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SENTENCES IN THE LANGUAGE OF REASONING

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Abstract

In the present paper we investigate a few grammatical aspects of texts in which a reasoning has been expressed. We are mainly interested in text units (called *sentences*) that are constituent parts of a reasoning.

Firstly we discuss the full conceptual framework reflected in the reasoning: its *underlying form*. Next we divide sentences into two main classes: *fundamental* and *informative* sentences. Each of these main classes will thereupon be subdivided. In classifying we are guided by the manner in which sentences function in a reasoning. The nature of each class is elucidated by a discussion and examples.

We subsequently deal with compound sentences and sentence groups. Finally we analyse two texts according to the grammatical classification principles explained in the paper.

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Introduction

Presently many attempts are being made to insert mathematical ideas into linguistic matters. Well-known topics in this area are the transformational grammars and the Montague-grammar. Examples of other fields in mathematics that bear a strong relation to linguistics are logic, traditionally connected to language, and the new field of artificial intelligence.

A fast developing area, inspired by computer science, is the field of mathematical languages. Such languages may serve a number of purposes. An example is the mathematical language Automath (see reference [2]), which is suited to represent mathematical texts in a most meticulous manner. The language that mathematicians use for written communication has been codified in De Bruijn's design of M.V., that is the Mathematical Vernacular (see [3]).

This paper analyses a few logical-mathematical aspects of a linguistic subfield, namely, the language of reasoning. The main purpose of this paper is to make a classification of sentences that occur in a reasoning text. In classifying we are guided by the logical background of a reasoning, which is its essential frame because it accounts for the cogency.

1. Underlying forms

A text expressing a reasoning is a - more or less adequate - reflection of a line of thought. Inspection shows that such texts are composed of elementary text units fitting into an overall structure. One may say that these text units represent units of thought, whereas the binding structure is a reflection of a logical pattern of some sort.

Nevertheless, a reasoning text is not more (and is usually not meant to be more) than a description: it only expresses a few links in the chain of reasoning, namely those links that are considered essential information. Arguments are often omitted, rules applied without mentioning them; substitutions are silently executed. The author of a text usually confines himself to giving a rough skeleton of structure and component parts of the reasoning, leaving the details to the reader.

One may assume, however, that each sound reasoning possesses some kind of *underlying form* that is perfectly complete. This underlying form is an abstraction of the integral reasoning; therefore it is more a philosophical idea than a tangible entity. One may think of an underlying form as being an idealization of the reasoning, in which alle details are elaborated, while the combining structure has been accounted for as well.

It is nevertheless possible to give underlying forms a concrete shape by agreeing upon a certain system suitable for completely expressing reasonings. Such a system can be a fully formalized language (for example De Bruijn's Automath; see reference [2]), but many other shapes are conceivable. Even an ordinary natural language could be the vehicle for conveying a perfect, impeccable version of any reasoning.

For a unique interpretation of a reasoning, and for the accepting of its validity, the existence of an underlying form (or rather, the belief in that existence) is indispensable. Henceforth we shall silently assume this existence. Moreover, we shall not distinguish between possible shapes of underlying forms. We even adopt the point of view that each reasoning has exactly one full completion: *the* underlying form, which is, of course, an abstraction.

A text expressing a reasoning then can be considered a *realization* of its underlying form. In investigating the language of reasoning, we shall occupy ourselves with such realizations. It will be clear, however, that the underlying forms will guide us in analysing reasoning texts.

2. Sentences

A reasoning text is, essentially, a finite sequence of elementary units, which we call *fundamental sentences*. A fundamental sentence can be described as being a unit of text needed for conveying an elementary link in the chain of reasoning, namely an assumption, argument, conclusion or definition. Such a unit of text is not necessarily a sentence in the ordinary meaning, or in the linguistic sense. For example, an argument can be rendered in a subordinate clause ("Since Mary sleeps"), or even in an adjunct ("Because of fire-risk").

In a reasoning text one also finds a second type of sentences, which we call *informative sentences*. These sentences do not express directly an elementary link of the chain of reasoning, as fundamental sentences do, but they convey information concerning these elementary links.

We shall discuss and classify both types of sentences in the following two sections.

3. Fundamental sentences

We distinguish three kinds of fundamental sentences: assumptive, declarative and defining sentences.

3.1. As the name suggests, *assumptive sentences* express assumptions. One may conceive of an assumption as a local addition of a certain proposition to the "reservoir of facts". The need for such an additional fact arises in a limited number of circumstances, depending on the *reasoning target* prevailing at a certain point in a reasoning. We shall explain this by discussing a few frequently occurring instances.

