Implicit-IMOR method for index-1 and index-2 linear constant DAEs

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N. Banagaaya∗, G. Ali†, W. H. A. Schilders ‡

Abstract

The index-aware model order reduction (IMOR) method for differential-algebraic equations (DAEs) is based on the decomposition of a DAE into differential and algebraic parts, depending on its tractability index. Then, the differential part is reduced by using existing MOR methods for ODE, and this reduction induces a reduction on the algebraic part, which can be further enhanced. However the IMOR method involves matrix inversions which may limit its practicability in some real-life examples. In this paper a modified IMOR method is presented, which does not involve matrix inversion.

1 Introduction

We consider a control problem of the form:

\[ Ex'(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \]
\[ y(t) = C^T x(t), \]

where \( E, A \in \mathbb{R}^{n,n} \), \( B \in \mathbb{R}^{n,m} \) and \( C \in \mathbb{R}^{n,\ell} \). \( x(t) \in \mathbb{R}^n \) is the state vector or unknown variables, \( u(t) \in \mathbb{R}^m \) is the input vector and \( y(t) \in \mathbb{R}^\ell \) is the desired output vector. If \( E \) is nonsingular, (1.1a) is a system of ordinary differential equations (ODEs), otherwise it is a system of differential algebraic equations (DAEs). We assume that \( E \) is singular, thus \( x_0 \in \mathbb{R}^n \) must be a consistent initial value and the matrix pencil \( \lambda E - A \) must be regular for at least one
\( \lambda \in \mathbb{C} \) in order to guarantee uniqueness and existence of solutions to the DAE.

In practice, the number of equations, \( n \), is very large, and one is usually interested in a small number of outputs, \( \ell \). This is an attractive feature to apply model-order reduction (MOR).

Model order reduction has been widely applied to the fast numerical simulation of large scale problems [8]. The most commonly used methods are the moment-matching or Krylov-subspace-based methods, which include PRIMA [11] and SPRIM [12]. These methods are able to reduce very large systems. However, they do not always lead to good reduced-order models for index greater than 1 [2]. ODE MOR methods which have been extended to reduce DAEs are the Balanced truncation method [19] and the interpolatory projection method [17]. The modification of these methods is based on using the spectral projections proposed in [16] which are numerically infeasible for general DAEs. However, these methods can be applied on DAEs with special structures.

In the MOR community, it is assumed that \( x_0 = 0 \), which leads to a transfer function \( H(s) = C^T(sE - A)^{-1}B \) if and only if the matrix pencil \( sE - A \) is regular [8]. Then this transfer function is used to approximate the reduced-order model. Unfortunately for the case of DAEs the initial condition \( x_0 \) cannot always be chosen arbitrarily. According to [17] the transfer function of the DAE can be decomposed as \( H(s) = H_{sp}(s) + P(s) \), where \( H_{sp}(s) \) and \( P(s) \) denote, respectively, the strictly proper part and the polynomial part of \( H(s) \). Since most of the conventional MOR methods were designed to approximate transfer function of ODE systems, thus they just approximate the strictly proper part and leave the polynomial part not approximated. This implies that the conventional moment-matching methods do not always approximate the complete solution space.

This motivated us to propose a new MOR technique for DAEs called index-aware MOR (IMOR) method which eliminates this inconvenience. This method was first introduced in [1]. In this technique, before applying MOR we first split the DAE (1.1) into differential and algebraic parts using the matrix and projector chains introduced by M"arz [4] in 1996. We then use the existing MOR techniques such as the conventional moment-matching methods on the differential part and develop new techniques for the algebraic part. This ensures a good approximation both of the strictly proper part and of the polynomial part of the transfer function. In this paper the strictly proper and polynomial part of the transfer function is what we call the transfer function of the differential and algebraic part of the transfer function \( H(s) \).
To define the tractability index of (1.1a), we define iteratively the following projector and matrix chains: setting $E_0 := E$, $A_0 := A$, then $E_{j+1} := E_j - A_j Q_j$, $A_{j+1} := A_j P_j$, $j \geq 0$, where $Q_j$ is a projector chosen so that $\text{Im } Q_j = \text{Ker } E_j$ and $P_j = I_n - Q_j$ is its complementary projector. The process is terminated if there exists a nonsingular $E_\mu$ while all $E_j$ are singular for all $0 \leq j < \mu - 1$. Then $\mu$ is called the tractability index of the DAE (1.1). We note that the tractability index coincides with other index concepts such as differentiation index for the case of LTI systems. In order to decompose higher index systems ($\mu > 1$), März [4] suggested an additional constraint $Q_j Q_i = 0$, $j > i$, on the projector construction. Using these chains we can rewrite Equation (1.1), of index-$\mu$, as [4]:

$$P_{\mu-1} \cdots P_0 x' + Q_0 x + \cdots + Q_{\mu-1} x = E_\mu^{-1} (A_\mu x + B u). \quad (1.2)$$

Then Equation (1.2) can be decomposed into 1 differential and $\mu$ algebraic parts. However, the März decomposition leads to a decoupled system of dimension $(\mu + 1)n$. Moreover it does not preserve the spectrum of the DAE. This motivated us to modify the März decomposition using special basis vectors as presented in papers [1, 2] for the case of index-1 and index-2, respectively. The modified decomposition leads to a decoupled system of the same dimension as that of the original DAE while preserving also its stability. Then we apply Krylov methods on the differential part and construct subspaces to reduce the algebraic parts.

The decoupling procedure, we presented in [1, 2] lead to an explicit differential and algebraic part. This is due to the fact that we used (1.2) as our starting system which involves inversion of non-singular matrix $E_\mu$. This inversion may be computationally expensive to some practical problems such as electric networks and problems from fluid dynamics. In this paper, we propose a strategy of decoupling (1.1) which does not involve inversion of $E_\mu$. Thus, this time our starting system is written in the equivalent form

$$E_\mu (P_{\mu-1} \cdots P_0 x' + Q_0 x + \cdots + Q_{\mu-1} x) = A_\mu x + B u. \quad (1.3)$$

This equation leads to an MOR procedure which yields implicit differential and algebraic parts.

The paper is organized as follows. In section 2 we present the decomposition procedure we adopt for DAEs. For clarity of exposition, we consider separately the application to index-1 and index-2 equations. Once these two cases are understood, the extension to higher-index equations is relatively simple. In section 3 we expound the implicit index-aware MOR method for DAEs. We conclude the paper with a section with numerical experiments, which show the computational advantage of the proposed method.
2 Implicit decoupling of DAEs

In this section, we expound a decomposition procedure based on the systematic use of projectors, starting from the work of M"arz \[4\]. For the sake of clarity, we consider separately index-1 and index-2 systems of the form (1.1), written in the equivalent form (1.3).

2.1 Implicit decoupling of index-1 systems

First we assume that (1.1) is an index-1 system, that is, \( \mu = 1 \). Thus, (1.1a) can be written in the equivalent form (1.3), with \( \mu = 1 \), that is,

\[
E_1 \left( P_0x' + Q_0x \right) = A_1x + Bu,
\]

(2.4)

where \( E_1 = E - AQ_0 \) and \( A_1 = AP_0 \) and \( Q_0 \) is a projector chosen such that \( \text{Im}Q_0 = \text{Ker}E \) and \( P_0 = I - Q_0 \) is its complementary projector. In [1], we showed that for the case of index-1 systems, \( x \) can be decomposed as

\[
x = \begin{pmatrix} p_0 & q_0 \end{pmatrix} \begin{pmatrix} \xi_p \\ \xi_q \end{pmatrix},
\]

where \( p_0 \in \mathbb{R}^{n,n_p} \) and \( q_0 \in \mathbb{R}^{n,n_q} \) are the linearly independent columns of projectors \( P_0 \) and \( Q_0 \), respectively. \( \xi_p \in \mathbb{R}^{n_p} \) and \( \xi_q \in \mathbb{R}^{n_q} \) are the projected differential and algebraic variables, respectively, and \( n = n_p + n_q \). We note that \( n_p \) is equal to the rank of the singular matrix \( E \). Using the decomposition of \( x \) and simplifying, (2.4) becomes

\[
\begin{pmatrix} E_1 p_0 & 0 \end{pmatrix} \begin{pmatrix} \xi_p \\ \xi_q \end{pmatrix}' = \begin{pmatrix} A_1 p_0 & -E_1 q_0 \end{pmatrix} \begin{pmatrix} \xi_p \\ \xi_q \end{pmatrix} + Bu.
\]

