On a partial symmetry of Faraday’s equations

by

J. de Graaf
On a Partial Symmetry of Faraday’s Equations

J. de GRAAF 15-6-2007

Faraday’s equations \(^1\) read

\[
\begin{align*}
\partial_t B + \nabla \times E & = 0, & \nabla \cdot E & = 0, \\
\nabla \times H & = j, & \nabla \cdot B & = 0.
\end{align*}
\] (0.1)

Here \(B = \mu H\). Further, as a consequence, \(\nabla \cdot j = 0\). Note that, if we assume \(\mu\) piecewise constant, Faraday’s equations are, piecewise, two successive Poisson equations: Put \(B = \nabla \times A\), with the gauge condition \(\nabla \cdot A = 0\). Then \(\Delta A = -j\). After having recovered \(B\) we can, in principle, find \(E\) from the equation \(\nabla \times E = -\partial_t B\). During this procedure the time \(t\) is just a parameter.

In Cartesian components Faraday’s equations read:

\[
\begin{align*}
\partial_t \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} + \begin{bmatrix} \partial_y E_z - \partial_z E_y \\ \partial_z E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{bmatrix} & = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, & \begin{bmatrix} \partial_y H_z - \partial_z H_y \\ \partial_z H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{bmatrix} & = \begin{bmatrix} j_x \\ j_y \\ j_z \end{bmatrix}, \\
\partial_x E_x + \partial_y E_y + \partial_z E_z & = 0, & \partial_x B_x + \partial_y B_y + \partial_z B_z & = 0. \quad (0.2)
\end{align*}
\]

We now assume a layered structure. Our cartesian coordinates are taken such that there is translation invariance in the \(y\)-direction. The permeability depends only on \(x\) and \(z\). So \(\mu_0 = \mu(x, z)\). Typically, one could think of a flat air gap perpendicular to the \(x\)-axis, which lies in a ferro material. If we want to consider solutions which only depend on \(x\) and \(z\), we assume \(j\) to depend on \(x\) and \(z\) only. Now Faraday’s equations reduce to

\[
\begin{align*}
\partial_t \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} + \begin{bmatrix} -\partial_z E_y \\ \partial_z E_x - \partial_x E_z \\ \partial_x E_y \end{bmatrix} & = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, & \begin{bmatrix} -\partial_z H_y \\ \partial_z H_x - \partial_x H_z \\ \partial_x H_y \end{bmatrix} & = \begin{bmatrix} j_x \\ j_y \\ j_z \end{bmatrix}, \\
\partial_x E_x + \partial_z E_z & = 0, & \partial_x B_x + \partial_z B_z & = 0. \quad (0.3)
\end{align*}
\]

As a consequence \(\partial_y j_y = 0\) and \(\partial_x j_x + \partial_z j_z = 0\). In many practical cases we will have \(j_x = j_z = 0\). Introduction of a ‘stream function’ \(x, z \mapsto \Psi(x, z)\) with \(H_x = \partial_z \Psi\), \(H_z = -\partial_x \Psi\) leads to the 2-dimensional Poisson equation

\[
(\partial_x^2 + \partial_z^2)\Psi = -j_y. \quad (0.4)
\]

\(^1\)Non-historically speaking: These are Maxwell’s equations with charge density 0 and the replacement current term ommitted
Next, a simple integration leads to $H_y$. Finally $E$ can be solved from the 1st set of equations with the same technique.

In cylindrical coordinates Faraday’s equations read

$\partial_t \begin{bmatrix} B_r \\ B_\theta \\ B_z \end{bmatrix} + \begin{bmatrix} \frac{1}{r} \partial_\theta E_z - \partial_z E_\theta \\ \frac{1}{r} \partial_\theta E_r - \partial_r E_\theta \\ \partial_r E_\theta + \frac{1}{r} E_\theta - \frac{1}{r} \partial_\theta E_r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{r} \partial_\theta H_z - \partial_z H_\theta \\ \partial_r H_r - \partial_\theta H_z \\ \partial_r H_\theta + \frac{1}{r} H_\theta - \frac{1}{r} \partial_\theta H_r \end{bmatrix} = \begin{bmatrix} j_r \\ j_\theta \\ j_z \end{bmatrix},$

$\partial_r E_r + \frac{1}{r} E_r + \frac{1}{r} \partial_\theta E_\theta + \partial_z E_z = 0, \quad \partial_r B_r + \frac{1}{r} B_r + \frac{1}{r} \partial_\theta B_\theta + \partial_z B_z = 0. \quad (0.5)$

We now assume a rotationally layered structure. Our cylindrical coordinates are taken such that there is rotational invariance in the $\theta$-direction. The permeability depends only on $r$ and $z$. So $\mu_0 = \mu(r, z)$. Typically, one could think of a cylindrical air gap, where the central axis coincides with the $z$-axis. The gap is surrounded by ferro material. If we want to consider solutions which only depend on $r$ and $z$, we assume $j$ to depend on $r$ and $z$ only. Now Faraday’s equations reduce to

$\partial_t \begin{bmatrix} rB_r \\ rB_\theta \\ rB_z \end{bmatrix} + \begin{bmatrix} -\partial_z (rE_\theta) \\ r(\partial_\theta E_r - \partial_r E_\theta) \\ \partial_r (rE_\theta) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -\partial_z (rH_\theta) \\ \partial_z H_r - \partial_\theta H_z \\ \partial_r (rH_\theta) \end{bmatrix} = \begin{bmatrix} rj_r \\ j_\theta \\ rj_z \end{bmatrix},$

$\partial_r (rE_r) + \partial_z (rE_z) = 0, \quad \partial_r (rB_r) + \partial_z (rB_z) = 0. \quad (0.6)$

As a consequence $\partial_\theta j_\theta = 0$ and $\partial_r (rj_r) + \partial_z (rj_z) = 0$. In many practical cases we will have $j_r = j_z = 0$. Introduction of a ‘stream function’ $r, z \mapsto \Psi(r, z)$ with $H_z = \partial_\theta \Psi$, $H_r = -\partial_z \Psi$ leads to the 2-dimensional Poisson equation

$$(\partial_r \partial_r + \partial_z \partial_z) \Psi = -j_\theta. \quad (0.7)$$

This equation is exactly the same as (0.4)! Mathematically.

Next, a simple integration leads to $r H_\theta$. Finally $E$ can be solved from the 1st set of equations with the same technique.

This note has been inspired by Elena Lomonova and her students.
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