THE SIGNAL VALUE OF WORDS IN MATHEMATICAL PHRASING

R.P. Nederpelt

Eindhoven University of Technology
Department of Mathematics and Computer Science
P.O. Box 513, 5600 MB Eindhoven
the Netherlands

to be published in

Proceedings of the 6th Conference ~~of the International~~ Group
'Psychology of Mathematics Education', Antwerpen, July 1982.

THE SIGNAL VALUE OF WORDS IN MATHEMATICAL PHRASING

R.P. Nederpelt

Eindhoven University of Technology, The Netherlands

Introduction

There are many words which, when used in mathematical texts, have a specific *signal value*. The word 'continuous', for instance, used in a mathematics book, will hardly ever be directly related to the natural language concept of continuity; on the contrary, the word brings about a reaction in the experienced reader such that he automatically chooses the 'right' (i.e. mathematical) meaning. Clearly, the signal value of the word 'continuous', when occurring in a mathematical text, decides on the concept that arises in the mind of the reader; for that reason we call it a *conceptual* signal. There are also certain words which emit a *contextual* signal by giving extra information on the environmental setting. For example, the use of the word 'suppose' in a mathematical text signals the start of an argument based on an assumption. Hence the word 'suppose' emits a signal concerning the context that can be expected: most probably the author intends to prove a universal statement or some form of an implication (cf. par. 2a of this paper).

Hereafter we shall discuss these two classes of mathematical signal words in more detail. Both classes will de subdivided, depending on the nature and the effects of these signals.

Right now we stress that the signal value of a word develops slowly in the mind, by repeated usage. For example, the word 'inequality' has a specific meaning in a mathematical context: it denotes an expression of the form $p \neq q$. However, in a learning situation, a student has to disengage himself from the notion 'inequality' as it already exists in his mind. In our opinion, this process consumes more time than teachers usually appreciate. In mathematical education one should be aware of this process and try to prevent conflicts in the development of understanding.

1. Conceptual signals

1a. *Natural language words*

In mathematical language one uses many natural language words with their normal meaning: 'example', 'the', 'if', 'use', 'with'. These words are 'signs' in the linguistic sense, according to their connotations (cf. the classification by Pierce (1931/1958)). However, we will not refer to these words as 'signals', because they do not evoke a special reaction in the reader, caused by the mathematical setting. In this respect we note that some natural language words are subject to a mathematical flavouring that causes them to drift away from their original meaning; in so doing they develop a signal value. For example, the word 'so' in mathematical phrasing has become, in the course of time, a logical-mathematical meta-connective; it signals a logically motivated conclusion, which is independent of any personal decision. Cf.: '$x > 1$, so $x > 0$', in contrast to 'I am tired, so I go home'.

A number of natural language words are given a <u>new</u> meaning in a mathematical context. In this way they acquire a mathematical signal value. This new meaning may agree with the usual one (e.g. 'maximum'), but there is often a considerable difference in meaning. Cf.: 'origin', 'complex', 'function', 'power', 'primitive'. This may give rise to conflicts. Some examples are: a limit of a sequence is not necessarily an (upper or lower) bound; with many mappings one cannot associate any depictable 'map'; one may have doubts about the 'reality' of real numbers.

1b. *Mathematical words*

In mathematics there is a standard notation for frequently used words: 0 for zero, Δ for triangle, etc. It is worth noting, however, that even standard notation can sometimes cause confusion. For example: $\mathbb{N}^2$ is not the set of all squares of natural numbers.

Some words occurring in mathematics are abbreviated directly from the original natural language word: lim for limit, ℓn for natural logarithm, etc. In usage, these abbreviations quickly detach themselves from their origins and they develop a signal value of their own. Standard abbreviations are not always unique. The symbol e, for example, can represent the base of the natural logarithm or the unit group element. One should not neglect the power of standard notation. When abbreviating 'length' by 'ℓn', one is asking for difficulties.

A number of mathematical words can be employed for various purposes; to this end one uses letters, possibly combined with numbers, subscripts, etc.: x, $a3$, $\ell\ell$, t_2, $M_{n,\ell}$, x'. Some of these words traditionally have restrictions in use: the letters i to n usually represent natural numbers or integers, x and y are real numbers, z a complex number, α an angle. Although there are no fixed rules for this usuage, one should respect these unwritten conventions because of the connected signal values. It is, for instance, didactically undesirable to define a function $f : \mathbb{R} \to \mathbb{R}$ by $f(n) := x^n$. Here n is a real variable and x a constant. (Try to find the derivative of f!)

2. <u>Contextual signals</u>

2a. *Essential signals*

A number of contextual signals are *essential* for the course of the mathematical reasoning being presented. First of all we mention the *assumption* signals, which establish the hypothetical nature of a sentence in an argument. The word 'suppose' in the sentence 'Suppose g.c.d.(a,b) = 1' is such an assumption signal.