(2.5)

In order to decouple (2.5), we construct full column rank matrices \( \hat{p}_0^T \in \mathbb{R}^{n,n_p} \) and \( \hat{q}_0^T \in \mathbb{R}^{n,n_q} \) such that \( \text{span}(\hat{p}_0) = \text{Ker} q_0^T E_1^T \) and \( \text{span}(\hat{q}_0) = \text{Ker} p_0^T E_1^T \), that is, \( \hat{p}_0^T E_1 q_0 = -\hat{p}_0^T Aq_0 = 0 \) and \( \hat{q}_0^T E_1 p_0 = \hat{q}_0^T E p_0 = 0 \). This implies that \( \text{span}(\hat{p}_0) = \text{Ker} q_0^T A^T \) and \( \text{span}(\hat{q}_0) = \text{Ker} E^T \). This is due to the fact that \( \text{Ker} E^T \subset \text{Ker} p_0^T E^T \subset \text{Ker} E_1^T E^T \). We note that \( \hat{q}_0 = q_0 \) if \( E \) is symmetric. Finally, left multiplying (2.5) by \( \begin{pmatrix} \hat{p}_0^T \\ \hat{q}_0^T \end{pmatrix} \in \mathbb{R}^n \), leads to the decoupled system of (1.1a) given by

\[
E_p \xi_p = A_p \xi_p + B_p u,
\]

(2.6a)

\[
E_q \xi_q = A_q \xi_q + B_q u,
\]

(2.6b)

\[
y = C_p \xi_p + C_q \xi_q,
\]

(2.6c)

where \( E_p = \hat{p}_0^T E p_0 \in \mathbb{R}^{n_p,n_p} \), \( A_p = \hat{p}_0^T A p_0 \in \mathbb{R}^{n_p,n_p} \), \( B_p = \hat{p}_0^T B \in \mathbb{R}^{n_p,m} \), \( E_q = -\hat{q}_0^T A q_0 \in \mathbb{R}^{n_q,n_q} \), \( A_q = \hat{q}_0^T A p_0 \in \mathbb{R}^{n_q,n_p} \), \( B_q = \hat{q}_0^T B \in \mathbb{R}^{n_q,m} \) and
\( C_p = p_0^T C \in \mathbb{R}^{n_p \cdot \ell}, \quad C_q = q_0^T C \in \mathbb{R}^{n_q \cdot \ell}. \) The implicit differential and algebraic part of (1.1), that is, of (2.5), are (2.6a) and (2.6b), respectively. The desired output solution (1.1) can be obtained using (2.6c).

It can be easily proved that \( \sigma(E_p, A_p) = \sigma(f, A). \) Thus the differential part inherits the stability of the undecoupled DAE (1.1). We note that if we rewrite (2.6a) and (2.6b) explicitly, they coincide with those presented in [1].

The decoupling procedure for index-1 system is illustrated in the Example 1 below.

**Example 1** Consider a semi-explicit index-1 DAE with the following system matrices:

\[
E = \begin{pmatrix} E_{11} & E_{12} \\ 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}. \quad (2.7)
\]

We assume \( E_{11} \in \mathbb{R}^{n_1 \cdot n_1} \) and \( A_{21} E_{11}^{-1} E_{12} - A_{22} \in \mathbb{R}^{n_2 \cdot n_2} \) are non-singular blocks due to index-1 property and \( n = n_1 + n_2 \) is the dimension of the DAE. We can choose projectors \( Q_0 = \begin{pmatrix} 0 & -Q_{12} \\ 0 & I \end{pmatrix} \) and \( P_0 = \begin{pmatrix} I & Q_{12} \\ 0 & 0 \end{pmatrix}, \)

where \( Q_{12} = E_{11}^{-1} E_{12} \). Next, we compute \( E_1 = E_0 - A_0 Q_0 \) given by:

\[
E_1 = \begin{pmatrix} E_{11} & (I + A_{11} E_{11}^{-1} E_{12} - A_{12}) \\ 0 & A_{21} Q_{12} - A_{22} \end{pmatrix}.
\]

Since \( E_1 \) is nonsingular, this DAE is indeed an index-1 system. The bases of the projectors \( P_0 \) and \( Q_0 \), and their respective inverses are given by

\[
p_0 = \begin{pmatrix} I \\ 0 \end{pmatrix} \in \mathbb{R}^{n_1}, \quad q_0 = \begin{pmatrix} -Q_{12} \\ I \end{pmatrix} \in \mathbb{R}^{n_2}, \quad \text{and} \quad p_0^T = \begin{pmatrix} I & Q_{12} \end{pmatrix} \in \mathbb{R}^{n_1 \cdot n_2}, \quad q_0^T = \begin{pmatrix} 0 & I \end{pmatrix} \in \mathbb{R}^{n_2 \cdot n_1}. \quad (2.8)
\]

Then we can construct decoupling bases \( \hat{p}_0 \) and \( \hat{q}_0 \) such that \( \text{span}(\hat{p}_0) = \text{Ker} q_0^T A^T \) and \( \text{span}(\hat{q}_0) = \text{Ker} E^T \), given by

\[
\hat{p}_0^T = (I - (A_{12} - A_{11} Q_{12}) (A_{22} - A_{21} Q_{12})^{-1}) \quad \hat{q}_0^T = (0 \quad I). \quad (2.9)
\]

Substituting (2.7)–(2.9) into (2.6), we obtain an implicit decoupled system given by

\[
E_{11} \xi_p' = (A_{11} - (A_{12} - A_{11} Q_{12}) (A_{22} - A_{21} Q_{12})^{-1} A_{21}) \xi_p \\
+ (B_1 - (A_{12} - A_{11} Q_{12}) (A_{22} - A_{21} Q_{12})^{-1} B_2) u,
\]

\[
(A_{21} Q_{12} - A_{22}) \xi_q = A_{21} \xi_p + B_2 u,
\]

\[
y = C_1^T \xi_p + (C_2^T - C_1^T Q_{12}) \xi_q.
\]

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2.2 Implicit decoupling of index-2 systems

Next, we assume that (1.1) is an index-2 system, that is, \( \mu = 2 \). Thus (1.1) can be written in the equivalent projected form (1.3), with \( \mu = 2 \), that is,

\[
E_2 (P_1 P_0 x' + Q_0 x + Q_1 x) = A_2 x + Bu,
\]

where \( E_2 = E_1 - A_1 Q_1 \) is a non-singular matrix and \( A_2 = A_1 P_1 \). The matrices \( Q_0, P_0, E_1 \), and \( A_1 \) are as defined in the previous subsection. The projector \( Q_1 \) is chosen such that \( \text{Im} Q_1 = \text{Ker} E_1 \) and \( P_1 = I - Q_1 \) is its complementary projector. In order to ensure that the projector products are also projectors, we enforce that \( Q_1 Q_0 = 0 \) holds [4, 7].

In [2], we discussed that for higher-index DAEs there is a possibility of obtaining a purely algebraic decoupled system depending on the nature of the spectrum of the matrix pencil \( \sigma(E,A) = \sigma_f(E,A) \cup \sigma_\infty(E,A) \), where \( \sigma_f(E,A) \) and \( \sigma_\infty(E,A) \) is the set of the finite and infinite eigenvalues, respectively. We get a purely algebraic system when the matrix spectrum has only infinite eigenvalues, i.e., \( \sigma_f(E,A) = \emptyset \). For higher-index DAEs two cases may occur, as described in the following two subsections.

2.2.1 Index-2 systems with a differential part

In the first case, we assume that the matrix pencil \((E,A)\) of (1.1) has at least one finite eigenvalue, i.e., \( \sigma_f(E,A) \neq \emptyset \). We can construct a matrix of basis vectors \((p_0, q_0)\) in \( \mathbb{R}^n \) with inverse \((p_0, q_0)^{-1} = (p_*^0, q_*^0)^T\), for the projectors \( P_0 \) and \( Q_0 \) respectively, where \( p_0 \in \mathbb{R}^{n_0, n}, q_0 \in \mathbb{R}^{n_0,k_0}, p_0^T \in \mathbb{R}^{n_0,n} \) and \( q_0^T \in \mathbb{R}^{k_0,n} \). This leads to the following theorem [2].

