Global compactness for a class of quasi-linear elliptic problems

by

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GLOBAL COMPACTNESS FOR A CLASS OF QUASI-LINEAR ELLIPTIC PROBLEMS

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Abstract. We prove a global compactness result for Palais-Smale sequences associated with a class of quasi-linear elliptic equations on exterior domains.

1. Introduction and main result

Let \( \Omega \) be a smooth domain of \( \mathbb{R}^N \) with a bounded complement and \( N > p > m > 1 \). The main goal of this paper is to obtain a global compactness result for the Palais-Smale sequences of the energy functional associated with the following quasi-linear elliptic equation

\[
- \text{div}(L_{\xi}(Du)) - \text{div}(M_{\xi}(u, Du)) + M_s(u, Du) + V(x)|u|^{p-2}u = g(u) \quad \text{in } \Omega,
\]

where \( u \in W_0^{1,p}(\Omega) \cap D^{1,m}_0(\Omega) \), meant as the completion of the space \( D(\Omega) \) of smooth functions with compact support, with respect to the norm \( \|u\|_{W^{1,p}(\Omega) \cap D^{1,m}(\Omega)} = \|u\|_p + \|u\|_m \), having set \( \|u\|_p := \|u\|_{W^{1,p}(\Omega)} \) and \( \|u\|_m := \|Du\|_{L^m(\Omega)} \). We assume that \( V \) is a continuous function on \( \Omega \),

\[
\lim_{|x| \to \infty} V(x) = V_\infty \quad \text{and} \quad \inf_{x \in \Omega} V(x) = V_0 > 0.
\]

As known, lack of compactness may occur due to the lack of compact embeddings for Sobolev spaces on \( \Omega \) and since the limiting equation on \( \mathbb{R}^N \)

\[
- \text{div}(L_{\xi}(Du)) - \text{div}(M_{\xi}(u, Du)) + M_s(u, Du) + V_\infty|u|^{p-2}u = g(u) \quad \text{in } \mathbb{R}^N,
\]

with \( u \in W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N) \), is invariant by translations. A particular case of (1.1) is

\[
- \Delta_p u - \text{div}(a(u)|Du|^{m-2}Du) + \frac{1}{m}a'(u)|Du|^m + V(x)|u|^{p-2}u = |u|^{\sigma-2}u \quad \text{in } \Omega,
\]

where \( \Delta_p u := \text{div}(|Du|^{p-2}Du) \), for a suitable function \( a \in C^1(\mathbb{R}; \mathbb{R}^+) \), or the even simpler case where \( a \) is constant, namely

\[
- \Delta_p u - \Delta_m u + V(x)|u|^{p-2}u = |u|^{\sigma-2}u \quad \text{in } \Omega.
\]

Since the pioneering work of Benci and Cerami [2] dealing with the case \( L(\xi) = |\xi|^2/2 \) and \( M(s, \xi) \equiv 0 \), many papers have been written on this subject, see for instance the bibliography of [12]. Quite recently, in [12], the case \( L(\xi) = |\xi|^p/p \) and \( M(s, \xi) \equiv 0 \) was investigated. The main point in the present contribution is the fact that we allow, under suitable assumptions, a quasi-linear term \( M(u, Du) \) depending on the unknown \( u \) itself. The typical tools exploited in [2, 12], in addition to the point-wise convergence of the gradients, are some decomposition (splitting) results both for the energy functional and for the equation, along a given bounded Palais-Smale sequence \((u_n)\). To this regard, the explicit dependence on \( u \) in the term \( M(u, Du) \) requires a rather careful analysis. In particular, we can handle it for

\[
\nu|\xi|^m \leq M(s, \xi) \leq C|\xi|^m, \quad p - 1 \leq m < p - 1 + p/N.
\]

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The restriction on $m$, together with the sign condition (1.9) provides, thanks to the presence of $L$, the needed a priori regularity on the weak limit of $(u_n)$, see Theorems 3.2 and 3.4. Besides the aforementioned motivations, which are of mathematical interest, it is worth pointing out that in recent years, some works have been devoted to quasi-linear operators with double homogeneity, which arise from several problems of Mathematical Physics. For instance, the reaction diffusion problem $u_t = -\text{div}(\mathbb{D}(u)Du) + \ell(x,u)$, where $\mathbb{D}(u) = d_p|Du|^{p-2} + d_m|Du|^{m-2}$, $d_p > 0$ and $d_m > 0$, admitting a rather wide range of applications in biophysics [10], plasma physics [16] and in the study of chemical reactions [1]. In this framework, $u$ typically describes a concentration and $\text{div}(\mathbb{D}(u)Du)$ corresponds to the diffusion with a coefficient $\mathbb{D}(u)$, whereas $\ell(x,u)$ plays the rôle of reaction and relates to source and loss processes. We refer the interested reader to [5] and to the reference therein. Furthermore, a model for elementary particles proposed by Derrick [9] yields to the study of standing wave solutions $\psi(x,t) = u(x)e^{i\omega t}$ of the following nonlinear Schrödinger equation

$$i\psi_t + \Delta_2\psi - b(x)\psi + \Delta_p\psi - V(x)|\psi|^{p-2}\psi + |\psi|^{\sigma-2}\psi = 0 \quad \text{in} \ \mathbb{R}^N,$$

for which we refer the reader e.g. to [3].

In order to state the first main result, assume $N > p > m \geq 2$ and

$$p - 1 \leq m < p - 1 + p/N, \quad p < \sigma < p^*,$$

and consider the $C^2$ functions $L : \mathbb{R}^N \to \mathbb{R}$ and $M : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ such that both the functions $\xi \mapsto L(\xi)$ and $\xi \mapsto M(s,\xi)$ are strictly convex and

$$\nu|\xi|^p \leq |L(\xi)| \leq C|\xi|^p, \quad |L_\xi(\xi)| \leq C|\xi|^{p-1}, \quad |L_{\xi\xi}(\xi)| \leq C|\xi|^{p-2},$$

for all $\xi \in \mathbb{R}^N$. Furthermore, we assume

$$\nu|\xi|^m \leq M(s,\xi) \leq C|\xi|^m, \quad |M_s(s,\xi)| \leq C|\xi|^m, \quad |M_\xi(s,\xi)| \leq C|\xi|^{m-1}, \quad |M_{\xi\xi}(s,\xi)| \leq C|\xi|^{m-1}, \quad |M_{\xi\xi}(s,\xi)| \leq C|\xi|^{m-2},$$

for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and that the sign condition (cf. [14])

$$M_s(s,\xi)s \geq 0,$$

holds for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$. Also, $G : \mathbb{R} \to \mathbb{R}$ is a $C^2$ function with $G'(s) := g(s)$ and

$$|G'(s)| \leq C|s|^\sigma - 1, \quad |G''(s)| \leq C|s|^\sigma - 2,$$

for all $s \in \mathbb{R}$. We define

$$j(s,\xi) := L(\xi) + M(s,\xi) - G(s),$$

and on $W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$ with $\|u\|_{W_0^{1,p}(\Omega) \cap D_1^{1,m}(\Omega)} = \|u\|_p + \|u\|_m$ the functional

$$\phi(u) := \int_\Omega j(u, Du) + \int_\Omega V(x)|u|^p/p.$$

Finally, on $W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)$ with $\|u\|_{W^{1,p}(\mathbb{R}^N) \cap D_1^{1,m}(\mathbb{R}^N)} = \|u\|_p + \|u\|_m$ we define

$$\phi_\infty(u) := \int_{\mathbb{R}^N} j(u, Du) + \int_{\mathbb{R}^N} V_\infty|u|^p/p.$$

See Section 2 for some properties of the functionals $\phi$ and $\phi_\infty$. The first main global compactness type result is the following
Theorem 1.1. Assume that (1.5)-(1.11) hold and let \((u_n) \subset W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)\) be a bounded sequence such that

\[
\phi(u_n) \to c \quad \phi'(u_n) \to 0 \quad \text{in} \quad (W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega))^*.
\]

Then, up to a subsequence, there exists a weak solution \(v_0 \in W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)\) of

\[-\div(L_\xi(Du)) - \div(M_\xi(u,Du)) + M_s(u,Du) + V(x)|u|^{p-2}u = g(u) \quad \text{in} \ \Omega,\]

a finite sequence \(\{v_1, ..., v_k\} \subset W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)\) of weak solutions of

\[-\div(L_\xi(Du)) - \div(M_\xi(u,Du)) + M_s(u,Du) + V_\infty|u|^{p-2}u = g(u) \quad \text{in} \ \mathbb{R}^N\]

and \(k\) sequences \((y^k_n) \subset \mathbb{R}^N\) satisfying

\[
|y^k_n| \to \infty, \quad |y^k_i - y^k_j| \to \infty, \quad i \neq j, \quad \text{as} \ n \to \infty,
\]

\[
\|u_n - v_0 - \sum_{i=1}^k v_i(-y^i_n)|\|_{W^{1,p}(\Omega) \cap D^{1,m}(\mathbb{R}^N)} \to 0, \quad \text{as} \ n \to \infty,
\]

\[
\|u_n\|_p^p \to \sum_{i=0}^k \|v_i\|_p^p, \quad \|u_n\|_m^m \to \sum_{i=0}^k \|v_i\|_m^m, \quad \text{as} \ n \to \infty,
\]

as well as

\[
\phi(v_0) + \sum_{i=1}^k \phi_\infty(v_i) = c.
\]

Let us now come to a statement for the cases \(1 < m \leq 2\) or \(1 < p \leq 2\). Let us define

\[
\mathcal{L}(\xi, h) := \frac{|L_\xi(\xi + h) - L_\xi(\xi)|}{|h|^{p-1}}, \quad \text{if} \ 1 < p < 2,
\]

\[
\mathcal{G}(s, t) := \frac{|G'(s + t) - G'(s)|}{|t|^{\sigma-1}}, \quad \text{if} \ 1 < \sigma < 2,
\]

\[
\mathcal{M}(s, \xi, h) := \frac{|M_\xi(s, \xi + h) - M_\xi(s, \xi)|}{|h|^{m-1}}, \quad \text{if} \ 1 < m < 2.
\]

