s}}{\partial \tilde{t}} + \tilde{v} \cdot \tilde{\nabla} \tilde{s} \right) = \tilde{\nabla} \cdot (K \tilde{\nabla} \tilde{T}) + \tilde{\Sigma} : \tilde{\nabla} \tilde{v}, 
\]

\[
\tilde{\rho} \frac{s}{\partial \tilde{t}} = \kappa_s \tilde{\nabla}^2 \tilde{T}_s, 
\]

Using the thermodynamic relations in (A.4.8) - (A.4.11) we can also add the following equations

\[
\tilde{\rho} c_p \left( \frac{\partial \tilde{T}}{\partial \tilde{t}} + \tilde{v} \cdot \tilde{\nabla} \tilde{T} \right) - \beta \tilde{T} \left( \frac{\partial \tilde{p}}{\partial \tilde{t}} + \tilde{v} \cdot \tilde{\nabla} \tilde{p} \right) = K \tilde{\nabla} \cdot (\tilde{\nabla} \tilde{T}) + \tilde{\Sigma} : \tilde{\nabla} \tilde{v}, 
\]

\[
\frac{\partial}{\partial \tilde{t}} \left( \tilde{\rho} \tilde{v}^2 + \tilde{\rho} \tilde{\varepsilon} \right) = -\tilde{\nabla} \cdot \left[ \tilde{\nu} \left( \frac{1}{2} \tilde{\rho} \tilde{v}^2 + \tilde{\rho} \tilde{\varepsilon} \right) - K \tilde{\nabla} \tilde{T} - \tilde{\nu} \cdot \tilde{\Sigma} \right],
\]

### A.6 Sub- and superscripts

- $\sim$ dimensional
- $C$ cold
- $H$ hot
- $k$ thermal
- $m$ mean
- $p$ isobaric
- $r$ real-valued
- $s$ solid, sink, source
- $v$ isochoric
- $\delta$ boundary layer
- $\nu$ viscous

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\(^4\)Sometimes in literature different definitions are used for ideal and perfect gases. Chapman [3] for example makes no distinction between ideal and perfect gases.
References