**Theorem 1** Let \( P_{01} = p_0^T P_1 p_0, Q_{01} = p_0^T Q_1 p_0 \), then \( P_{01}, Q_{01} \in \mathbb{R}^{n_0,n_0} \) are projectors in \( \mathbb{R}^{n_0} \) provided the constraint condition \( Q_1 Q_0 = 0 \) holds.

Next, we construct another basis matrix \((p_{01}, q_{01})\) in \( \mathbb{R}^{n_0} \) made of \( n_p \) independent columns of projector \( P_{01} \) and \( k_1 \) independent columns of its complementary projector \( Q_{01} \) such that \( n_0 = n_p + k_1 \). According to [2], the dependent variable \( x \) can be decomposed as

\[
x = (p_0 p_{01} \quad p_0 q_{01} \quad q_0) \left( \begin{array}{c} \xi_p \\ \xi_{q,1} \\ \xi_{q,0} \end{array} \right),
\]
where $\xi_p \in \mathbb{R}^{n_p}, \xi_{q,1} \in \mathbb{R}^{k_1}, \xi_{q,0} \in \mathbb{R}^{k_0}$ and $n = n_p + k_1 + k_0$. Using this decomposition in (2.10) and simplifying, we obtain:

$$
\begin{pmatrix}
E_{2p_0p_01} - E_2Q_0Q_1p_0q_01 & 0
\end{pmatrix}
\begin{pmatrix}
\xi_p \\
\xi_{q,1} \\
\xi_{q,0}
\end{pmatrix}
\begin{pmatrix}
E_{2p_0p_01} & -E_2Q_0Q_1p_0q_01 & -E_2q_0
\end{pmatrix}
\begin{pmatrix}
\xi_p \\
\xi_{q,1} \\
\xi_{q,0}
\end{pmatrix}
+ Bu. \tag{2.12}
$$

We first construct $\hat{p}_0^T \in \mathbb{R}^{n_{o,n}}, \hat{q}_0^T \in \mathbb{R}^{k_{o,n}}$, such that $\hat{p}_0^TE_2q_0 = 0$ and $\hat{q}_0^TE_2p_0 = 0$. This implies that span $\hat{p}_0 = \text{Ker} q_0^TE_2$ and span $\hat{q}_0 = \text{Ker} p_0^TE_2$. We then construct $\hat{p}_0^T \in \mathbb{R}^{k_{1,n}}, \hat{q}_0^T \in \mathbb{R}^{k_{0,n}}$, such that $\hat{p}_0^T \hat{p}_0^TE_2p_0q_01 = 0$ and $\hat{q}_0^T \hat{q}_0^TE_2p_0q_01 = 0$. This implies that span $\hat{p}_0 = \text{Ker} (p_0^T E_2p_0q_01)^T$ and span $\hat{q}_0 = \text{Ker} (\hat{p}_0^T E_2p_0q_01)^T$. Multiplying (2.12) by $(\hat{p}_0 \hat{p}_01 \hat{q}_0 \hat{q}_01)^T$ and simplifying, we obtain the decoupled version of (1.1), given by:

$$
\begin{align*}
E_p\xi_p' &= A_p\xi_p + B_pu, \tag{2.13a} \\
E_{q,1}\xi_{q,1} &= A_{q,1}\xi_{q,1} + B_{q,1}u, \tag{2.13b} \\
E_{q,0}\xi_{q,0} &= A_{q,0}\xi_{q,0} + B_{q,0}u + A_{q,01}\left[\xi_{q,1}' - \xi_{q,1}\right], \tag{2.13c} \\
y &= C_p\xi_p + C_{q,1}\xi_{q,1} + C_{q,0}\xi_{q,0}. \tag{2.13d}
\end{align*}
$$

where

$$
\begin{align*}
E_p &= \hat{p}_0^T \hat{p}_0^T E_p p_01 \in \mathbb{R}^{n_p,n_p}, \quad A_p = \hat{p}_0^T \hat{p}_0^T A p_01 \in \mathbb{R}^{n_p,n_p}, \\
B_p &= \hat{p}_0^T \hat{p}_0^T B \in \mathbb{R}^{n_p,m}, \quad E_{q,1} = -\hat{q}_0^T \hat{q}_0^T A p_0 q_01 \in \mathbb{R}^{k_{1,k_{1}}}, \\
A_{q,1} &= \hat{q}_0^T \hat{q}_0^T A q_01 \in \mathbb{R}^{k_{1,n_p}}, \quad B_{q,1} = \hat{q}_0^T \hat{q}_0^T B \in \mathbb{R}^{k_{1,m}}, \\
E_{q,0} &= -\hat{q}_0^T A q_0 \in \mathbb{R}^{k_{0,k_{0}}}, \quad A_{q,0} = \hat{q}_0^T A p_0 q_01 \in \mathbb{R}^{k_{0,n_p}}, \\
B_{q,0} &= \hat{q}_0^T B \in \mathbb{R}^{k_{0,m}}, \quad A_{q,01} = -\hat{q}_0^T A p_0 q_01 \in \mathbb{R}^{k_{0,k_{1}}}, \\
C_p &= p_0^T \hat{p}_0^T C \in \mathbb{R}^{n_p,\ell}, \quad C_{q,1} = q_01 \hat{p}_0^T C \in \mathbb{R}^{k_{1,\ell}} \text{ and } C_{q,0} = q_0^T C \in \mathbb{R}^{k_{0,\ell}}.
\end{align*}
$$

Equations (2.13a), (2.13b) and (2.13c) are of dimension $n_p$, $k_1$ and $k_0$ respectively, where $n = n_p + k_1 + k_0$ is the dimension of DAE (1.1a). We observe that $n_p$ and $k_1 + k_0$ is the number of differential and algebraic equations, respectively. Thus system (2.13) preserves the dimension and stability of the DAE (1.1) since it can also be easily proved that $\sigma(E_p, A_p) = \sigma_f(E, A)$.
2.2.2 Index-2 systems without a differential part

In the second case, we assume that the matrix pencil \((E, A)\) of (1.1) has no finite eigenvalues, i.e., \(\sigma_f(E, A) = \emptyset\). This happens when \(P_0P_1 = 0\), thus (2.10) simplifies to

\[
E_2 \left( P_1 P_0 x' + Q_0 x + Q_1 x \right) = B u,
\]

since \(A_2 = A_0P_0P_1 = 0\). Thus \(x\) can be decomposed as

\[
x = \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \begin{pmatrix} \xi_{q,1} \\ \xi_{q,0} \end{pmatrix},
\]

where \(p_0\) and \(q_0\) form a matrix of basis vectors \((p_0, q_0)\) in \(\mathbb{R}^n\) with inverse \((p_0, q_0)^{-1} = (p_0^*, q_0^*)^T\), for the projectors \(P_0Q_1 = P_0\) and \(Q_0\) respectively, where \(p_0 \in \mathbb{R}^{n_0}, q_0 \in \mathbb{R}^{k_0}, p_0^T \in \mathbb{R}^{n_0,n}\) and \(q_0^T \in \mathbb{R}^{k_0,n}\). Then substituting (2.15) into (2.14) and simplifying, we obtain:

\[
(E_2 P_1 p_0 \ 0) \begin{pmatrix} \xi_{q,1} \\ \xi_{q,0} \end{pmatrix}' = -(E_2 Q_1 p_0 \ E_2 q_0) \begin{pmatrix} \xi_{q,1} \\ \xi_{q,0} \end{pmatrix} + B u.
\]