If either \(p < 2, \sigma < 2\) or \(m < 2\), we shall weaken the twice differentiability assumptions, by requiring \(L_\xi \in C^1(\mathbb{R}^N \setminus \{0\}), \ G' \in C^1(\mathbb{R} \setminus \{0\}), \ M_\xi \in C^1(\mathbb{R} \times (\mathbb{R}^N \setminus \{0\})), \ M_\xi \in C^0(\mathbb{R} \times \mathbb{R}^N)\) and \(M_{ss} \in C^0(\mathbb{R} \times \mathbb{R}^N)\). Moreover we assume the same growth conditions for \(L, M, G\) and their derivatives, replacing only the growth assumptions for \(L_\xi, M_\xi, G''\) by the following hypotheses:

\[
\sup_{h \neq 0, \xi \in \mathbb{R}^N} \mathcal{L}(\xi, h) < \infty, \quad \sup_{t \neq 0, s \in \mathbb{R}} \mathcal{G}(s, t) < \infty, \quad \sup_{h \neq 0, (s, \xi) \in \mathbb{R} \times \mathbb{R}^N} \mathcal{M}(s, \xi, h) < \infty.
\]

Conditions (1.12)-(1.13), in some more concrete situations, follow immediately by homogeneity of \(L_\xi\) and \(G'\) (see, for instance, [12, Lemma 3.1]). Similarly, (1.14) is satisfied for instance when \(M_\xi\) is of the form \(M(s, \xi) = a(s)\mu(\xi)\), being \(a : \mathbb{R} \to \mathbb{R}^+\) a bounded function and \(\mu : \mathbb{R}^N \to \mathbb{R}^+\) a \(C^1\) strictly convex function such that \(\mu_\xi\) is homogeneous of degree \(m - 1\).

Theorem 1.2. Under the additional assumptions (1.12)-(1.14) in the sub-quadratic cases, the assertion of Theorem 1.1 holds true.

As a consequence of the above results we have the following compactness criterion.
Corollary 1.3. Assume (2.1) below for some $\delta > 0$ and $\mu > p$. Under the hypotheses of Theorem 1.1 or 1.2, if $c < c^*$, then $(u_n)$ is relatively compact in $W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)$ where
\[
\begin{align*}
c^* := \min \left\{ \frac{\delta}{\mu}, \frac{\mu - p}{\mu p} V_\infty \right\} \left[ \frac{\min\{\nu, V_\infty\}}{C_g S_{p,\sigma}} \right]^\frac{\mu}{\mu - p},
\end{align*}
\] and $S_{p,\sigma}$ and $C_g$ are constants such that $S_{p,\sigma} \|u\|_p^\sigma \geq \|u\|^\sigma_{L^\sigma(\mathbb{R}^N)}$ and $|g(s)| \leq C_g |s|^{\sigma - 1}$.

Remark 1.4. It would be interesting to get a global compactness result in the case $L = 0$ and $p = m$, namely for the model case
\[
(1.15) \quad -\text{div}(a(u)|Du|^{m-2}Du) + \frac{1}{m} a'(u)|Du|^m + V(x)|u|^{m-2}u = |u|^{\sigma - 2}u \quad \text{in } \Omega.
\] Notice that, even assuming $a'$ bounded, $a'(u)|Du|^m$ is merely in $L^1(\Omega)$ for $W^{1,m}_0(\Omega)$ distributional solutions. In general, in this setting, the splitting properties of the equation are hard to formulate in a reasonable fashion.

Remark 1.5. The restriction of between $m$ and $p$ in assumption (1.5) is no longer needed in the case where $M$ is independent of the first variable $s$, namely $M_s \equiv 0$.

Remark 1.6. We prove the above theorems under the a-priori boundedness assumption of $(u_n)$. This occurs in a quite large class of problems, as Proposition 2.2 shows.

Remark 1.7. With no additional effort, we could cover the case where an additional term $W(x)|u|^{m-2}u$ appears in (1.1) and the functional framework turns into $W^{1,p}_0(\Omega) \cap W^{1,m}_0(\Omega)$.

In the spirit of [11], we also get the following

Corollary 1.8. Let $N > p \geq m > 1$ and assume that $\xi \mapsto L(\xi)$ is $p$-homogeneous, $\xi \mapsto M(\xi)$ is $m$-homogeneous, $L(\xi) \geq p|\xi|^p$, $M(\xi) \geq m|\xi|^m$ and set
\[
(1.16) \quad S_\Omega := \inf_{\|u\|_{L^p(\Omega)} = 1} \int_\Omega \frac{L(Du)}{p} + \frac{M(Du)}{m} + \frac{V(x)}{p} |u|^p,
\]
\[
S_{\mathbb{R}^N} := \inf_{\|u\|_{L^p(\mathbb{R}^N)} = 1} \int_{\mathbb{R}^N} \frac{|Du|^p}{p} + \frac{|u|^p}{p},
\]
with $V(x) \to 1$ as $|x| \to \infty$. Assume furthermore that
\[
(1.17) \quad S_\Omega < \left( \frac{\sigma - p}{\sigma - m} \right)^{\frac{\sigma - p}{\sigma}} S_{\mathbb{R}^N}.
\]
Then (1.16) admits a minimizer.

Remark 1.9. We point out that, some conditions guaranteeing the nonexistence of nontrivial solutions in the star-shaped case $\Omega = \mathbb{R}^N$ can be provided. For the sake of simplicity, assume that $L$ is $p$-homogeneous and that $\xi \mapsto M(s, \xi)$ is $m$-homogeneous. Then, in view of [13, Theorem 3], that holds for $C^1$ solutions by virtue of the results of [8], we have that (1.1) admits no nontrivial $C^1$ solution well behaved at infinity, namely satisfying condition (19) of [13], provided that there exists a number $a \in \mathbb{R}^+$ such that a.e. in $\mathbb{R}^N$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$
\[
\begin{align*}
(N - p(a+1))L(\xi) + (N - m(a+1))M(s, \xi) + (asg(s) - NG(s)) + \frac{(N - ap)V(x) + x \cdot DV(x)}{p} |s|^p - aM_s(s, \xi)s \geq 0,
\end{align*}
\] holding, for instance, if there exists $0 \leq a \leq \frac{N - p}{p}$ such that $asg(s) - NG(s) \geq 0$, $(N - ap)V(x) + x \cdot DV(x) \geq 0$, $M_s(s, \xi)s \leq 0$, $M_s(s, \xi)s \leq 0$,
for a.e. \( x \in \mathbb{R}^N \) and for all \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^N \). Also, in the more particular case where \( g(s) = |s|^{\sigma-2}s \) and \( V(x) = V_\infty > 0 \), then the above conditions simply rephrase into

\[
\sigma \geq p^*, \quad M_s(s, \xi)s \leq 0,
\]

for every \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^N \). In fact, in (1.9), we consider the opposite assumption on \( M_s \).

2. Some preliminary facts

It is a standard fact that, under condition (1.6) and (1.10), the functionals

\[
u \mapsto \int_{\Omega} L(Du), \quad \nu \mapsto \int_{\Omega} V(x)|\nu|^p, \quad \nu \mapsto \int_{\Omega} G(\nu)
\]

are \( C^1 \) on \( W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega) \). Analogously, although \( M \) depends explicitly on \( s \), the functional

\[
\mathcal{M} : W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega) \to \mathbb{R}, \quad \mathcal{M}(u) = \int_{\Omega} M(u, Du),
\]

admits, thanks to condition (1.5), directional derivatives along any \( v \in W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega) \) and

\[
\mathcal{M}'(u)(v) = \int_{\Omega} M_s(u, Du) \cdot Dv + \int_{\Omega} M_s(u, Du)v,
\]

as it can be easily verified observing that \( p = \frac{p}{p-m} \leq p^* \) and that, for \( u \in W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega) \), by Young’s inequality, for some constant \( C \) it holds

\[
|M_s(u, Du) \cdot Dv| \leq C|Du|^m + C|Dv|^m \in L^1(\Omega),
\]

\[
|M_s(u, Du)v| \leq C|Du|^p + C|v|^{\frac{p}{p-m}} \in L^1(\Omega).
\]

Furthermore, if \( u_k \to u \) in \( W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega) \) as \( k \to \infty \) then \( \mathcal{M}'(u_k) \to \mathcal{M}'(u) \) in the dual space \( (W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega))^* \), as \( k \to \infty \). Indeed, for \( \|v\|_{W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)} \leq 1 \), we have

\[
|\mathcal{M}'(u_k)(v) - \mathcal{M}'(u)(v)|
\]

\[
\leq \int_{\Omega} |M_s(u_k, Du_k) - M_s(u, Du)| |Dv| + \int_{\Omega} |M_s(u_k, Du_k) - M_s(u, Du)| |v|
\]

\[
\leq \|M_s(u_k, Du_k) - M_s(u, Du)\|_{L^{p'}} \|Dv\|_{L^p} + \|M_s(u_k, Du_k) - M_s(u, Du)\|_{L^{p/(p-m)}} \|v\|_{L^{p/(p-m)}}
\]

\[
\leq \|M_s(u_k, Du_k) - M_s(u, Du)\|_{L^{p'}} + \|M_s(u_k, Du_k) - M_s(u, Du)\|_{L^{p/(p-m)}}.
\]

This yields the desired convergence, using (1.7) and the Dominated Convergence Theorem. Notice that the same argument carried out before applies either to integrals defined on \( \Omega \) or on \( \mathbb{R}^N \). Hence the following proposition is proved.

**Proposition 2.1.** In the hypotheses of Theorems 1.1 and 1.2, the functionals \( \phi \) and \( \phi_\infty \) are \( C^1 \).