(1a) One has to demonstrate, at a certain point in a reasoning, an implication; for example: "If Herbert grows rich, then he can take the apartment" (KM p. 30[1]). A natural manner of showing this is to propose: "Assume that Herbert grows rich", subsequently proving, "under" this assumption, that "Herbert can take the apartment". In general, in the case that the reasoning target is to demonstrate the implication: "If P_1 then Q_1", one may continue by making the assumption: "Assume P_1" and setting a new reasoning target, namely to demonstrate Q_1. (Such a reasoning target is an essential guide in a reasoning; yet, unfortunately, a target will not often be explicitly expressed in the reasoning text.)

(1b) A comparable deductive pattern may arise when the local target, somewhere in a reasoning, is to demonstrate a negation: "It is not the case that P_2". In such a case one naturally continues the reasoning by making

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1 A few examples are taken from Kalish and Montague, [5]; we indicate such places by the letters KM and the page number. Other examples will be taken from Anderson and Johnstone, [1]; they will be marked by the letters AJ.

the assumption: "Assume P_2", the new target being to derive a contradiction. (This deductive pattern is known as *reductio ad absurdum.*)

(1c) A third, less frequently occurring pattern is applicable when a disjunction has to be demonstrated: "P_3 or Q_3 (or both)". The added assumption then can be: "Assume that it is not the case that P_3", whereupon Q_3 becomes the new target.

(2) An essentially different situation arises when one has to demonstrate, at a certain point, a generalization, for example: "Every husband has a spouse "(KM p. 147). In this case one may start the reasoning at this point with the sentence: "Take any (arbitrary) husband", and subsequently one goes about showing that this husband must unquestionably have a spouse. One often wishes to *identify* the arbitrary subject (c.q. object) in question by the use of a devised name; for example: instead of the last-mentioned sentence, one writes: "Let Marc be a husband". A reasoning thereupon shows that Marc has a spouse. In order to avoid confusion between this Marc, only existing in the mind of the reasoning person, and real flesh-and-blood Marcs, one often contrives fantasy names for identifying the arbitrary subject. These names may be no more than one-letter words like x. (In the last-mentioned cases one speaks of *variables* rather than names.)

The first type of sentences, expressing an assumption and discussed in (1a) to (1c), obviously are assumptive sentences. In the second type of sentences, arising in (2), an arbitrary representative of a certain class of objects is introduced; we call these sentences assumptive sentences too. When wishing to distinguish between the two types of assumptive sentences, one may call the first type: *purely assumptive sentences*, and the second type: *introductive sentences*.

A certain kind of introductive sentences deserve special mention, namely those arising from an existential proposition. For example, the proposition: "Some teacher is able to solve all problems" (KM p. 169) asserts the existence of at least one teacher with the described ability. If one wishes to demonstrate, taking this existential proposition for granted, that a specific problem p actually possesses a solution, one might reason as follows:

"Let T be such a teacher; then T is able to solve all problems; in particular, she (or he) can solve problem p". Now we consider the sentence: "Let T be such a teacher" an assumptive sentence (to be precise: an introductive sentence), because the logical framework underlying the beginning of the deduction previously described is identical to that in all other cases in which an introductive sentence is involved (see also [6]).

It is, however, slightly confusing for us to state one exception to this agreement concerning existential propositions, namely in the case that the existence is *unique* (i.e.: there demonstrably is *exactly one* object having a given property). Example: if we have obtained irrefutable information confirming that there is one and only one teacher that is able to solve all problems, we may start a reasoning with the sentence: "Let T be this teacher"; we call such a sentence a defining sentence (see 3.3), because it identifies a well-described, unique subject (c.q. object).

Finally we note that all assumptions, including introductions, only live in a limited environment. It is regrettable that there is, in the normal manner of expressing reasonings, no common way of indicating the place in which an assumption is "discharged", i.e. no longer valid.

3.2. We call a sentence that represents a link in the chain of thoughts, hence obtaining its justification either by axiom (because of its "universal truth") or from preceding links, a *declarative* sentence. It will be obvious that sentences with a *concluding* character are a special type of declarative sentences. Example: "Therefore Alfred succumbs to temptation" (KM p. 78). The word "therefore", just like "so", "hence", "thus" etc., is a clear indication of the concluding aspects of this sentence.

Another type of declarative sentences are the *causal* sentences, which justify other sentences. Causal sentences are introduced by words like "since" or "because". The core statement of a causal sentence is universally true or follows from preceding sentences, so that we may consider causal sentences a special type of declarative sentences, as well. We note that causal sentences may be expressed without a verb. Example: "Because of Alexander's kindness". (This sentence is, however, linguistically equivalent to a sentence *with* verb: "Since Alexander is kind".)

3.3. In *defining sentences* one names an object, a notion, a situation etc. As a consequence, one may discern three parts in a defining sentence: the defined part (containing the new name), the defining part (a description of the object etc. being named) and a relational part (expressing the definitional relation, generally by means of a verb).