We then introduce \(\hat{p}_0^T \in \mathbb{R}^{n_0,n}\), \(\hat{q}_0^T \in \mathbb{R}^{k_0,n}\), such that \(\hat{p}_0^T E_2 q_0 = 0\) and \(\hat{q}_0^T E_2 Q_1 p_0 = 0\). This implies that \(\text{span} \hat{p}_0 = \text{Ker} q_0^T E_2^T\) and \(\text{span} \hat{q}_0 = \text{Ker} p_0^T Q_1^T E_2^T\). Multiplying (2.16) by \((\hat{p}_0 \ \hat{q}_0)^T\), leads to an implicit decoupled system, equivalent to (1.1), given by:

\[
\begin{align*}
E_{q,1} \xi_{q,1} &= B_{q,1} u, \\
E_{q,0} \xi_{q,0} &= B_{q,0} u + A_{q_0,1} \xi_{q,1}, \\
y &= C_{q,0} \xi_{q,1} + C_{q,0} \xi_{q,0},
\end{align*}
\]

where \(E_{q,1} = \hat{p}_0^T E_2 Q_1 p_0 \in \mathbb{R}^{n_0,n_0}, B_{q,1} = \hat{p}_0^T B \in \mathbb{R}^{n_0,m}, E_{q,0} = \hat{q}_0^T E_2 q_0 \in \mathbb{R}^{k_0,k_0}, B_{q,0} = \hat{q}_0^T B \in \mathbb{R}^{k_0,m}, A_{q_0,1} = -\hat{q}_0^T E_2 P_1 p_0 \in \mathbb{R}^{k_0,n_0}, C_{q,1} = \hat{p}_0^T C \in \mathbb{R}^{n_0,l}\) and \(C_{q,0} = \hat{q}_0^T C \in \mathbb{R}^{k_0,l}\). Here, \(n = n_0 + k_0\) is the dimension of the DAE (1.1a).

**Example 2** Consider a semi-explicit index-2 DAE with the following system matrices:

\[
E = \begin{pmatrix} E_{11} & 0 \\ 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & 0 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.
\]
We assume $E_{11} \in \mathbb{R}^{n_1 \times n_1}$ and $A_{12}^T E_{11}^{-1} A_{12} \in \mathbb{R}^{n_2 \times n_2}$ are non-singular blocks in order to satisfy the index-2 property and $n = n_1 + n_2$ is the dimension of the DAE. Since the matrix pencil $(E, A)$ is regular, we can choose projectors

\[ Q_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad Q_1 = \begin{pmatrix} E_{11}^{-1} A_{12} (A_{12}^T E_{11}^{-1} A_{12})^{-1} & 0 \\ 0 & 0 \end{pmatrix} \]  

such that $Q_1 Q_0 = 0$ and their respective complementary projectors are given by $P_i = I - Q_i$, $i = 0, 1$. The final matrix chain is given by

\[ E_2 = \begin{pmatrix} E_{11} - A_{11} E_{11}^{-1} A_{12} (A_{12}^T E_{11}^{-1} A_{12})^{-1} A_{12}^T & -A_{12} \\ -A_{12}^T E_{11}^{-1} A_{12} (A_{12}^T E_{11}^{-1} A_{12})^{-1} A_{12}^T & 0 \end{pmatrix} \]

(2.20)
since $E_2$ is non-singular thus, this DAE is indeed an index-2 system.

We have discussed how index-2 systems can be decoupled in two ways, at variance with the spectrum of the matrix pencil $(E, A)$. Therefore we shall consider both cases for this example. We have already seen that $P_0 P_1 = 0$ if and only if the spectrum of the matrix pencil has only infinite spectrum, otherwise it has both finite and infinite spectrum. For the case of system (2.18), $P_0 P_1 = 0$ if and only if $A_{12} (A_{12}^T E_{11}^{-1} A_{12})^{-1} A_{12}^T = E_{11}$. In order to decouple (2.18), we need to first construct the basis vectors for projector $Q_0$ and its complementary projector $P_0$ with their respective inverses given by

\[ q_0 = q_0^* = (0 \ 1)^T \quad \text{and} \quad p_0 = p_0^* = (1 \ 0)^T. \]

We decouple the system (1.1a), with matrices (2.18), in the following two possible cases:

(a) System with both differential and algebraic parts. In this case we assume that $A_{12} (A_{12}^T E_{11}^{-1} A_{12})^{-1} A_{12}^T \neq E_{11}$. Using (2.19), (2.21) and Theorem 1, we obtain:

\[ Q_{01} = E_{11}^{-1} A_{12} (A_{12}^T E_{11}^{-1} A_{12})^{-1} A_{12}^T, \quad P_{01} = I - Q_{01}. \]

(2.22)

It can be easily proved that $Q_{01}$ and $P_{01}$ are indeed projectors as expected. Then, we can obtain their respective vector bases given by $q_{01}$ and $p_{01}$, respectively. Following the procedure discussed in section 2.2.1, we obtain the decoupling basis vectors given by

\[ \text{span } \hat{p}_0 = \text{Ker } q_0^T E_2^T \quad \text{and} \quad \text{span } \hat{q}_0 = \text{Ker } p_0^T E_2^T. \]

(2.23)

Then finally we can obtain the decoupling projectors using $\text{span } \hat{p}_0 = \text{Ker } (\hat{p}_0^T E_2 p_0 q_0)^T$ and $\text{span } \hat{q}_0 = \text{Ker } (\hat{q}_0^T E_2 p_0 p_0)^T$. Hence substituting (2.18)-(2.23) into (2.13), we obtain the decoupled system of DAE (2.18) with a differential and algebraic part.
(b) System with only algebraic parts. This time, we assume the equality
\[ A_{12} (A_{12}^T E_{11}^{-1} A_{12})^{-1} A_{12}^T = E_{11}, \]
and (2.20) simplifies to
\[ E_2 = \begin{pmatrix} E_{11} - A_{11} & -A_{12} \\ -A_{12}^T & 0 \end{pmatrix}. \] (2.24)

In this case we just need the decoupling basis vector \( \hat{p}_0 \) and \( \hat{q}_0 \) which can be deduced from the previous case:
\[ \hat{p}_0 = \begin{pmatrix} \hat{p}_1 \\ 0 \end{pmatrix}, \quad \hat{p}_1 = \ker(A_{12}^T) \quad \text{and} \quad \hat{q}_0 = \begin{pmatrix} 0 \\ \hat{q}_2 \end{pmatrix}, \quad \hat{q}_2 = \ker(A_{12}). \] (2.25)

Hence substituting system (2.18), (2.24) and 2.25 into (2.17), we obtain the decoupled system of DAEs (2.18) with only an algebraic part.

3 Implicit IMOR method for DAEs

In this section, we describe what we call the Implicit IMOR method which we can abbreviate as IMOR method. This reduction follows the same procedure as the IMOR method which we proposed in [1, 2]. The main difference is in the splitting since the IMOR method leads to implicit systems while the IMOR method leads to explicit systems. The IMOR has the computational advantage that it does not involve matrix inversions which may be expensive for some practical problems. The IMOR method can be derived as follows.

We can observe that the decoupled systems (2.6) and (2.13) can be rewritten as:
\[ E_p \xi_p' = A_p \xi_p + B_p u \] (3.26a)
\[ -L \xi_q' = A_q \xi_p - L_q \xi_q + B_q u, \] (3.26b)
\[ y = C_p^T \xi_p + C_q^T \xi_q, \] (3.26c)

where \( L \) is a nilpotent matrix of index \( \mu = 1 \) and \( \mu = 2 \) for (2.6) and (2.13), respectively, and \( L_q \) is a non-singular lower block-triangular matrix, for \( \mu > 1 \). In this paper we consider index-1 and -2 systems, but also higher-index DAEs have the same structure (3.26).