In addition to the assumptions on \( L, M \) and \( g,G \) set in the introduction, assume now that there exist positive numbers \( \delta > 0 \) and \( \mu > p \) such that

\[
(2.1) \quad \mu M(s, \xi) - M_s(s, \xi)s - M_s(s, \xi) \cdot \xi \geq \delta|\xi|^m, \quad \mu L(\xi) - L_s(\xi) \cdot \xi \geq \delta|\xi|^p, \quad sg(s) - \mu G(s) \geq 0,
\]

for any \( s \in \mathbb{R} \) and all \( \xi \in \mathbb{R}^N \). This hypothesis is rather well established in the framework of quasi-linear problems (cf. [14]) and it allows an arbitrary Palais-Smale sequence \( (u_n) \) to be bounded in \( W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega) \), as shown in the following.
Proposition 2.2. Let \( j \) be as in (1.11) and assume that (1.5) holds. Let \((u_n) \subset W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)\) be a sequence such that

\[
\phi(u_n) \to c \quad \phi'(u_n) \to 0 \quad \text{in } (W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega))^*
\]

Then, if condition (2.1) holds, \((u_n)\) is bounded in \(W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)\).

Proof. Let \((w_n) \subset (W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega))^*\) with \(w_n \to 0\) as \(n \to \infty\) be such that \(\phi'(u_n)(v) = \langle w_n, v \rangle\), for every \(v \in W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)\). Whence, by choosing \(v = u_n\), it follows

\[
\int_{\Omega} j_c(u_n, Du_n) \cdot Du_n + \int_{\Omega} j_s(u_n, Du_n) u_n + \int_{\Omega} V(x)|u_n|^p = \langle w_n, u_n \rangle.
\]

Combining this equation with \(\mu \phi(u_n) = \mu c + o(1)\) as \(n \to \infty\), namely

\[
\int_{\Omega} \mu j(u_n, Du_n) + \frac{\mu}{p} \int_{\Omega} V(x)|u_n|^p = \mu c + o(1),
\]

recalling the definition of \(j\), and using condition (2.1), yields

\[
\frac{\mu - p}{p} \int_{\Omega} V(x)|u_n|^p + \delta \int_{\Omega} |Du_n|^p + \delta \int_{\Omega} |Du_n|^m \leq \mu c + \|w_n\| W^{1,p}(\Omega) \cap D^{1,m}(\Omega) + o(1),
\]

as \(n \to \infty\), implying, due to \(V \geq V_0\) that

\[
\|u_n\| W^{1,p}(\Omega) + \|u_n\|^m D^{1,m}(\Omega) \leq C + C\|u_n\| W^{1,p}(\Omega) + C\|u_n\| D^{1,m}(\Omega) + o(1),
\]

as \(n \to \infty\). The assertion then follows immediately. \(\square\)

From now on we shall always assume to handle bounded Palais-Smale sequences, keeping in mind that condition (2.1) can guarantee the boundedness of such sequences.

Proposition 2.3. Let \( j \) be as in (1.11) and assume that \(1 < m < p < N\) and \(p < \sigma < p^*\). Let \((u_n) \subset W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)\) bounded be such that

\[
\phi(u_n) \to c \quad \phi'(u_n) \to 0 \quad \text{in } (W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega))^*.
\]

Then, up to a subsequence, \((u_n)\) converges weakly to some \(u\) in \(W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)\), \(u_n(x) \to u(x)\) and \(Du_n(x) \to Du(x)\) for a.e. \(x \in \Omega\).

Proof. It is sufficient to justify that \(Du_n(x) \to Du(x)\) for a.e. \(x \in \Omega\). Given an arbitrary bounded subdomain \(\omega \subset \subset \Omega\) of \(\Omega\), from the fact that \(\phi'(u_n) \to 0\) in \((W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega))^*\), we can write

\[
\int_{\omega} a(u_n, Du_n) \cdot Du = \langle w_n, v \rangle + \langle f_n, v \rangle + \int_{\omega} v \, d\mu_n, \quad \text{for all } v \in D(\omega),
\]

where \((w_n) \subset (W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega))^*\) is vanishing, and hence in particular \(w_n \in W^{-1,p'}(\omega)\), with \(w_n \to 0\) in \(W^{-1,p'}(\omega)\) as \(n \to \infty\) and we have set

\[
a_n(x, s, \xi) := L_\xi(\xi) + M_\xi(s, \xi), \quad \text{for all } (s, \xi) \in \mathbb{R} \times \mathbb{R}^N,
\]

\[
f_n := -V(x)|u_n|^{p-2}u_n + g(u_n) \in W^{-1,p'}(\omega), \quad n \in \mathbb{N},
\]

\[
\mu_n := -M_\xi(u_n, Du_n) \in L^1(\omega), \quad n \in \mathbb{N}.
\]

Due to the strict convexity assumptions on the maps \(\xi \mapsto L(\xi)\) and \(\xi \mapsto M(s, \xi)\) and the growth conditions on \(L_\xi, M_\xi, M_s\) and \(g\), all the assumptions of [6, Theorem 1] are fulfilled. Precisely,

\[
|a_n(x, s, \xi)| \leq |L_\xi(\xi)| + |M_\xi(s, \xi)| \leq C|\xi|^{p-1} + C|\xi|^{m-1} \leq C + C|\xi|^{p-1},
\]
for a.e. \( x \in \omega \) and all \((s, \xi) \in \mathbb{R} \times \mathbb{R}^N\), and
\[
\begin{align*}
f_n & \to f, \quad f := -V(x)|u|^{p-2}u + g(u), \quad \text{strongly in } W^{-1,p'}(\omega), \\
\mu_n & \rightharpoonup \mu, \quad \text{weakly* in } M(\omega), \quad \text{since } \sup_{n \in \mathbb{N}} \|M_s(u_n, Du_n)\|_{L^1(\omega)} < +\infty.
\end{align*}
\]

Then, it follows that \( Du_n(x) \to Du(x) \) for a.e. \( x \in \omega \). Finally, a simple Cantor diagonal argument allows to recover the convergence over the whole domain \( \Omega \).

Next we prove a regularity result for the solutions of equation (1.1).

**Proposition 2.4.** Let \( j \) be as in (1.11) and assume (1.5) and (1.9). Let \( u \in W^{1,p}_0(\Omega) \cap D_0^{1,m}(\Omega) \) be a solution of (1.1). Then
\[
L^q(\Omega), \quad u \in L^\infty(\Omega) \quad \text{and} \quad \lim_{|x| \to \infty} u(x) = 0.
\]

**Proof.** Let \( k, i \in \mathbb{N} \). Then, setting \( v_{k,i}(x) := (u_k(x))^i \) with \( u_k(x) := \min\{u^+(x), k\} \), it follows that \( v_{k,i} \in W^{1,p}_0(\Omega) \cap D_0^{1,m}(\Omega) \) can be used as a test function in (1.1), yielding
\[
\begin{align*}
\int_\Omega L_\xi(Du) \cdot Du_{k,i} + \int_\Omega M_\xi(u, Du) \cdot Du_{k,i} & \quad + \int_\Omega M_s(u, Du)v_{k,i} + \int_\Omega V(x)|u|^{p-2}uv_{k,i} = \int_\Omega g(u)v_{k,i}.
\end{align*}
\]

Taking into account that \( Du_{k,i} \) is equal to \( iu^{i-1}Du \chi_{\{0 < u < k\}} \), by convexity and positivity of the map \( \xi \mapsto M(s, \xi) \) we deduce that \( M_\xi(u, Du) \cdot Du_{k,i} \geq 0 \). Moreover, by the sign condition (1.9) it follows \( M_s(u, Du)v_{k,i} \geq 0 \) a.e. in \( \Omega \). Then, we reach
\[
\begin{align*}
\int_\Omega i(u_k)^{i-1}L_\xi(Du_k) \cdot Du_k + \int_\Omega V(x)|u|^{p-2}u(u_k(x))^i \leq \int_\Omega g(u)(u_k(x))^i,
\end{align*}
\]
yielding in turn, by (1.10), that for all \( k, i \geq 1 \)
\[
(2.2) \quad \nu i \int_\Omega (u_k)^{i-1}|Du_k|^p \leq C \int_\Omega (u^+(x))^{\sigma-1+i}.
\]

If \( \hat{u}_k := \min\{u^-(x), k\} \), a similar inequality
\[
(2.3) \quad \nu i \int_\Omega (\hat{u}_k)^{i-1}|Du_k|^p \leq C \int_\Omega (u^-(x))^{\sigma-1+i},
\]
can be obtained by using \( \hat{v}_{k,i} := - (\hat{u}_k)^i \) as a test function in (1.1), observing that by (1.9),
\[
M_s(u, Du)\hat{v}_{k,i} = -M_s(u, Du)\chi_{\{-k < u \leq 0\}}(-u)^i \geq 0, \quad M_\xi(u, Du) \cdot Du_{k,i} = i(-u)^{i-1}\chi_{\{-k < u < 0\}}M_\xi(u, Du) \cdot Du \geq 0.
\]

Once (2.2)-2.3) are reached, the assertion follows exactly as in [15, Lemma 2, (a) and (b)].

We now recall the following version of [7, Lemma 4.2] which turns out to be a rather useful tool in order to establish convergences in our setting. Roughly speaking, one needs some kind of sub-criticality in the growth conditions.

**Lemma 2.5.** Let \( \Omega \subset \mathbb{R}^N \) and \( h : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \) be a Carathéodory function, \( p, m > 1, \mu \geq 1, \), \( p \leq \sigma \leq p^* \) and assume that, for every \( \varepsilon > 0 \) there exist \( \varepsilon : \in \mathbb{L}^p(\Omega) \) such that
\[
(2.4) \quad |h(x, s, \xi)| \leq \varepsilon \chi_{\{s \leq \varepsilon \sigma / \mu \}} \xi |^{p / \mu} + \varepsilon \xi |^{m / \mu},
\]
a.e. in \( \Omega \) and for all \((s, \xi) \in \mathbb{R} \times \mathbb{R}^N\). Assume that \( u_n \to u \) a.e. in \( \Omega \), \( Du_n \to Du \) a.e. in \( \Omega \) and \( (u_n) \) is bounded in \( W^{1,p}_0(\Omega) \), \( (u_n) \) is bounded in \( D_0^{1,m}(\Omega) \).
Then \( h(x, u_n, Du_n) \) converges to \( h(x, u, Du) \) in \( L^p(\Omega) \).