As De Bruijn observed, one may distinguish four kinds of defining sentences, depending on the nature of the defining part. We shall illustrate this by means of examples.

(1) The defining part describes an object that is uniquely determined. The defined part has the nature of a "constant". Examples: " The North Star is the bright star closest to the north celestial pole"; "The span of a bridge is the distance between upright supports". The latter example shows that definitions may depend on a certain context, because when using the word "span" as defined here, one has to refer to some (real or imaginary) bridge: "The span of the Bayonne-State Island bridge is 1675 ft", "There exists no wooden bridge with a span of over 100 meters".

(2) The defining part describes any member of a class of objects. The defined part is then a "generating name" for this class. It has the character of a noun. Example: "A rectangle is a quadrangle having four right angles". There are many quadrangles of this type, "rectangle" being a name for a representative of the whole class of right-angled quadrangles.

(3) The defining part describes a property common to a class of objects. The defined part then has the character of an adjective. Example: "A natural number is called even when it is divisible by 2". This example again shows that defined parts may depend on a certain context; in this case: "even" is only defined for "natural numbers". A second noteworthy fact concerning the kind of definitions now under consideration is, that definitions of "adjectives" bear a strong relation to definitions of "nouns". For example, instead of (or besides) defining the noun "rectangle", one may also define the adjective "rectangular". ("A quadrangle is called rectangular, when ..."). Moreover, the definition of the adjective "even" could be replaced by a definition of a

noun-like word: "An even number is ...", determining the compound word (or noun phrase) "even number".

(4) The defining part is essentially a full sentence, and the defined part has the nature of a sentence as well. Both defining and defined part have a verb as core. Example: "We say that two sets coincide, when they contain the same members". This definition accounts for the meaning of the sentence: "Two sets coincide", by stating its equivalence to another sentence: "The sets contain the same members". Note that there is an implicit context in this definition, namely "the (two) sets". This can be made visible by a reformulation: "Let A and B be sets. We say that A and B coincide, when A and B contain the same members".

4. Informative sentences

When inserting a non-fundamental sentence in a reasoning, an author usually wishes to inform the reader about local aspects of the reasoning in question. Such informative sentences are often used for one of the following two purposes:

- replacing a number of fundamental units of thought, or
- giving advance information on the course or the structure of the reasoning.

The first kind of informative sentences we call *replacing*, the second kind *anticipating*. By means of examples we shall elucidate these concepts.

A replacing sentence takes the place of a number of fundamental sentences that are not incorporated in the text, but are supposed to be part of the reasoning. Example: "From these observations one may derive the conclusion desired".

An author of a text often extends a replacing sentence with information about the actual replacement, in case the reader should like to elaborate the proof at that point. One way is by explaining the *manner of replacement*: "Analogously it may be shown that ...". Another way is to give the author's opinion about the *degree of difficulty* of the actual replacement: "it is easy to see that ...", or: "The result follows by simple verification".

Anticipating sentences often determine the *reasoning target* for the time to come: "We now show that line a is not tangent to circle c". Sometimes one refers in these sentences to the logical *proof structure* of the reasoning that follows: "By means of an indirect proof we shall show that there is no greatest prime number".

5. Compound sentences

When attempting to divide a reasoning text into a number of consecutive sentences without disturbing the logical structure, one sometimes comes across complicated sentences that hardly can be considered "elementary units of thought". Regard the following example: "Anything that contradicts a law of nature is incredible, for, since laws of nature are universally true, anything that contradicts a law of nature must be false, and that which must be false is incredible" (AJ p. 177).

From the outside, this example contains only two declarative sentences, the first being: "Anything that contradicts a law of nature is incredible", the second being all the rest. For the second sentence is a causal sentence, justifying the first one, so in first instance it may not be split up.

Yet one observes a number of shorter sentences contained inside the second sentence. So one may conceive of the second sentence as being a *compound* one. Its structure may be elucidated by the addition of pairs of brackets, as follows: "for, [[[since laws of nature are universally true], [anything that contradicts a law of nature must be false]], [and that which must be false is incredible]]". We observe three "embedded" sentences, which we call B, C and D respectively, for the sake of reference.

Let us refer to the original first sentence: "Anything ... incredible" with letter A, and let us denote a causal relation by an arrow, and a (conjunctive) coordination by a comma. Then the example can symbolically be rendered by the sentential combination $A \leftarrow [[B \rightarrow C], D]$. Clearly, sentence $E \equiv [B \rightarrow C]$ and the original second sentence $F \equiv [E,D]$ are compound sentences.

In reasonings one often encounters compound sentences. We already came across sentences that are compound due to a causal relation (sentence E) or a coordination (sentence F). Other relations that connect sentences are implication ("If there are no guilty people, then all food is salted", AJ p. 176), equivalence ("Any triangle is isosceles if and only if it has two equal sides", AJ p. 175) and disjunction ("Either savings must return to circulation or incomes must decline", AJ p. 71).