For the case of index-1 systems, we have \( \xi_q \in \mathbb{R}^{n_q}, A_q \in \mathbb{R}^{n_q \times n_q}, B_q \in \mathbb{R}^{n_q \times m}, C_q \in \mathbb{R}^{n_q \times \ell}, L = 0 \in \mathbb{R}^{n_q \times n_q} \) (nilpotent matrix of index 1) and \( L_q := E_q \in \mathbb{R}^{n_q \times n_q} \).
For the case of index-2 systems, posing \( n_q = k_0 + k_1 \), we have
\[
\xi_q := (\xi_{q,1}^T, \xi_{q,0}^T)^T \in \mathbb{R}^{n_q}, A_q := (A_{q,1}^T, A_{q,0}^T)^T \in \mathbb{R}^{n_q,n_p}, B_q := (B_{q,1}^T, B_{q,0}^T)^T \in \mathbb{R}^{n_q,m}, C_q := (C_{q,1}^T, C_{q,0}^T)^T \in \mathbb{R}^{n_q,\ell},
\]
and
\[
L := \begin{pmatrix} 0 & 0 \\ A_{q,01} & 0 \end{pmatrix} \in \mathbb{R}^{n_q,n_q}, \quad L_q := \begin{pmatrix} E_{q,1}^T & 0 \\ A_{q,01}^T & E_{q,0} \end{pmatrix} \in \mathbb{R}^{n_q,n_q}.
\]
(3.27)

For comparison with (1.1), we can rewrite Equation (3.26) in descriptor form:
\[
\tilde{E} \xi' = \tilde{A} \xi + \tilde{B} u, \quad \xi(0) = \xi_0,
\]
(3.28a)
\[
y = C^T \xi,
\]
(3.28b)
where \( \tilde{E} := \begin{pmatrix} E_p & 0 \\ 0 & -L \end{pmatrix}, \tilde{A} := \begin{pmatrix} A_p & 0 \\ A_q & -L_q \end{pmatrix}, \tilde{B} := \begin{pmatrix} B_p \\ B_q \end{pmatrix}, \tilde{C} := \begin{pmatrix} C_p \\ C_q \end{pmatrix} \)
and \( \xi := \begin{pmatrix} \xi_p \\ \xi_q \end{pmatrix} \). It can be easily proved that (1.1) and (3.28) are equivalent. Thus their solutions coincide, although the latter is easier to solve and to reduce than the former, since it reveals the interconnection of the DAEs.

We take the Laplace transform of (3.28), using capital letters for the Laplace transform of variables with corresponding small letters, and set \( \xi_p(0) = 0 \), since it can be chosen arbitrarily. Then we can compute explicitly the transform of the output, \( Y(s) \), at variance with the transform of the input, \( U(s) \):
\[
Y(s) = H(s)U(s) + Y_0(s).
\]
(3.29)

\( H(s) \) is known as the transfer function of the system, and \( Y_0(s) \) is a term which is usually neglected, because it does not occur for systems with index less than 2. In details, \( H(s) = H_p(s) + H_q(s) \), where
\[
H_p(s) := C_p^T h_p(s), \quad \text{with} \quad h_p(s) := (sE_p - A_p)^{-1}B_p,
\]
(3.30)
is the differential component, and
\[
H_q(s) := C_q^T (L_q - sL)^{-1} [A_q h_p(s) + B_q]
\]
(3.31)
the algebraic component of the transfer function. We note that this transfer function coincides with that of system (1.1). As for the additional term \( Y_0(s) \), we have
\[
Y_0(s) = -C_q^T (L_q - sL)^{-1}L \xi_q(0).
\]
(3.32)
From (3.29), we observe that for index-1 systems, we can always predict the input-output behavior of the DAE if we know the transfer function $H(s)$, since $L = 0$, thus $Y_0(s) = 0$. This is not true for index-2 systems, unless $L\xi_q(0) = 0$, which implies that $Y_0(s) = 0$. The initial data $\xi_q(0)$ cannot be imposed arbitrarily, but they are given by the algebraic part of the system. A bit of algebra yields $L\xi_q(0) = LL^{-1}B_qu(0)$ (see (3.38) below), so the most critical matrix is $LL^{-1}B_q$. This explains the failure of conventional MOR methods when this matrix is not zero. This is well discussed in [2] in the context of IMOR, with $L_q = I$. For this reason instead of using the transfer function $H(s)$ as the validity tool for accuracy of reduced-order model for DAEs, we need to use (3.29) written as $Y(s) = H(s)U(s) + Y_0(s)$, which we call the output-transfer function. This output-transfer function idea was proposed in [3].

3.1 Reduction procedure

Systems of DAEs contain interconnected subsystems and these subsystems have to be reduced separately. As in the IMOR method [1, 2], also in this approach we first split (3.26) into:

\begin{align}
E_p\xi'_p &= A_p\xi_p + B_pu,
\quad \text{(3.33a)} \\
y_p &= C_p^T\xi_p,
\quad \text{(3.33b)}
\end{align}

and

\begin{align}
-L\xi'_q &= A_q\xi_p - L_q\xi_q + B_qu,
\quad \text{(3.34a)} \\
y_q &= C_q^T\xi_q,
\quad \text{(3.34b)}
\end{align}

where (3.33) and (3.34) are the differential and algebraic subsystems of the control problem (1.1). Then the output equation can be reconstructed using:

\[ y = y_p + y_q. \quad \text{(3.35)} \]

Next, we reduce the differential and algebraic parts separately.

3.1.1 Reduction of the differential part

Consider the subsystem (3.33). We can use the conventional MOR methods as follows: choose an expansion point $s_0 \in \mathbb{C} \setminus \sigma(E_p, A_p)$ and then construct an order-$r$ Krylov subspace generated by $M_p$ and $R_p$ given by:

\[ V_p := \mathcal{K}_r(M_p, R_p) = \text{span}\{R_p, M_pR_p, \ldots, M_p^{r-1}R_p\}, \quad r \leq n_p, \]
where $M_p := (s_0E_p - A_p)^{-1}E_p$, $R_p := (s_0E_p - A_p)^{-1}B_p$. Then, $V_p \in \mathbb{R}^{n_p \times r}$ denotes the orthonormal basis matrix of the above Krylov subspace, so that $V_p^T V_p = I$. The reduced-order subsystem is obtained by using the approximation $\xi_p = V_p \hat{\xi}_p$, leading to a reduced-order subsystem:

\begin{align}
\hat{E}_p \hat{\xi}_p' &= \hat{A}_p \hat{\xi}_p + \hat{B}_p u, \\
\hat{y}_p &= \hat{C}_p^T \hat{\xi}_p,
\end{align}

where $\hat{E}_p = V_p^T E_p V_p$, $\hat{A}_p = V_p^T A_p V_p \in \mathbb{R}^{r \times r}$, $\hat{B}_p = V_p^T B_p \in \mathbb{R}^{r \times m}$ and $\hat{C}_p = V_p^T C_p \in \mathbb{R}^{r \times p}$. $\hat{\xi}_p \in \mathbb{R}^r$ is the reduced state vector, $\hat{y}_p \in \mathbb{R}^p$ is the approximated output. Thus the dimension of the differential part is reduced to $r \leq n_p$. The transfer function of this reduced-order model of the differential part (3.33) is given by

$$
\hat{H}_p(s) = \hat{C}_p^T \hat{h}_p(s), \quad \text{with} \quad \hat{h}_p(s) = (s\hat{E}_p - \hat{A}_p)^{-1}\hat{B}_p.
$$

We can observe that the differential part of the DAE is reduced but the order of the algebraic part is still unchanged.

### 3.1.2 Reduction of the algebraic part

We now intend to show that the reduction on the differential part induces a reduction on the algebraic part, obtained by (3.34). We start from this equation, which we write as

$$
L_q \xi_q = N_q L_q \xi_q' + b_q := (LL_q^{-1}) L_q \xi_q' + (A_q \xi_p + B_q u).
$$

The matrix $N_q$ is nilpotent with the same index $\mu$ of $L$, so we obtain recursively

$$
L_q \xi_q = N_q (N_q L_q \xi_q' + b_q)' + b_q = \cdots = \sum_{k=0}^{\mu-1} N_q^k b_q^{(k)}
$$

$$
= \sum_{k=0}^{\mu-1} N_q^k (A_q \xi_p^{(k)} + B_q u^{(k)}).
$$

Recalling the reduction of the differential part of the system, which confines $\xi_p$ to the subspace $V_p$, spanned by $V_p$, then also $\xi_p^{(k)}$, $k = 1, \ldots, \mu - 1$, belongs to the same subspace. Thus, for the algebraic variable $\xi_q$ we have the restriction

$$
L_q \xi_q \in W_\mu(N_q, R_q),
$$

13
with \( R_q = (A_qV_p \quad B_q) \in \mathbb{R}^{n_q \times r + m} \). Since \( N_q = LL^{-1} \), it follows that

\[
\xi_q \in \mathcal{V}_q = L_q^{-1}\mathcal{W}_q = \mathcal{K}_\mu(L_q^{-1}L, L_q^{-1}R_q).
\]