**Proof.** The proof follows as in [7, Lemma 4.2] and we shall sketch it here for self-containedness. By Fatou’s Lemma, it immediately holds that \( u \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega) \). Furthermore, there exists a positive constant \( C \) such that

\[
|h(x, s_1, \xi_1) - h(x, s_2, \xi_2)|^\mu \leq C(a_\varepsilon(x))^\mu + C\varepsilon^m|s_1|^{\sigma} + C\varepsilon^m|s_2|^{\sigma} + C\varepsilon^m|\xi_1|^m + C\varepsilon^m|\xi_2|^m + C\varepsilon^m|\xi_1|^{p} + C\varepsilon^m|\xi_2|^{p},
\]
a.e. in \( \Omega \) and for all \((s_1, \xi_1) \in \mathbb{R} \times \mathbb{R}^N \) and \((s_2, \xi_2) \in \mathbb{R} \times \mathbb{R}^N \). Then, taking into account the boundedness of \( (Du_n) \) in \( L^p(\Omega) \cap L^m(\Omega) \) and of \((u_n)\) in \( L^p(\Omega) \) by interpolation being \( p \leq \sigma \leq p^* \), the assertion follows by applying Fatou’s Lemma to the sequence of functions \( \psi_n : \Omega \to [0, +\infty] \)

\[
\psi_n(x) := -|h(x, u_n, Du_n) - h(x, u, Du)|^\mu + C(a_\varepsilon(x))^\mu + C\varepsilon^mu|u_n|^{\sigma} + C\varepsilon^mu|u|^{\sigma} + C\varepsilon^m|Du_n|^m + C\varepsilon^m|Du|^m + C\varepsilon^m|Du_n|^{p} + C\varepsilon^m|Du|^{p},
\]
and, finally, exploiting the arbitrariness of \( \varepsilon \).

\( \square \)

### 3. Proof of the result

#### 3.1. Energy splitting

The next result allows to perform an energy splitting for the functional

\[
J(u) = \int_\Omega j(u, Du), \quad u \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega),
\]
along a bounded Palais-Smale sequence \( (u_n) \subset W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega) \). The result is in the spirit of the classical Brezis-Lieb Lemma [4].

**Lemma 3.1.** Let the integrand \( j \) be as in (1.11) and

\[
p - 1 \leq m < p - 1 + p/N, \quad p \leq \sigma \leq p^*.
\]

Let \( (u_n) \subset W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega) \) with \( u_n \rightharpoonup u \), \( u_n \to u \) a.e. in \( \Omega \) and \( Du_n \to Du \) a.e. in \( \Omega \). Then

\[
(3.1) \quad \lim_{n \to \infty} \int_\Omega j(u_n - u, Du_n - Du) - j(u_n, Du_n) + j(u, Du) = 0.
\]

**Proof.** We shall apply Lemma 2.5 to the function

\[
h(x, s, \xi) := j(s - u(x), \xi - Du(x)) - j(s, \xi), \quad \text{for a.e. } x \in \Omega \text{ and all } (s, \xi) \in \mathbb{R} \times \mathbb{R}^N.
\]

Given \( x \in \Omega, s \in \mathbb{R} \) and \( \xi \in \mathbb{R}^N \), consider the \( C^1 \) map \( \varphi : [0, 1] \to \mathbb{R} \) defined by setting

\[
\varphi(t) := j(s - tu(x), \xi - tDu(x)), \quad \text{for all } t \in [0, 1].
\]

Then, for some \( \tau \in [0, 1] \) depending upon \( x \in \Omega, s \in \mathbb{R} \) and \( \xi \in \mathbb{R}^N \), it holds

\[
\begin{align*}
\varphi(1) - \varphi(0) &= \varphi'(\tau) \\
&= -j_s(s - \tau u(x), \xi - \tau Du(x))u(x) - j_\xi(s - \tau u(x), \xi - \tau Du(x)) \cdot Du(x) \\
&= -L_s(\xi - \tau Du(x)) \cdot Du(x) \\
&\quad - M_s(s - \tau u(x), \xi - \tau Du(x))u(x) \\
&\quad - M_\xi(s - \tau u(x), \xi - \tau Du(x)) \cdot Du(x) + G_s(s - \tau u(x))u(x).
\end{align*}
\]
Hence, for a.e. \( x \in \Omega \) and all \((s, \xi) \in \mathbb{R} \times \mathbb{R}^N\), it follows that
\[
|h(x, s, \xi)| \leq |L_\xi (\xi - \tau Du(x))| |Du(x)| + |M_\xi (s - \tau u(x), \xi - \tau Du(x))| |u(x)|
\]
\[
+ |M_\xi (s - \tau u(x), \xi - \tau Du(x))| |Du(x)| + |G' (s - \tau u(x))| |u(x)|
\]
\[
\leq C (|\xi|^{p-1} + |Du(x)|^{p-1}) |Du(x)| + C (|\xi|^m + |Du(x)|^m) |u(x)|
\]
\[
+ C (|\xi|^{m-1} + |Du(x)|^{m-1}) |Du(x)| + C (|s|^{\sigma-1} + |u(x)|^{\sigma-1}) |u(x)|
\]
\[
\leq \varepsilon |\xi|^p + C_\varepsilon |Du(x)|^p + \varepsilon |\xi|^p + C_\varepsilon |Du(x)|^p + C_\varepsilon |u(x)|^{p/(p-m)}
\]
\[
+ \varepsilon |\xi|^m + C_\varepsilon |Du(x)|^m + \varepsilon |s|^\sigma + C_\varepsilon |u(x)|^\sigma
\]
\[
= a_\varepsilon (x) + \varepsilon |s|^\sigma + \varepsilon |\xi|^p + \varepsilon |\xi|^m,
\]
where \( a_\varepsilon : \Omega \to \mathbb{R} \) is defined a.e. by
\[
a_\varepsilon (x) := C_\varepsilon |Du(x)|^p + C_\varepsilon |Du(x)|^m + C_\varepsilon |u(x)|^{p/(p-m)} + C_\varepsilon |u(x)|^\sigma.
\]
Notice that, as \( p - 1 \leq m < p - 1 + p/N \) it holds \( p \leq p/(p-m) \leq p^* \), yielding \( u \in L^{p/(p-m)}(\Omega) \)
and in turn, \( a_\varepsilon \in L^1(\Omega) \). The assertion follows directly by Lemma 2.5 with \( \mu = 1 \).

We have the following splitting result

**Theorem 3.2.** Let the integrand \( j \) be as in (1.11) and
\[
p - 1 \leq m \leq p - 1 + p/N, \quad p < \sigma < p^*.
\]
Assume that \((u_n) \subset W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)\) is a bounded Palais-Smale sequence for \( \phi \) at the level
\( c \in \mathbb{R} \) weakly convergent to some \( u \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega) \). Then
\[
\lim_{n \to \infty} \left( \int_\Omega j(u_n - u, Du_n - Du) + \int_\Omega V_\infty \frac{|u_n - u|^p}{p} \right) = c - \int_\Omega j(u, Du) - \int_\Omega V(x) \frac{|u|^p}{p},
\]
namely
\[
\lim_{n \to \infty} \phi_\infty (u_n - u) = c - \phi(u),
\]
being \( u_n \) and \( u \) regarded as elements of \( W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N) \) after extension to zero out of \( \Omega \).

**Proof.** In light of Proposition 2.3, up to a subsequence, \((u_n)\) converges weakly to some function \( u \) in \( W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega) \), \( u_n(x) \to u(x) \) and \( Du_n(x) \to Du(x) \) for a.e. \( x \in \Omega \). Also, recalling that by assumption \( V(x) \to V_\infty \) as \( |x| \to \infty \), we have [4,17]
\[
\lim_{n \to \infty} \int_\Omega V(x) |u_n - u|^p = V_\infty |u_n - u|^p = 0,
\]
\[
\lim_{n \to \infty} \int_\Omega V(x) |u_n - u|^p = V(x) |u_n|^p + V(x) |u|^p = 0.
\]
Therefore, by virtue of Lemma 3.1, we conclude that
\[
\lim_{n \to \infty} \phi_\infty (u_n - u) = \lim_{n \to \infty} \left( \int_\Omega j(u_n - u, Du_n - Du) + \int_\Omega V_\infty \frac{|u_n - u|^p}{p} \right)
\]
\[
= \lim_{n \to \infty} \left( \int_\Omega j(u_n, Du_n) + \int_\Omega V(x) \frac{|u_n|^p}{p} \right) - \int_\Omega j(u, Du) - \int_\Omega V(x) \frac{|u|^p}{p}
\]
\[
= \lim_{n \to \infty} \phi(u_n) - \phi(u) = c - \phi(u),
\]
concluding the proof. \qed
Remark 3.3. In order to shed some light on the restriction (1.5) of \( m \), it is readily seen that it is a sufficient condition for the following local compactness property to hold. Assume that \( \omega \) is a smooth domain of \( \mathbb{R}^n \) with finite measure. Then, if \( (u_h) \) is a bounded sequence in \( W^{1,p}_0(\omega) \), there exists a subsequence \( (u_{h_k}) \) such that

\[
\mathcal{Y}(x, u_{h_k}, Du_{h_k}) \text{ converges strongly to } \mathcal{Y}_0 \text{ in } W^{-1,p'}(\omega),
\]

where \( \mathcal{Y}(x, s, \xi) = g(s) - M_s(s, \xi) - V(x)|s|^{p-2}s \). In fact, taking into account the growth condition on \( g \) and \( M_s \), this can be proved observing that for every \( \varepsilon > 0 \), there exists \( C_\varepsilon \) such that

\[
|\mathcal{Y}(x, s, \xi)| \leq C_\varepsilon + \varepsilon|s|^{N(p-1)+p \over N-p} + \varepsilon|\xi|^{p-1+p/N},
\]

for a.e. \( x \in \omega \) and all \((s, \xi) \in \mathbb{R} \times \mathbb{R}^N\).