6. Sentence groups

In a reasoning text one may often observe coherent subtexts consisting of consecutive sentences. The coherence in such subtexts originates from the structure of the reasoning. When, for example, distinguishing several cases, each case gives rise to a coherent subtext. We call such subtexts *sentence groups*.

Sentence groups may occur disjointly in a reasoning, but one sentence group may also be contained in the other. We shall describe a few classes of sentence groups below.

An *assumptive-sentence group* contains all sentences depending on a certain assumption (the assumptive sentence included). For example, the following sentence may occur somewhere in a reasoning: "Let us assume that the population of the world doubles every twenty-five years" (AJ p. 72). Then this assumption will still "hold" in a following part of the reasoning. All sentences in this portion of text, plus the assumptive sentence itself, form an assumptive-sentence group.

An *existential-sentence group* contains a reasoning demonstrating an existential proposition, such as: "Some impartial seekers of truth eschew philosophy" (KM p. 118). Such a sentence group usually falls apart into two subgroups, one showing the existence by itself ("There is an impartial seeker of truth, namely X ... "), the other proving the predicate (" ... and X eschews philosophy").

Case-proving sentence groups occur in a reasoning where a certain proposition is shown by case analysis: one distinguishes several cases, and in each of these cases the proposition is proved. When the cases are exhaustive, that is to say: when all the cases covered in the original statement are covered by the cases considered, one may conclude that the proposition holds generally. Each case is dealt with in a case-proving sentence group; the collection of these groups, together with the final conclusion, may be regarded as a sentence group itself: a *case analysis group*.

One comes across *co-ordinative sentence groups* in a reasoning in which a conjunction of several independent propositions has to be demonstrated. This may, for example, occur when an equivalence is to be proved: statement P_1 holds if and only if statement P_2 holds. The proof in this case may fall apart into two parts, one showing that P_1 implies P_2, the other showing the reverse. The collection of the sentences in co-ordinate sentence groups, together with a possible summarizing conclusion, is a sentence group in itself: *a co-ordination group*.

7. Sentences and sentence groups: two examples

The first example is taken from Anderson and Johnstone [1], p. 73, exercise II.7.

1. Let us assume that we raise the gasoline tax one cent per gallon.
2. It follows that the net income from taxes will increase by three million dollars.
3. But if we increase the price of auto registrations,
4. the net income will increase only two and a half millions.
5. Furthermore, if we increase the gasoline tax,
6. the burden will not fall entirely on state residents,
7. since travelers must also buy gasoline,
8. and if we increase the price of registrations,
9. the burden will fall entirely on state residents.
10. Now, certainly the latter is not desirable,
11. and, furthermore, two and a half millions is not sufficient to cover current appropriations.
12. Since we must either increase the gasoline tax or the price of automobile registrations,

13. it is obvious
14. that we must increase the gasoline tax
15. and (we must) not (increase) the price of automobile registrations.

Classification of sentences and sentence groups:

Fundamental, purely assumptive sentences: 1, 3, 5, 8;
fundamental, declarative sentences: 2, 4, 6, 7, 9, 10, 11, 12, 14, 15 (concluding: 2, 4, 6, 9, 14, 15; causal: 7, 12);
informative sentence, replacing, stating degree of difficulty: 13.
Assumptive-sentence groups: 1-2, 3-4, 5-7, 8-9.

A second example originates from Hall Partee: Fundamentals of Mathematics for Linguistics [4], p. 216, answer to problem III.C.7.(iii).

1. Taking 0 as one of the elements in the trichotomy law for <,
2. we have that for any element a, either a < 0, a = 0 or a > 0.
3. To prove (iii)', it remains only to show that a < 0 if and only if -a > 0.
4. Suppose a < 0.
5. Then since -a = -a,
6. it follows
7. by the addition law for <

6 (continued). that a + (-a) < 0 + (-a).

8. But since a + (-a) = 0 and 0 + (-a) = -a,
9. this gives us the result that 0 < -a,
10. i.e. -a > 0.
11. In an exactly parallel manner we can show that
12. if -a > 0, then a < 0.
13. This establishes that a < 0 if and only if -a > 0,
14. completing the proof of (iii)'.

Classification of sentences and sentence groups:

Fundamental, purely assumptive sentence: 4;
fundamental, declarative sentences: 1, 2, 5, 6, 7, 8, 9, 10, 12, 13 (concluding: 2, 6, 9, 10, 12, 13; causal: 1, 5, 7, 8);

informative sentences: 3, 11, 14 (anticipating, determining the reasoning target: 3; replacing, explaining the manner of replacement: 11).
Assumptive-sentence group: 4-10;
co-ordination group: 4-13 (co-ordinating 4-10 and 11-12, 13 being a summarizing conclusion).

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