We denote by \( V_q \) an orthonormal basis of \( \mathcal{V}_q \), so that \( V_q^TV_q = I \), and we write \( \xi_q = V_q\hat{\xi}_q \). Then, the reduced algebraic system is:

\[
\begin{align*}
\hat{L}\hat{\xi}'_q &= \hat{A}_q\hat{\xi}_q - \hat{L}_q\hat{\xi}_q + \hat{B}_qu, \\
\hat{y}_q &= \hat{C}_q^T\hat{\xi}_q,
\end{align*}
\]

(3.41)

with \( \hat{L}_q = V_q^TL_qV_q \), \( \hat{L} = V_q^TLV_q \), \( \hat{A}_q = V_q^TA_qV_p \), \( \hat{B}_q = V_q^TB_q \) and \( \hat{C}_q = V_q^TC_q \). The transfer function of this reduced-order model of the algebraic part (3.34) is defined by the relation

\[
\hat{Y}_q(s) = \hat{H}_q(s)U(s) + \hat{Y}_0(s),
\]

(3.42)

with

\[
\begin{align*}
\hat{H}_q(s) &:= \hat{C}_q^T(\hat{L}_q - s\hat{L})^{-1}(\hat{A}_q\hat{h}_p(s) + \hat{B}_q), \\
\hat{Y}_0(s) &:= -\hat{C}_q^T(\hat{L}_q - s\hat{L})^{-1}\hat{L}\xi_q(0).
\end{align*}
\]

(3.43) (3.44)

**Remark 1** The main difference between the IMOR and IIMOR methods is that IMOR method leads to explicit decoupled reduced-order models while IIMOR method leads to implicit decoupled reduced-order models. We note that these two methods coincide if and only if \( E_p = I \) and \( L_q = I \) in (3.26).

The IIMOR method is computationally cheaper than the IMOR method since its decoupling procedure does not involve matrix inversions as discussed in section 2. Although the IMOR models are sparser than the IIMOR models. Hence we need to trade off between sparsity and computational cost.

If we combine the reduced-order subsystems (3.36) and (3.41), we can obtain the reduced model of the decoupled system (3.28). We observe that the projected state space solution is approximated by \( \xi = V\hat{\xi} \), where \( V = \begin{pmatrix} V_p & 0 \\ 0 & V_q \end{pmatrix} \). Hence substituting \( \xi = V\hat{\xi} \) into (3.28) and left multiplying by \( V^T \), we obtain the IIMOR model for DAE (1.1) given by:

\[
\begin{align*}
\hat{E}\hat{\xi}' &= \hat{A}\hat{\xi} + \hat{B}u, \\
\hat{y} &= \hat{C}^T\hat{\xi},
\end{align*}
\]

(3.45a) (3.45b)
where $\hat{E} = V^T \bar{E} V$, $\hat{A} = V^T \bar{A} V$, $\hat{B} = V^T \bar{B}$ and $\hat{C} = V^T \bar{C}$. As expected, its output-transfer function can be written as:

$$\hat{Y}(s) = \hat{H}(s)U(s) + \hat{Y}_0(s),$$

(3.46)

where $\hat{H}_q(s) = \hat{H}_p(s) + \hat{H}_q(s)$. Thus the reduced-order model is accurate if and only if approximation error in the output transfer function $\|Y - \hat{Y}\|$ is very small in the suitable norm. This can be written as

$$\|Y(s) - \hat{Y}(s)\| \leq \|H(s) - \hat{H}(s)\| \|U(s)\| + \|Y_0(s) - \hat{Y}_0(s)\|. \quad (3.47)$$

In [3], it is proposed that the output-transfer function of the IMOR reduced-order model has a small approximation error if and only if

(a) $\|H - \hat{H}\|$ is very small

(b) and $\|Y_0(s) - \hat{Y}_0(s)\|$ is also very small.

The above same conditions have to be satisfied by the IIMOR reduced-order model. We note that for the case of index-1 systems the reduced-order models just need to satisfy condition (a) since $Y_0(s) = 0$ always. As we show in the next subsection, this is also true for the case of index-2 system, if we choose $r \geq 2$, which is reasonable.

### 3.2 Properties of the implicit IMOR method

The IIMOR method possesses many key MOR properties, such as stability and moment-matching properties, as it did its counterpart the IMOR method. The theoretical prove for these properties can be done as for the case of IMOR method in [3]. In this paper, we restrict ourselves to the moment matching property. This can be done as follows.

By construction, the first $r$ moments of the differential part of the transfer function, $H_p(s)$, are preserved. To see this, we write the expansions

$$H_p(s) = C_p^T \sum_{k=0}^{\infty} h_{p,k}(s_0)(s-s_0)^k, \quad \hat{H}_p(s) = \hat{C}_p^T \sum_{k=0}^{\infty} \hat{h}_{p,k}(s_0)(s-s_0)^k, \quad (3.48)$$

$$h_{p,k}(s_0) = (-1)^k M_p^k R_p, \quad \hat{h}_{p,k}(s_0) = (-1)^k \hat{M}_p^k \hat{R}_p, \quad (3.49)$$

with $\hat{M}_p := (s_0 \hat{E}_p - \hat{A}_p)^{-1} \hat{E}_p$, $\hat{R}_p := (s_0 \hat{E}_p - \hat{A}_p)^{-1} \hat{B}_p$.

**Theorem 2**

$$\hat{C}_p^T \hat{h}_{p,k}(s_0) = C_p^T h_{p,k}(s_0), \quad k = 0,1,\ldots,r-1. \quad (3.50)$$
Proof. We only give a sketch of the proof, since it can be found in literature [11]. It is possible to prove that
\[ \hat{h}_{p,k} = V_p^T h_{p,k}, \] (3.51)
and by construction, since \( h_{p,k} \in \mathcal{V}_p \) for \( k = 0, 1, \ldots, r-1 \), we have
\[ V_p V_p^T h_{p,k} = h_{p,k}, \quad k = 0, 1, \ldots, r-1, \] (3.52)
which implies (3.50). \( \square \)

Next, we need to show that the first moments of the transfer function \( \hat{H}_q(s) \) around \( s_0 \) of the approximated system (3.41) are equal to the first moments of the transfer function \( H_q(s) \) given by (3.31).

First, we need to write explicitly the moments of \( \hat{H}_q(s) \) and \( H_q(s) \) around \( s = s_0 \). The two computations follow along the same line, since the original and reduced system have the same structure.

To write explicitly the moments of \( H_q(s) \) around \( s = s_0 \), we start from the definition (3.31), and use the expansion of \( (sE_p - A_p)^{-1}B_p \) found in (3.48)-(3.49), and the identity
\[ (L_q - sL)^{-1} = L_q^{-1}(I - sN_q)^{-1} = L_q^{-1}\sum_{j=0}^{\mu-1} N_q^j s^j. \] (3.53)

In this way we find
\[ H_q(s) = C_q^T L_q^{-1}\sum_{j=0}^{\mu-1} N_q^j s^j \left( A_q \sum_{k=0}^{\infty} h_{p,k}(s_0)(s - s_0)^k + B_q \right). \] (3.54)

Finally, expanding \( s^j = (s_0 + (s - s_0))^j \), we obtain
\[ H_q(s) = C_q^T L_q^{-1}\sum_{j=0}^{\mu-1} N_q^j \sum_{i=0}^{\mu-1} \binom{j}{i} s_0^i(s - s_0)^{j-i} \left( A_q \sum_{k=0}^{\infty} h_{p,k}(s_0)(s - s_0)^k + B_q \right), \]
which gives
\[ H_q(s) = \sum_{\ell=0}^{\infty} H_{q,\ell}(s_0)(s - s_0)^\ell, \] (3.55)
with
\[
H_{q,\ell}(s_0) = C_q^T h_{q,\ell}(s_0) = C_q^T (h'_{q,\ell}(s_0) + h''_{q,\ell}(s_0)),
\]
(3.56)
\[
h'_{q,\ell}(s_0) = \sum_{k+j-i=\ell}^{0 \leq i \leq j \leq \mu-1} \binom{j}{i} s_0^{i} L_{q}^{-1} N_{q}^{j} A q h_{p,k}(s_0)
\]
(3.57)
\[
h''_{q,\ell}(s_0) = \begin{cases} 
\sum_{j-i=\ell}^{0 \leq i \leq j \leq \mu-1} \binom{j}{i} s_0^{i} L_{q}^{-1} N_{q}^{j} B q, & \text{if } \ell \leq \mu - 1, \\
0, & \text{otherwise}.
\end{cases}
\]
(3.58)