3.2. Equation splitting I (super-quadratic case). We shall assume that \( m, p \geq 2 \) and that conditions (1.7)-(1.8) hold. The following Theorem 3.4 and the forthcoming Theorem 3.5 (see next subsection) are in the spirit of the Brezis-Lieb Lemma [4], in a dual framework. For the particular case

\[
M(s, \xi) = 0 \quad \text{and} \quad L(\xi) = \frac{|\xi|^p}{p},
\]

we refer the reader to [12].

Theorem 3.4. Assume that (1.5)-(1.11) hold and that

\[
p - 1 \leq m < p - 1 + p/N, \quad p < \sigma < p^*.
\]

Assume that \( (u_n) \subset W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega) \) is such that \( u_n \rightharpoonup u, u_n \to u \) a.e. in \( \Omega \), \( Du_n \to Du \) a.e. in \( \Omega \) and there is \( (w_n) \) in the dual space \( (W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega))^* \) such that \( w_n \to 0 \) as \( n \to \infty \) and, for all \( v \in W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega) \),

\[
\int_\Omega j_\xi(u_n, Du_n) \cdot Dv + \int_\Omega j_s(u_n, Du_n)v + \int_\Omega V(x)|u_n|^{p-2}u_nv = \langle w_n, v \rangle.
\]

Then \( \phi'(u) = 0 \). Moreover, there exists a sequence \( (\xi_n) \) that goes to zero in \( (W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega))^* \), such that

\[
\langle \xi_n, v \rangle := \int_\Omega j_s(u_n - u, Du_n - Du)v + \int_\Omega j_\xi(u_n - u, Du_n - Du) \cdot Dv - \int_\Omega j_s(u_n, Du_n)v - \int_\Omega j_\xi(u_n, Du_n) \cdot Dv + \int_\Omega j_s(u, Du)v + \int_\Omega j_\xi(u, Du) \cdot Dv,
\]

for all \( v \in W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega) \).

Furthermore, there exists a sequence \( (\zeta_n) \) in \( (W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega))^* \) such that

\[
\int_\Omega j_\xi(u_n - u, Du_n - Du) \cdot Dv + \int_\Omega j_s(u_n - u, Du_n - Du)v + \int_\Omega V_\infty|u_n - u|^{p-2}(u_n - u)v = \langle \zeta_n, v \rangle
\]

for all \( v \in W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega) \) and \( \zeta_n \to 0 \) as \( n \to \infty \), namely \( \phi'(u_n - u) \to 0 \) as \( n \to \infty \).

Proof. Fixed some \( v \in W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega) \), let us define for a.e. \( x \in \Omega \) and all \((s, \xi) \in \mathbb{R} \times \mathbb{R}^N\),

\[
f_v(x, s, \xi) := j_s(s - u(x), \xi - Du(x))v(x)
\]

\[
+ j_\xi(s - u(x), \xi - Du(x)) \cdot Dv(x) - j_s(s, \xi)v(x) - j_\xi(s, \xi) \cdot Dv(x).
\]

In order to prove (3.6) we are going to show that

\[
\lim_{n \to \infty} \sup_{\|v\|_{W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)} \leq 1} \left| \int_\Omega f_v(x, u_n, Du_n) - f_v(x, u, Du) \right| = 0.
\]
As it can be easily checked, there holds
\[-f_v(x, s, \xi) = \int_0^1 j_{ss}(s - \tau u(x), \xi - \tau Du(x))u(x)v(x)d\tau + \int_0^1 j_{s\xi}(s - \tau u(x), \xi - \tau Du(x)) \cdot [Du(x)v(x) + Dv(x)u(x)]d\tau + \int_0^1 [j_{\xi\xi}(s - \tau u(x), \xi - \tau Du(x)) Du(x)] \cdot Dv(x)d\tau.
\]
Hence, by plugging the particular form of \( j \) in the above equation yields
\[-f_v(x, s, \xi) = a(x, s, \xi)v(x) + b(x, s)v(x) + c_1(x, s, \xi) \cdot Du(x) + c_2(x, s, \xi) \cdot Dv(x) + d(x, \xi) \cdot Dv(x)
\]
where
\[
a(x, s, \xi) := \int_0^1 [M_{ss}(s - \tau u(x), \xi - \tau Du(x))u(x) + M_{s\xi}(s - \tau u(x), \xi - \tau Du(x)) \cdot Du(x)]d\tau,
b(x, s) := -\int_0^1 G''(s - \tau u(x))u(x)d\tau,
c_1(x, s, \xi) := \int_0^1 M_{s\xi}(s - \tau u(x), \xi - \tau Du(x))u(x)d\tau,
c_2(x, s, \xi) := \int_0^1 M_{\xi\xi}(s - \tau u(x), \xi - \tau Du(x)) Du(x)d\tau,
d(x, \xi) := \int_0^1 L_{\xi\xi}(\xi - \tau Du(x)) Du(x)d\tau.
\]
We claim that, as \( n \to \infty \), it holds
\[
a(\cdot, u_n, Du_n) \to a(\cdot, u, Du) \quad \text{in } L^{(p')'}(\Omega),
b(\cdot, u_n) \to b(\cdot, u) \quad \text{in } L^{p'}(\Omega),
c_1(\cdot, u_n, Du_n) \to c_1(\cdot, u, Du) \quad \text{in } L^{p'}(\Omega),
c_2(\cdot, u_n, Du_n) \to c_2(\cdot, u, Du) \quad \text{in } L^{m'}(\Omega),
d(\cdot, Du_n) \to d(\cdot, Du) \quad \text{in } L^{p'}(\Omega).
\]
Then, using Hölder’s inequality and the embeddings of \( W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega) \) into \( L^s(\Omega) \) and \( L^{p'}(\Omega) \) we obtain
\[
\sup_{\|v\|_{W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)} \leq 1} \left| \int_\Omega f_v(x, u_n, Du_n) - f_v(x, u, Du) \right| \leq C\|a(\cdot, u_n, Du_n) - a(\cdot, u, Du)\|_{L^{(p')'}(\Omega)} + C\|b(\cdot, u_n) - b(\cdot, u)\|_{L^{p'}(\Omega)} + C\|c_1(\cdot, u_n, Du_n) - c_1(\cdot, u, Du)\|_{L^{p'}(\Omega)} + C\|c_2(\cdot, u_n, Du_n) - c_2(\cdot, u, Du)\|_{L^{m'}(\Omega)} + C\|d(\cdot, Du_n) - d(\cdot, Du)\|_{L^{p'}(\Omega)},
\]
yielding the desired conclusion (3.6). It remains to prove the convergences we claimed above. For each term, we shall exploit Lemma 2.5. Since \( m < p - 1 + p/N \), we can set
\[
\alpha := \frac{m}{p^* - 1}, \quad \beta := \frac{pN}{pN - N + p - mN}
\]
it follows $\beta > 0$ and $m < m + \alpha < p$. Young's inequality yields in turn
\[
y^{(m+\alpha)/(p^*)'} \leq Cy^{m/(p^*)'} + Cy^{p/(p^*)'}, \quad \text{for all } y \geq 0.
\]
Since $\beta/(p^*)' > 1$ and $(m + \alpha)/(p^*)' > 1$, by the growths of $M_{ss}$ and $M_{s\xi}$, we have
\[
|a(x, s, \xi)| \leq C(|\xi|^m + |Du(x)|^m)|u(x)| + C(|\xi|^{m-1} + |Du(x)|^{m-1})|Du(x)|
\leq \varepsilon|\xi|^{p/(p^*)'} + C_{\varepsilon}|u(x)|^{(p^*)/p} + C_c|Du(x)|^{p/(p^*)'} + \varepsilon|\xi|^{(m+\alpha)/(p^*)'} + C_{\varepsilon}|Du(x)|^{(m+\alpha)/(p^*)'}.
\]
Furthermore,
\[
|b(x, s)| \leq C(|s|^{\sigma-2} + |u(x)|^{\sigma-2})|u(x)| \leq \varepsilon|s|^{\sigma/\sigma'} + C_{\varepsilon}|u|^{\sigma/\sigma'},
\]
\[
|c_1(x, s, \xi)| \leq C(|\xi|^{m-1} + |Du(x)|^{m-1})|u(x)|
\leq \varepsilon|\xi|^{p/p'} + C_{\varepsilon}|u(x)|^{p/(p-m)p'} + C_{\varepsilon}|Du(x)|^{p/p'},
\]
\[
|c_2(x, s, \xi)| \leq C(|\xi|^{m-2} + |Du(x)|^{m-2})|Du(x)|
\leq \varepsilon|\xi|^{m/m'} + C_{\varepsilon}|Du(x)|^{m/m'},
\]
\[
|d(x, \xi)| \leq C(|\xi|^{p-2} + |Du(x)|^{p-2})|Du(x)| \leq \varepsilon|\xi|^{p/p'} + C_{\varepsilon}|Du(x)|^{p/p'}.
\]
From the point-wise convergence of the gradients and the growth estimates of $j_\xi, j_s$ and $g$ that $u$ is a week solution to the problem, namely for all $v \in W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)$
\[
\int_\Omega L_\xi(Du) \cdot Dv + \int_\Omega f(u)Dv + \int_\Omega M(u, Du)v + \int_\Omega V(x)|u|^{p-2}uv = \int_\Omega g(u)v.
\]
To get this, recall that $v \in L^{p/(p-m)'}(\Omega)$ and the sequence $(M_s(u_n, Du_n))$ is bounded in $L^{p/m}(\Omega)$ and hence it converges weakly to $M_s(u, Du)$ in $L^{p/m}(\Omega)$. Thanks to Proposition 2.4 (recall that $\beta \geq p$ if and only if $m \geq p - 2 + p/N$ and this is the case since $m \geq p - 1$), we have $L^\beta(\Omega)$. Hence, $u \in L^\sigma(\Omega) \cap L^{p/(p-m)}(\Omega) \cap L^{\beta}(\Omega)$, being $p \leq p/(p-m) < p^*$ and $p < \sigma < p^*$. By the previous inequalities the claim follows by Lemma 2.5 with the choice $\mu = (p^*)', \sigma', p', m'$ and $p'$ respectively. Let us now recall a dual version of properties (3.2)-(3.3) (cf. [17]), namely there exist two sequences $(\mu_n)$ and $(\nu_n)$ in $(W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega))^*$ which converge to zero as $n \to \infty$ and such that
\[
\int_\Omega V_\infty|u_n - u|^{p-2}(u_n - u)v = \int_\Omega V(x)|u_n - u|^{p-2}(u_n - u)v + \langle \mu_n, v \rangle,
\]
\[
\int_\Omega V(x)|u_n - u|^{p-2}(u_n - u)v = \int_\Omega V(x)|u_n|^{p-2}u_nv - \int_\Omega V(x)|u|^{p-2}uw + \langle \mu_n, v \rangle,
\]
for every $v \in W_0^{1,p}(\Omega) \cap D_0^{1,m}(\Omega)$. Whence, by collecting (3.4), (3.5), (3.6), (3.7), we get
\[
\int_\Omega j_\xi(u_n - u, Du_n - Du) \cdot Dv + \int_\Omega j_s(u_n - u, Du_n - Du)v + \int_\Omega V_\infty|u_n - u|^{p-2}(u_n - u)v
\leq \int_\Omega j_\xi(u_n, Du_n) \cdot Dv + \int_\Omega j_s(u_n, Du_n)v + \int_\Omega V(x)|u_n|^{p-2}u_nv
- \int_\Omega j_\xi(u, Du) \cdot Dv - \int_\Omega j_s(u, Du)v - \int_\Omega V(x)|u|^{p-2}uw + \langle \xi_n + \mu_n + \nu_n, v \rangle = \langle \zeta_n, v \rangle,
\]
where $\langle \zeta_n, v \rangle := \langle w_n + \xi_n + \mu_n + \nu_n, v \rangle$ and $\zeta_n \to 0$ as $n \to \infty$. This concludes the proof. □
3.3. Equation splitting II (sub-quadratic case). We assume that (1.12)-(1.14) hold.