There are two kinds of assumptions, hence two kinds of assumption signals. Apart from the 'pure' assumption, as in the above sentence, there is a 'free' assumption, introducing a new name for an arbitrary object of some specific type, as is the case in 'Let x be a real number'. These two kinds of assumptions serve different purposes. The first one is used to prove an implication (e.g. 'If g.c.d.(a,b) = 1, then there exist integers x and y such that $ax + by = 1$'), the second serves to prove a generalization (e.g. 'For all real numbers x it holds that $x^2 - x + 1 > 0$').For a further discussion, see Donkers (1981). Since assumptions play an important role in arguments, it is regrettable that in common usage some assumption signals can have several possible interpretations. For example, the assumption signal 'assume' can be used for both kinds of assumptions. This fact diminishes its signal value. We note that this lack of distinctive power occurs with many contextual signals. We mention, for instance, the signal 'let...be', that serves as a definition signal in 'Let n be the smallest number for which $2^n > 1000$' (viz. n := 10), and not as an assumption signal.

Definition signals, too, are essential signals. Apart from 'let...be', we have, among others, 'take', 'is called', 'define...as', 'becomes', 'we indicate...with', 'consider ...to be'. In a sentence expressing 'x is defined by A', definition signals connect two parts, the definiens A and the definiendum x. .

Next, there are contextual signals expressing a logical connection between one part of an argument and another. These *connecting* signals occur in two versions: the *causal* signals and the *concluding* signals. Of the causal signals we mention: 'for', 'because', 'while', 'since', 'inasmuch as', 'namely', 'in consequence of', 'as a result of'. Examples of concluding signals are: 'thus', 'therefore', 'consequently', 'so that', 'hence', 'apparently', 'evidently', 'obviously' (cf. Anderson and Johnstone (1963), p. 60-61).

We note that concluding signals separate two parts of the reasoning which must necessarily occur in the natural deductive order: 'A, hence B'. With causal signals the order is mostly free: 'Since A, B' occurs next to 'B, since A'. (The signal 'for' always brings about the inverse order: 'B, for A'.) The logical connection caused by connecting signals can occur directly between two adjacent sentences ('$x > 1$, hence $x > 0$'), but a more indirect connection is possible as well.

2b. *Textual signals*

In mathematical texts one uses the same signals for denoting the textual structure as one does in ordinary phrasing. Although these signals are not special to mathematical texts, we will nevertheless mention them, since they often express the deductive structure implicitly present in a portion of text. For example, the signals 'on the one hand' and 'on the other hand', used in mathematical phrasing, will most probably signal a (logical) case analysis.

To begin with, we mention the *separation* signals. These signals interrupt the line of (directly connected) thoughts, and announce a new line. Some examples are: 'moreover', 'also', 'besides', 'likewise', 'now', 'on the contrary', 'but', 'however', 'yet', 'although'.

Next, there are *grouping* signals, dividing a text into connected parts. They often occur in pairs. Examples are: 'firstly', 'secondly', 'on the one hand', 'on the other hand', 'further', 'next', 'hereafter', 'finally', 'conversely'.

A *commenting* signal indicates that the author of a text is inserting a comment for the reader that stands apart from the actual reasoning. These commenting signals form an amorphous group, containing e.g. 'we shall prove', 'it suffices to show that', 'analogously', 'it is trivial that', 'apparently', 'it follows easily that', 'we recall'. The signal value of these word combinations is not very impressive.

A signal that is not expressed by a specific word, but by the mood used, occurs in proofs by *reductio ad absurdum*. Example: 'If n *were* a prime number, ...'.

Final remarks

The conventions and habits associated with mathematical signals have important consequences for the understanding of mathematical phrasing. Moreover, there are a considerable number of signals occurring in such texts. In our opinion, it would be interesting to know how a mathematical signal value evolves in the mind.

The development of conceptual signal values is connected to the building of *concept images* (see the description in Tall and Vinner (1981)). Contextual signals help to develop a *structure image* of an argument; this is an essential device for the understanding of complex logical arguments like proofs. As a side effect, knowledge concerning mathematical signals may be of aid in the production of didactically sound mathematics courses.

References

Anderson, J.M. and H.W. Johnstone, Jr.: Natural Deduction. Belmont, 1963.

Donkers, Jan: Mathematical Vernacular: an Introduction. Proceedings of the 5th Conference of the International Group 'Psychology of Mathematics Education', Vol. 2, p. 78-85. Grenoble, 1981.

Peirce, C.S.: The Collected Papers of Charles Sanders Peirce, edited by Charles Hartshorne and Paul Weiss. Cambridge, Mass., 1931-1958.

Tall, David and Shlomo Vinner: Concept Image and Concept Definition in Mathematics, with particular reference to limits and continuity. Educational Studies in Mathematics, Vol.12, no.2, 1981.