For the reduced algebraic system (3.41) we get a similar expression for the moments of the algebraic part of the transfer function, with the matrices replaced by their hatted counterpart. Thus the moments of the reduced algebraic system are
\[
\hat{H}_{q,\ell}(s_0) = \hat{C}_q^T \hat{h}_{q,\ell}(s_0) = \hat{C}_q^T (\hat{h}'_{q,\ell}(s_0) + \hat{h}''_{q,\ell}(s_0)),
\]
(3.59)
\[
\hat{h}'_{q,\ell}(s_0) = \sum_{k+j-i=\ell}^{0 \leq i \leq j \leq \mu-1} \binom{j}{i} s_0^{i} \hat{L}_{q}^{-1} \hat{N}_{q}^{j} \hat{A} q \hat{h}_{p,k}(s_0)
\]
(3.60)
\[
\hat{h}''_{q,\ell}(s_0) = \begin{cases} 
\sum_{j-i=\ell}^{0 \leq i \leq j \leq \mu-1} \binom{j}{i} s_0^{i} \hat{L}_{q}^{-1} \hat{N}_{q}^{j} \hat{B} q, & \text{if } \ell \leq \mu - 1, \\
0, & \text{otherwise}.
\end{cases}
\]
(3.61)

As for the differential part, we can prove that the moments of the algebraic part of the transfer function are preserved.

**Theorem 3**
\[
\hat{H}_{q,\ell}(s_0) = H_{q,\ell}(s_0), \quad \ell = 0, 1, \ldots, r - 1.
\]
(3.62)

**Proof.** It is possible to show that
\[
\hat{h}_{q,\ell}(s_0) = V_q^T h_{q,\ell}(s_0), \quad \ell = 0, 1, \ldots, r - 1.
\]
(3.63)
Then, observing that \( h_{q,\ell}(s_0) \in \mathcal{V}_q \), for \( \ell = 0, 1, \ldots, r - 1 \), we have by construction
\[
V_q V_q^T h_{q,\ell}(s_0) = h_{q,\ell}(s_0), \quad \ell = 0, 1, \ldots, r - 1,
\]
(3.64)
which, together with (3.63), implies (3.62).
To prove (3.63), we need to consider separately $h'_{q,\ell}(s_0)$ and $h''_{q,\ell}(s_0)$. We observe that (3.64), implies that

\begin{align}
V_q V_q^T L_q^{-1} N_q^j A_q h_{p,k}(s_0) &= L_q^{-1} N_q^j A_q h_{p,k}(s_0), \quad k+j \leq r-1, \\
V_q V_q^T L_q^{-1} N_q^j B_q &= L_q^{-1} N_q^j B_q, \quad j \leq \mu - 1.
\end{align}

(3.65) \quad (3.66)

First we show that

\begin{equation}
\hat{L}_q^{-1} \hat{N}_q^j \hat{A}_q \hat{h}_{p,k}(s_0) = V_q T_q L_q^{-1} N_q^j A_q h_{p,k}(s_0). \tag{3.67}
\end{equation}

The above equality is equivalent to

\begin{equation}
\hat{N}_q^j \hat{A}_q \hat{h}_{p,k}(s_0) = \hat{L}_q V_q T_q L_q^{-1} N_q^j A_q h_{p,k}(s_0),
\end{equation}

that is, using (3.65), equivalent to

\begin{align}
\hat{L} \hat{L}_q^{-1} \hat{N}_q^j \hat{A}_q \hat{h}_{p,k}(s_0) &= V_q^T L_q V_q V_q^T L_q^{-1} N_q^j A_q h_{p,k}(s_0) \\
&= V_q^T L_q V_q V_q^T L_q^{-1} N_q^j A_q h_{p,k}(s_0) \\
&= V_q^T L_q V_q V_q^T L_q^{-1} N_q^j A_q h_{p,k}(s_0).
\end{align}

(3.68)

Then, by induction, (3.67) holds for arbitrary $j$ (provided the limitations for the validity of (3.65)) if it holds for $j = 0$. The induction basis is thus

\begin{equation}
\hat{L}_q^{-1} \hat{A}_q \hat{h}_{p,k}(s_0) = V_q T_q L_q^{-1} A_q h_{p,k}(s_0), \tag{3.69}
\end{equation}

which is equivalent to

\begin{align}
\hat{A}_q \hat{h}_{p,k}(s_0) &= \hat{L}_q V_q T_q L_q^{-1} A_q h_{p,k}(s_0) \\
&= V_q^T L_q V_q V_q^T L_q^{-1} A_q h_{p,k}(s_0) \\
&= V_q^T A_q h_{p,k}(s_0),
\end{align}

which is satisfied due to (3.51). This concludes the proof of (3.67), which implies

\begin{equation}
\hat{h}'_{q,\ell}(s_0) = V_q^T h'_{q,\ell}(s_0), \quad \ell = 0,1,\ldots,r-1. \tag{3.69}
\end{equation}

Proceeding in a similar way it is possible to prove

\begin{equation}
\hat{h}''_{q,\ell}(s_0) = V_q^T h''_{q,\ell}(s_0), \quad \ell = 0,1,\ldots,r-1, \tag{3.70}
\end{equation}

and thus (3.63). \hfill \Box
As a byproduct of the decomposition of the algebraic part, we get also a control of the output-transfer function, including the term $Y_0(s)$. We find

$$Y_0(s) = -C_q^T(L_q - sL)^{-1}N_qB_qu(0)$$

$$= -C_q^T \sum_{j=0}^{\mu-2} \sum_{i=0}^{j} \binom{j}{i} s_0^i L_q^{-1} N_q^{j+1} B_q(s - s_0)^{j-i} u(0), \quad (3.71)$$

and in a similar way

$$\hat{Y}_0(s) = -\hat{C}_q^T \sum_{j=0}^{\mu-2} \sum_{i=0}^{j} \hat{\binom{j}{i}} s_0^i \hat{L}_q^{-1} \hat{N}_q^{j+1} \hat{B}_q(s - s_0)^{j-i} u(0), \quad (3.72)$$

**Corollary 1** If $r \geq \mu$, we have

$$\hat{Y}_0(s) = Y_0(s). \quad (3.73)$$

### 4 Numerical experiments

In this section, we numerically compare the IMOR method and newly proposed IIMOR method which is an implicit version of the IMOR method previously proposed in [1, 2]. We use the PRIMA method to reduce the differential part, thus both methods can have the same order of the differential part but different reduced order of the algebraic part. We tested these methods using the index-1 Example 3 and the index-2 Example 4 below. Both examples originate from the computational fluid dynamics (CFD) community.

**Example 3** (Active control of a supersonic engine inlet.)

Consider the Euler equations modelling the unsteady flow through a supersonic diffuser as described in [18]. Linearization around a steady-state solution and spatial discretization using a finite volume method leads to a semi-explicit descriptor system of the form (2.7) of dimension $n = 11730$ and the CFD model had 3078 grid points. This is an index-1 system of $m = 2$ inputs and $\ell = 1$ output. According to [18], the reduced-order model must capture the dynamics of the output: the average Mach number at the diffuser throat in response to two inputs: the incoming flow disturbance and the bleed actuation. According to [18], are two transfer functions of interest in this problem. Thus the problem can be viewed as 2 single input single output (SISO) subsystems and the frequencies of practical interest lie in the range $\frac{f}{f_0} = 0$ to $\frac{f}{f_0} = 2$, where $f_0 = \frac{a_0}{h}$, $a_0$ is the freestream speed of sound.
and $h$ is the height of the diffuser. We decoupled this subsystems system into $n_p = 11323$ differential equations and $n_q = 407$ algebraic equations using both implicit decoupling method discussed in section 2.1 and the explicit decoupling method presented in [1]. We note the decoupled systems of this DAE subsystems takes the form (2.6) since they have the same matrix pencil $(E, A)$. We observed that the implicit decoupling procedure is computationally cheaper than it counter part because it does not involves computing the inverse of $E_1$ which is computationally very expensive. Figure 1 and 2 show the sparsity of the matrix pencil of the implicit and explicit decoupled system in descriptor form. We observe that the implicit decoupling procedure leads to a sparser matrix $\tilde{A}$ than the matrix $\hat{A}$ of the explicit decoupling procedure.