**Theorem 3.5.** Assume (1.9), let the integrand \( j \) be as in (1.11) and \( p \leq 2 \) or \( m \leq 2 \) or \( \sigma \leq 2 \),
\[ p - 1 \leq m < p - 1 + \frac{p}{N}, \quad p < \sigma < p^*. \]
Assume that \( (u_n) \subset W^{1,p}(\Omega) \cap D^{1,m}_0(\Omega) \) is such that \( u_n \rightharpoonup u \) in \( \Omega \), \( Du_n \rightharpoonup Du \) a.e. in \( \Omega \) and there exists \( (w_n) \) in \( (W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega))^* \) such that \( w_n \rightharpoonup 0 \) as \( n \to \infty \) and, for every \( v \in \Omega \),
\[ \int_{\Omega} j(u_n, Du_n) \cdot Dv + \int_{\Omega} j_s(u_n, Du_n) v + \int_{\Omega} V(x) |u_n|^{p-2} u_n v = \langle w_n, v \rangle. \]
Then \( \phi'(u) = 0 \). Moreover, there exists a sequence \( (\hat{\xi}_n) \) that goes to zero in \( (W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega))^* \), such that
\[ \langle \hat{\xi}_n, v \rangle := \int_{\Omega} j_s(u_n - u, Du_n - Du) v + \int_{\Omega} j_s(u_n - u, Du_n) \cdot Dv \]
\[ - \int_{\Omega} j_s(u_n, Du_n) v - \int_{\Omega} j_s(u_n, Du_n) \cdot Dv + \int_{\Omega} j_s(u, Du) v + \int_{\Omega} j_s(u, Du) \cdot Dv, \]
for all \( v \in \Omega \).
Furthermore, there exists a sequence \( (\hat{\xi}_n) \) in \( W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega) \) with
\[ \int_{\Omega} j_s(u_n - u, Du_n - Du) \cdot Dv + \int_{\Omega} j_s(u_n - u, Du_n - Du) v + \int_{\Omega} V(x) |u_n - u|^{p-2} (u_n - u) v = \langle \hat{\xi}_n, v \rangle \]
for all \( v \in \Omega \) and \( \hat{\xi}_n \rightharpoonup 0 \) as \( n \to \infty \), namely \( \phi'(u_n - u) \rightharpoonup 0 \) as \( n \to \infty \).

**Proof.** Keeping in mind the argument in proof of Theorem 3.4, here we shall be more sketchy. For every \( s \in \mathbb{R} \) and \( \xi \in \mathbb{R}^N \) we plug \( L, M, G \) into the equation
\[ f_v(x, s, \xi) = j_s(u(x) - u(x), \xi - Du(x)) v(x) \]
\[ + j_s(u(x) - u(x), \xi - Du(x)) \cdot Du(x) - j_s(s, \xi) v(x) - j_s(s, \xi) \cdot Du(x), \]
thus obtaining
\[ f_v(x, s, \xi) = (M_s(s - u(x), \xi - Du(x)) - M_s(s, \xi)) v(x) - (G'(s - u(x)) - G'(s)) v(x) \]
\[ + (M_s(s - u(x), \xi - Du(x)) - M_s(s, \xi)) \cdot Du(x) + (L_s(\xi - Du(x)) - L_s(\xi)) \cdot Du(x) \]
\[ = a' v(x) + b' v(x) + c' \cdot Du(x) + d' \cdot Du(x). \]
We write the term \( M_s(s - u(x), \xi - Du(x)) - M_s(s, \xi) \) in a more suitable form, namely
\[ c' = M_s(s - u(x), \xi - Du(x)) - M_s(s, \xi) \]
\[ = M_s(s - u(x), \xi - Du(x)) - M_s(s, \xi - Du(x)) + M_s(s, \xi - Du(x)) - M_s(s, \xi), \]
so that
\[ f_v(x, s, \xi) = a' v(x) + b' v(x) + (c'_1(x, s, \xi) + c'_2(x, s, \xi)) \cdot Du(x) + d'v(x) \cdot Du(x). \]
The term \( a' \) admits the same growth condition of \( a \), cf. the proof of Theorem 3.4. Also, since
\[ c'_1(x, s, \xi) = - \int_0^1 M_{\xi_s}(s - \tau u(x), \xi - Du(x)) u(x) d\tau, \]
as for the term \( c_1 \) in the proof of Theorem 3.4 we obtain
\[ |c'_1(x, s, \xi)| \leq \epsilon |\xi|^{p/p'} + C_\epsilon |u(x)|^{p/(p-m)p'} + C_\epsilon |Du(x)|^{p/p'}. \]
On the other hand, directly from assumptions (1.12)-(1.14) we get
\[ |b'(x, s)| \leq C|u(x)|^{\sigma'/\sigma}, \quad |c'_2(x, s, \xi)| \leq C|Du(x)|^{m/m'}, \quad |d'(x, \xi)| \leq C|Du(x)|^{p/p'}.\]
The conclusion follows then by the same argument carried out in Theorem 3.4. □

In the spirit of [17, Lemma 8.3], we have the following

**Lemma 3.6.** Under the hypotheses of Theorem 1.1 or 1.2, let \((y_n) \subset \mathbb{R}^N\) with \(|y_n| \to \infty\),

\[
\begin{align*}
&u_n(\cdot + y_n) \rightharpoonup u \quad \text{in } W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N), \\
n_n(\cdot + y_n) \to u &\quad \text{a.e. in } \mathbb{R}^N, \\
Dn_n(\cdot + y_n) &\to Du \quad \text{a.e. in } \mathbb{R}^N, \\
\phi_{\infty}(u_n) &\to c, \\
\phi'_{\infty}(u_n) &\to 0 \quad \text{in } (W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega))^*,
\end{align*}
\]

Then \(\phi'_{\infty}(u) = 0\) and, setting \(v_n := u_n - u(\cdot - y_n)\), we have
\[
\begin{align*}
(3.9) &\quad \phi_{\infty}(v_n) \to c - \phi_{\infty}(u), \\
(3.10) &\quad \phi'_{\infty}(v_n) \to 0 \quad \text{in } (W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega))^*,
\end{align*}
\]
and \(\|v_n\|_p^p = \|u_n\|_p^p - \|u\|_p^p + o(1)\) and \(\|v_n\|_m^m = \|u_n\|_m^m - \|u\|_m^m + o(1)\) as \(n \to \infty\).

**Proof.** The energy splitting (3.9) follows by Theorem 3.2 applied with \(\Omega = \mathbb{R}^N\) and the sequence \((u_n)\) replaced by \((u_n(\cdot + y_n))\). Take now \(\varphi \in D(\Omega)\) with \(\|\varphi\|_{W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)} \leq 1\) and define \(\varphi_n := \varphi(\cdot + y_n)\). Then \(\varphi_n \in D(\Omega_n)\), where \(\Omega_n = \Omega - \{y_n\} \subset \Omega\) for \(n\) large. For any \(n \in \mathbb{N}\), we get
\[
\langle \phi'_{\infty}(v_n), \varphi \rangle = \langle \phi'_{\infty}(u_n(\cdot + y_n) - u), \varphi_n \rangle.
\]
By the splitting argument in the proof of Theorem 3.4, it follows that
\[
\langle \phi'_{\infty}(u_n(\cdot + y_n) - u), \varphi_n \rangle = \langle \phi'_{\infty}(u_n(\cdot + y_n)), \varphi_n \rangle - \langle \phi'_{\infty}(u), \varphi_n \rangle + \langle \zeta_n, \varphi_n \rangle,
\]
where \(\zeta_n \to 0\) in the dual of \(W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)\). If we prove that \(u\) is critical for \(\phi_{\infty}\), then the right-hand side reads as \(\langle \phi'_{\infty}(u_n), \varphi \rangle + \langle \zeta_n, \varphi \rangle\), and also the second limit (3.10) follows. To prove that \(\phi'_{\infty}(u) = 0\) we observe that, for all \(\varphi \in D(\mathbb{R}^N)\),
\[
\langle \phi'_{\infty}(u_n(\cdot + y_n)), \varphi \rangle - \langle \phi'_{\infty}(u), \varphi \rangle = \left| \langle \phi'_{\infty}(u_n(\cdot + y_n)), \varphi \rangle - \|\phi'_{\infty}(u_n)\|_2\|\varphi\|_{W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)} \right| \to 0.
\]
Indeed, defining \(\hat{\varphi}_n := \varphi(\cdot - y_n)\), since \(|y_n| \to \infty\) as \(n \to \infty\), we have \(\text{supp } \hat{\varphi}_n \subset \Omega\), for \(n\) large enough and \(\|\hat{\varphi}_n\|_{W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)} = \|\varphi\|_{W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)}\). The last assertion follows by using Brezis-Lieb Lemma [4]. □

We can finally come to the proof of the main results.