Next we compared the IMOR, IIMOR and PRIMA method on these two subsystems, in both case we used $s_0 = 0$ as the expansion point. We reduced both subsystems into 15 differential and 16 algebraic equations, using IMOR and IIMOR methods. Thus the DAEs were reduced from 11730 to 31 dimension. We then compared these reduced-order models with the PRIMA reduced-order models reduced to 15 dimensions.

![Figure 1: Sparsity of matrix pencil (̂E, ̂A)](image)

In Figure 3, we compare the magnitude of the transfer function and its approximation error from bleed actuation to average throat Mach number for supersonic diffuser. We observed that all reduced-order models are accurate in the desired frequencies but the IIMOR and PRIMA reduced-order models are more accurate than the IMOR method.

In Figure 5, we compare the magnitude of the transfer function and its approximation error from the incoming flow disturbance to average throat Mach number for supersonic diffuser. We also observed that all the reduced-order models are accurate in the desired low frequencies. The IIMOR method is still slightly more accurate than the IMOR methods but the
Figure 2: Sparsity of matrix pencil $(\tilde{E}, \tilde{A})$

Figure 3: Transfer function from bleed actuation to average throat Mach number for supersonic diffuser.

Figure 4: Approximation error of the Transfer function from bleed actuation to average throat Mach number for supersonic diffuser.
PRIMA method is the most accurate for this example. However the PRIMA method leads to ODE reduced-order models, thus it does not always preserve the index of the DAEs. The IIMOR and IMOR methods always preserve the index of the DAE, thus we can always guarantee the accuracy of the reduced-order models.

Figure 5: Transfer function from incoming flow disturbance to average throat Mach number for supersonic diffuser.

![Figure 5: Transfer function from incoming flow disturbance to average throat Mach number for supersonic diffuser.](image)

Figure 6: Approximation error of the Transfer function from incoming flow disturbance to average throat Mach number for supersonic diffuser.

![Figure 6: Approximation error of the Transfer function from incoming flow disturbance to average throat Mach number for supersonic diffuser.](image)
Example 4 In this example, we apply the IMOR and IIMOR methods on the semidiscretized Stokes problem as described in [19]. Consider the instationary stokes equation describing the flow of an incompressible fluid

\[
\frac{\partial \mathbf{v}}{\partial t} = \Delta \mathbf{v} - \nabla \rho + \mathbf{f}, \quad (\zeta, t) \in \Omega \times (0, T)
\]

\[
0 = \text{div} \mathbf{v},
\]

(4.74)

with appropriate initial condition and boundary condition. Here \( \mathbf{v}(\zeta, t) \in \mathbb{R}^d \) is the velocity vector (\( d = 2 \) or \( 3 \) is the dimension of the spatial domain), \( \rho(\zeta, t) \in \mathbb{R} \) is the pressure, \( \mathbf{f}(\zeta, t) \in \mathbb{R} \) is the vector of external forces, \( \Omega \in \mathbb{R}^d \) is a bounded open domain and \( T > 0 \) is the endpoint of the time interval.

The spatial discretization of the Stokes equation (4.74) by either the finite difference or finite element methods on a uniform staggered grid leads to a DAE of the form

\[
\mathbf{v}_h(t)' = A_{11} \mathbf{v}_h(t) + A_{12} \rho_h(t) + B_1 u(t),
\]

\[
0 = A_{12}^T \mathbf{v}_h(t) + B_2 u(t),
\]

\[
y = C_1^T \mathbf{v}_h(t) + C_2^T \rho_h(t),
\]

(4.75)

where \( \mathbf{v}_h \in \mathbb{R}^{n_1} \) and \( \rho_h \in \mathbb{R}^{n_2} \) are the semidiscretized vectors of velocity and pressure, respectively, see [19]. We can observe that if we write (4.75) in descriptor form, it takes the form (2.18), where \( E_{11} = I \in \mathbb{R}^{n_1,n_1}, A_{11} \in \mathbb{R}^{n_1,n_1} \) is the discrete Laplace operator, \( -A_{12} \in \mathbb{R}^{n_1,n_2} \) and \( -A_{12}^T \in \mathbb{R}^{n_2,n_1} \) are the discrete gradient and divergence operators, respectively. Due to the non-uniqueness of the pressure, the matrix \( A_{12} \) has a rank defect one. In this case, instead of \( A_{12} \) we can take a full column rank matrix obtained from \( A_{12} \) by discarding the last column. Therefore, in the following we will assume without loss of generality that \( A_{12} \) has full column rank. In this case system with matrix coefficients (4.75) is of index-2. The matrices \( B_1 \in \mathbb{R}^{n_1,m}, B_2 \in \mathbb{R}^{n_2,m} \) and the control input \( u(t) \in \mathbb{R}^m \) are the resulting from the boundary condition and external forces, the output \( y(t) \in \mathbb{R}^\ell \) is the vector of interest. The order \( n = n_1 + n_2 \) of system (4.75) depends on the level of refinement of the discretization and is usually very large, whereas the number \( m \) of inputs and the number \( \ell \) of outputs are typically small.

We performed a spatial discretization of the Stokes equation (4.74) on a square domain \( \Omega = [0, 1] \times [0, 1] \) by the finite volume method on a uniform staggered grid. In order to compare the computational cost of the implicit and explicit decoupling methods, we carried out experiments on different grid sizes as shown in Table 1. From Table 1, we observe that as the mesh...
becomes finer the larger the problem becomes and hence solving the problem becomes computationally more expensive. We can also observe that both methods were able to decouple the problem but the implicit method is computationally cheaper than the explicit method as expected since it does not involve matrix inversion of matrix $E_2$.

Table 1: Comparison of the computational cost

<table>
<thead>
<tr>
<th>Grid</th>
<th>Order</th>
<th>Decoupled model</th>
<th>Computational cost</th>
</tr>
</thead>
<tbody>
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<td>$n$</td>
<td>$n_p$</td>
<td>$k_1$</td>
</tr>
<tr>
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<td>9295</td>
<td>3025</td>
<td>3135</td>
</tr>
<tr>
<td>52 x 52</td>
<td>8007</td>
<td>2601</td>
<td>2703</td>
</tr>
</tbody>
</table>

We then reduced the decoupled Stokes problems using the IMOR and IIMOR methods. We applied the IMOR and IIMOR method, at the explicit and implicit decoupled problems, respectively as shown in Table 2. We used the PRIMA method and using $s_0 = 0$ as expansion point to reduced the differential part. The differential and algebraic equations are reduced to order $r$ and $\tau$, respectively and $r + \tau$ is the order of the reduced -order DAE. We observe that the IMOR method takes less time than the IIMOR method this is due to the inversion of $L_q$ but this is small compared to the time it takes to generate the explicit decoupled system. We used the system matrices from grid $52 \times 52$ to compare the transfer function and the phase angle of the IMOR model and IIMOR model with that of the original as shown in Figure 7. We can observe that the transfer function and phase angle of the IMOR, IIMOR and original models coincides. However the IMOR model is more accurate than the IIMOR model as shown by the approximation error plot in Figure 9.

Table 2: Comparison of the IMOR methods

<table>
<thead>
<tr>
<th>Grid</th>
<th>Order</th>
<th>Decoupled model</th>
<th>IMOR model</th>
<th>IIMOR model</th>
</tr>
</thead>
<tbody>
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<td>$n_p$</td>
<td>$n_q$</td>
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<td>5406</td>
<td>3599</td>
<td>22</td>
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</tbody>
</table>
We finally compared the solutions of the reduced-order models with that of the original model. From Figure 9, we observe that the solutions of the reduced-order models coincides with that of the original model with a small approximation error as shown in Figure 10. Both reduced model took 10 seconds while the original model took 148 seconds. Thus the decoupling techniques also makes solving much cheaper.
5 Conclusion

In conclusion, the IIMOR method is computationally cheaper than the IMOR method. However it is less accurate than the IMOR method. Hence one has to trade off between accuracy and computational cost.
References


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