### 4. Proof of Theorems 1.1 and 1.2 completed

We follow the scheme of the proof given in [17, p.121]. Let \((u_n) \subset W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)\) be a bounded Palais-Smale sequence for \(\phi\) at the level \(c \in \mathbb{R}\). Hence, there exists a sequence \((w_n)\) in the dual of \(W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)\) such that \(w_n \to 0\) and \(\phi(u_n) \to c\) as \(n \to \infty\) and, for all \(v \in W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)\), we have
\[
\int_{\Omega} L_\xi(Du_n) \cdot Du + \int_{\Omega} M_\xi(u_n, Du_n) \cdot Du + \int_{\Omega} M_s(u_n, Du_n) + \int_{\Omega} V(x) |u_n|^{p-2} u_n v = \int_{\Omega} g(u_n) v + \langle w_n, v \rangle.
\]
Since \((u_n)\) is bounded in \(W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)\), up to a subsequence, it converges weakly to some function \(v_0 \in W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)\) and, by virtue of Proposition 2.3, \((u_n)\) and \((Du_n)\) converge to \(v_0\) and \(Dv_0\) a.e. in \(\Omega\), respectively. In turn (see also the proof of Theorem 3.4) it follows

\[
\int_\Omega L_\xi(Dv_0) \cdot Dv + \int_\Omega M_\xi(v_0, Du_0) \cdot Dv + \int_\Omega M_s(v_0, Du_0)v + \int_\Omega V(x)|v_0|^{p-2}v_0v = \int_\Omega g(v_0)v,
\]

for any \(v \in W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)\). By combining Theorem 3.2 and Theorem 3.4, setting \(u_n^1 := u_n - v_0\) and thinking the functions on \(\mathbb{R}^N\) after extension to zero out of \(\Omega\), get

\[
\phi_\infty(u_n^1) \to c - \phi(v_0), \quad n \to \infty,
\]

\[
\int_{\mathbb{R}^N} L_\xi(Du_n^1) \cdot Dv + \int_{\mathbb{R}^N} M_\xi(u_n^1, Du_n^1) \cdot Dv + \int_{\mathbb{R}^N} M_s(u_n^1, Du_n^1)v
\]

\(+ \int_{\mathbb{R}^N} V(x)|u_n^1|^{p-2}u_n^1v = \int_{\mathbb{R}^N} g(u_n^1)v + \langle w_n^1, v \rangle.
\]

where \((w_n^1)\) is a sequence in the dual of \(W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)\) with \(u_n^1 \to 0\) as \(n \to \infty\). In turn, it follows that \((u_n^1)\) is Palais-Smale sequence for \(\phi_\infty\) at the energy level \(c - \phi(v_0)\). In addition,

\[
\|u_n^1\|_p = \|u_n\|_p - \|v_0\|_p + o(1), \quad \|u_n^1\|_m = \|u_n\|_m - \|v_0\|_m + o(1), \quad \text{as } n \to \infty,
\]

by the Brezis-Lieb Lemma [4]. Let us now define

\[
\varpi := \limsup_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B(y,1)} |u_n^1|_p.
\]

If it is the case that \(\varpi = 0\), then, according to [11, Lemma I.1], \((u_n^1)\) converges to zero in \(L^r(\mathbb{R}^N)\) for every \(r \in (p, p^*)\). Then, one obtains that

\[
\lim_{n \to \infty} \int_\Omega g(u_n^1)u_n^1 = 0, \quad \int_\Omega M_s(u_n^1, Du_n^1)u_n^1 \geq 0,
\]

where the inequality follows by the sign condition (1.9). In turn, testing equation (4.2) with \(v = u_n^1\), by the coercivity and convexity of \(\xi \mapsto L(\xi), M(s, \xi)\), we have

\[
\limsup_{n \to \infty} \left[ \nu \int_{\mathbb{R}^N} |Du_n^1|_p + \nu \int_{\mathbb{R}^N} |Du_n^1|_m + V_\infty \int_{\mathbb{R}^N} |u_n^1|_p \right]
\]

\[
\leq \limsup_{n \to \infty} \left[ \int_{\mathbb{R}^N} L_\xi(Du_n^1) \cdot Du_n^1 + \int_{\mathbb{R}^N} M_\xi(u_n^1, Du_n^1) \cdot Du_n^1 + \int_{\mathbb{R}^N} V(u_n^1)|u_n^1|_p \right] \leq 0,
\]

yielding that \((u_n^1)\) strongly converges to zero in \(W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)\), concluding the proof in this case. If, on the contrary, it holds \(\varpi > 0\), then, there exists an unbounded sequence \((y_n^1) \subset \mathbb{R}^N\) with \(\int_{B(y_n^1,1)} |u_n^1|_p > \varpi/2\). Whence, let us consider \(v_n^1 := u_n^1(- + y_n^1)\), which, up to a subsequence, converges weakly and pointwise to some \(v_1 \in W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)\), which is nontrivial, due to the inequality \(\int_{B(0,1)} |v_1|_p \geq \varpi/2\). Notice that, of course,

\[
\lim_{n \to \infty} \phi_\infty(v_n^1) = \lim_{n \to \infty} \phi_\infty(u_n^1) = c - \phi(v_0).
\]
Moreover, since $|y_n^1| \to \infty$ and $\Omega$ is an exterior domain, for all $\varphi \in \mathcal{D}(\mathbb{R}^N)$ we have $\varphi(\cdot - y_n^1) \in \mathcal{D}(\Omega)$ for $n \in \mathbb{N}$ large enough. Whence, in light of equation (4.2), for every $\varphi \in \mathcal{D}(\mathbb{R}^N)$ we get

$$
\begin{align*}
&\int_{\mathbb{R}^N} L_\xi(Dv_n^1) \cdot D\varphi + \int_{\mathbb{R}^N} M_\xi(v_n^1, Dv_n^1) \cdot D\varphi + \int_{\mathbb{R}^N} M_s(v_n^1, Dv_n^1)\varphi \\
&\quad + \int_{\mathbb{R}^N} V_\infty|v_n^1|_{p-2}(v_n^1)\varphi - \int_{\mathbb{R}^N} g(v_n^1)\varphi = \int_{\mathbb{R}^N} L_\xi(Du_1^1) \cdot D\varphi(\cdot - y_n^1) \\
&\quad + \int_{\mathbb{R}^N} M_\xi(u_1^1, Du_1^1) \cdot D\varphi(\cdot - y_n^1) + \int_{\mathbb{R}^N} M_s(u_1^1, Du_1^1)\varphi(\cdot - y_n^1) + \int_{\mathbb{R}^N} V_\infty|u_n^1|_{p-2}(u_n^1)\varphi(\cdot - y_n^1) \\
&\quad - \int_{\mathbb{R}^N} g(u_n^1)\varphi(\cdot - y_n^1) = \langle w_n^1, \varphi(\cdot + y_n^1) \rangle.
\end{align*}
$$

Defining the form $\langle \tilde{w}_n^1, \varphi \rangle := \langle u_n^1, \varphi(\cdot - y_n^1) \rangle$ for all $\varphi \in \mathcal{D}(\mathbb{R}^N)$, we conclude that

$$
\begin{align*}
&\int_{\mathbb{R}^N} L_\xi(Dv_n^1) \cdot D\varphi + \int_{\mathbb{R}^N} M_\xi(v_n^1, Dv_n^1) \cdot D\varphi + \int_{\mathbb{R}^N} M_s(v_n^1, Dv_n^1)\varphi \\
&\quad + \int_{\mathbb{R}^N} V_\infty|v_n^1|_{p-2}(v_n^1)\varphi - \int_{\mathbb{R}^N} g(v_n^1)\varphi = \langle \tilde{w}_n^1, \varphi \rangle, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^N).
\end{align*}
$$

Since $(\tilde{w}_n^1)$ converges to zero in the dual of $W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)$, it follows by Proposition 2.3 (with $V = V_\infty$ and $\Omega = \mathbb{R}^N$) that the gradients $Du_n^1$ converge point-wise to $Dv_1$, namely

$$
Dv_n^1(x) \to Dv_1(x), \quad \text{a.e. in } \mathbb{R}^N. \tag{4.3}
$$

Setting $u_n^2 := u_n^1 - v_1(\cdot - y_1^1)$, in light of (4.1)-(4.2) and (4.3), we can apply Lemma 3.6 to the sequence $(v_n^1)$, getting

$$
\lim_{n \to \infty} \phi_\infty(u_n^2) = c - \phi(v_0) - \phi_\infty(v_1),
$$

as well as $\phi_\infty(v_1) = 0$ and, furthermore, for every $v \in W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)$, we have

$$
\begin{align*}
&\int_{\mathbb{R}^N} L_\xi(Du_n^2) \cdot Dv + \int_{\mathbb{R}^N} M_\xi(u_n^2, Du_n^2) \cdot Dv + \int_{\mathbb{R}^N} M_s(u_n^2, Du_n^2)v \\
&\quad + \int_{\mathbb{R}^N} V_\infty|u_n^2|_{p-2}u_n^2v - \int_{\mathbb{R}^N} g(u_n^2)v = \langle c_n^1, v \rangle,
\end{align*}
$$

where $(\zeta_n^1)$ goes to zero in the dual of $W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega)$. In turn, $(u_n^2) \subset W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)$ is a Palais-Smale sequence for $\phi_\infty$ at the energy level $c - \phi(v_0) - \phi(v_1)$. Arguing on $(u_n^2)$ as it was done for $(u_n^1)$, either $u_n^2$ goes to zero strongly in $W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N)$ or we can generate a new $(\zeta_n^2)$. By iterating the above procedure, one obtains diverging sequences $(y_n^i)$, $i = 1, \ldots, k - 1$, solutions $v_i$ on $\mathbb{R}^N$ to the limiting problem, $i = 1, \ldots, k - 1$ and a sequence

$$
u_n^k = u_n - v_0 - v_1(\cdot - y_1) - v_2(\cdot - y_2) - \cdots - v_{k-1}(\cdot - y_{k-1}),$$

such that (recall again Lemma 3.6) as $n \to \infty$

$$
\begin{align*}
&\|u_n^k\|_p = \|u_n\|_p - \|v_0\|_p - \|v_1\|_p - \cdots - \|v_{k-1}\|_p + o(1), \\
&\|u_n^k\|_m = \|u_n\|_m - \|v_0\|_m - \|v_1\|_m - \cdots - \|v_{k-1}\|_m + o(1),
\end{align*}
$$

as well as $\phi'_\infty(u_n^k) \to 0$ in $(W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega))^*$ and

$$
\phi_\infty(u_n^k) \to c - \phi(v_0) - \sum_{j=1}^{k-1} \phi_\infty(v_j).\tag{4.4}
$$
Notice that the iteration is forced to end up after a finite number \( k \geq 1 \) of steps. Indeed, for every nontrivial critical point \( v \in W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N) \) of \( \phi_\infty \) we have,

\[
\int_{\mathbb{R}^N} L_\xi(Dv) \cdot Dv + \int_{\mathbb{R}^N} M_\xi(v, Dv) \cdot Dv + \int_{\mathbb{R}^N} M_\xi(v, Dv)v + \int_{\mathbb{R}^N} \nu|v|^p = \int_{\mathbb{R}^N} g(v)v,
\]

yielding by the sign condition, the coercivity-convexity conditions and the growth of \( g \),

\[
\min\{\nu, \sqrt{V}\}||v||_p^p + ||Dv||_L^m(\mathbb{R}^N) \leq C_g||v||_{L^\sigma(\mathbb{R}^N)} \leq C_gS_{p, \sigma}||v||_p^\sigma,
\]

so that, due to \( \sigma > p \), it holds

\[
||v||_p^p \geq \left[ \frac{\min\{\nu, \sqrt{V}\}}{C_gS_{p, \sigma}} \right] \overline{\sigma}_p =: \Gamma_\infty > 0,
\]

thus yielding from (4.4)

\[
||u_n^k||_p^p \leq ||u_n||_p^p - ||v_0||_p^p - (k - 1)\Gamma_\infty + o(1).
\]

By boundedness of \((u_n)\), \( k \) has to be finite. Hence \( u_n^k \to 0 \) strongly in \( W^{1,p}(\mathbb{R}^N) \cap D^{1,m}(\mathbb{R}^N) \) at some finite index \( k \in \mathbb{N} \). This concludes the proof.

5. Proof of Corollary 1.3

As a byproduct of the proof of the Theorems 1.1 and 1.2, since the \( p \) norm is bounded away from zero on the set of nontrivial critical points of \( \phi_\infty \), cf. (4.5), we can estimate \( \phi_\infty \) from below on that set. In order to do so, we use condition (2.1). For any nontrivial critical point of the functional \( \phi_\infty \), we have (see the proof of Proposition 2.2)

\[
\mu\phi_\infty(v) \geq \delta \int_{\Omega} |Dv|^p + \frac{\mu - p}{p} V_\infty \int_{\mathbb{R}^N} |v|^p \geq \min\left\{ \delta, \frac{\mu - p}{p} V_\infty \right\} ||v||_p^p.
\]

An analogous argument applies to \( \phi \), yielding for any nontrivial critical point

\[
\mu\phi(u) \geq \delta \int_{\Omega} |Du|^p + \frac{\mu - p}{p} V_0 \int_{\Omega} |u|^p \geq \min\left\{ \delta, \frac{\mu - p}{p} V_0 \right\} ||u||_p^p.
\]

Now notice that, recalling (4.6) and a similar variant for the norm of the critical points of \( \phi \) in place of \( \phi_\infty \), setting also

\[
e_\infty := \min\left\{ \frac{\delta}{\mu}, \frac{\mu - p}{\mu \rho} V_\infty \right\} \Gamma_\infty, \quad e_0 := \min\left\{ \frac{\delta}{\mu}, \frac{\mu - p}{\mu \rho} V_0 \right\} \Gamma_0, \quad \Gamma_0 := \left[ \frac{\min\{\nu, V_0\}}{C_gS_{p, \sigma}} \right] \overline{\sigma}_p > 0,
\]

from Theorems 1.1 or 1.2 we have \( c \geq \ell e_0 + k e_\infty \) for some \( \ell \in \{0, 1\} \) and non-negative integer \( k \). Condition \( c < c^* := e_\infty \) implies necessarily \( k < 1 \), namely \( k = 0 \). This provides the desired compactness result, using Theorems 1.1 or 1.2.

6. Proof of Corollary 1.8

Defining the functionals \( J, Q : W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega) \to \mathbb{R} \) by

\[
J(u) := \frac{1}{p} \int_{\Omega} L(Du) + \frac{1}{m} \int_{\Omega} M(Du) + \frac{1}{p} \int_{\Omega} V(x)|u|^p, \quad Q(u) := \frac{S_{\Omega}}{\sigma} \int_{\Omega} |u|^\sigma,
\]

and given a minimization sequence \((u_n)\) for problem (1.16), by Ekeland’s variational principle, without loss of generality we can replace it by a new minimization sequence, still denoted by \((u_n)\) for which there exists a sequence \((\lambda_n) \subset \mathbb{R} \) such that for all \( v \in W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega) \)

\[
J'(u_n)(v) - \lambda_n Q'(u_n)(v) = \langle w_n, v \rangle, \quad \text{with} \ w_n \to 0 \ \text{in the dual of} \ W^{1,p}_0(\Omega) \cap D^{1,m}_0(\Omega).
\]
Taking into account the homogeneity of $L$ and $M$, choosing $v = u_n$ this means
\[ \int_{\Omega} L(Du_n) + \int_{\Omega} M(Du_n) + \int_{\Omega} V(x)|u_n|^p - S_{\Omega} \lambda_n \int_{\Omega} |u_n|^\sigma = \langle w_n, u_n \rangle. \]
Up to a subsequence, let us define
\[ \lambda := \frac{1}{S_{\Omega}} \lim_{n \to \infty} \int_{\Omega} L(Du_n) + M(Du_n) + V(x)|u_n|^p. \]
Since $\|u_n\|_{L^\sigma(\Omega)} = 1$ for all $n$ and $\int_{\Omega} L(Du_n)/p + M(Du_n)/m + V(x)|u_n|^p/p \to S_{\Omega}$ as $n \to \infty$, this means that $\lambda \in [m, p]$ and $(u_n)$ is a Palais-Smale sequence for the functional $I(u) := J(u) - \lambda Q(u)$ at an energy level
\[ (6.1) \]
\[ c \leq \frac{\sigma - m}{\sigma} S_{\Omega}, \]
\[ \text{since it holds (recall that } p \geq m, \text{ as } n \to \infty, \]
\[ I(u_n) = \frac{1}{p} \int_{\Omega} L(Du_n) + \frac{1}{m} \int_{\Omega} M(Du_n) + \frac{1}{p} \int_{\Omega} V(x)|u_n|^p - S_{\Omega} \lambda \]
\[ \leq S_{\Omega} - \frac{S_{\Omega}}{\sigma} \lambda + o(1) \leq S_{\Omega} \frac{\sigma - m}{\sigma} + o(1). \]
From Corollary 1.3 (applied with $L(Du)$ replaced by $L(Du)/p$, $M(u, Du)$ replaced by $M(Du)/m$ and $G(u) = \frac{S_{\Omega}}{\sigma} \lambda |s|^\sigma$), the compactness of $(u_n)$ holds provided (in the notations of Corollary 1.3)
\[ c < \min \left\{ \delta, \frac{\mu - p}{\mu p} \right\} \left[ \frac{\min \{\nu, V_{\infty}\}}{C_g S_{p, \sigma}} \right]^{\frac{\nu}{\sigma-p}}. \]
In our case, we can take $\mu = \sigma$, $\delta = \frac{\sigma - p}{p}$, $C_g = p S_{\Omega}$, $V_{\infty} = 1$, $\nu = 1$, $S_{p, \sigma} = (p S_{\mathbb{R}^N})^{-\sigma/p}$, yielding
\[ c < \frac{\sigma - p}{\sigma} \frac{S_{\mathbb{R}^N}}{S_{\Omega}^{\frac{\sigma}{\sigma-p}}}. \]
Hence, finally, by combining this conclusion with (6.1) the compactness (and in turn the solvability of the minimization problem) holds if (1.17) holds, concluding the proof.

References


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<thead>
<tr>
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<th>Author(s)</th>
<th>Title</th>
<th>Month</th>
</tr>
</thead>
<tbody>
<tr>
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<td>June '11</td>
</tr>
<tr>
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<td>V. Chalupecký, A. Muntean</td>
<td>Semi-discrete finite difference multiscale scheme for a concrete corrosion model: approximation estimates and convergence</td>
<td>June '11</td>
</tr>
<tr>
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<td>July '11</td>
</tr>
<tr>
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<td>July '11</td>
</tr>
</tbody>
</table>