Location of slaughterhouses under economies of scale ^{*}

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Abstract

The facility location problem described in this paper comes from an industrial application in the slaughterhouse industry of Norway. Investigations show that the slaughterhouse industry experiences economies of scale in the production facilities. We examine a location-allocation problem focusing on the location of slaughterhouses, their size and the allocation of animals in the different farming districts to these slaughterhouses. The model is general and has applications within other industries that experience economies of scale.

We present an approach based on linearisation of the facility costs and Lagrangean relaxation. We also develop a greedy heuristic to find upper bounds. We use the method to solve a problem instance for the Norwegian Meat Co-operative and compare our results to previous results achieved using standard branch-and-bound in commercial software.

Keywords: Location, Integer Programming, Non-linear Programming, Branch and Bound, Economies of scale

1 Introduction

In this paper we investigate a facility location problem with linear transportation costs and economies of scale in the operation of facilities. The problem arose at the Norwegian Meat Co-operative when they did a strategic restructuring of their business in year 2000. When this work started there were 25 cattle slaughterhouses in the company. Our task was to investigate the saving potential of reducing this number and investing in capacity in the remaining facilities in order to profit from economies of scale. As this was part of a long term strategic analysis, we were asked to consider all municipalities in Norway as possible locations of slaughterhouses without giving preference to existing slaughterhouses: the meat co-operative wished to know the saving potential between today's solution and the ideal solution if they were free to build up their structure from scratch. They were also interested in finding out if there are many alternative solutions with about the same cost as the optimal solution. Our results have been used in a strategic restructuring of the network of slaughterhouses for cattle in Norway.

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The Norwegian Meat Co-operative is owned by a majority (37000) of the Norwegian farmers. The annual turnover is about 1200 million Euro. The market share for the company in Norway was in 2000 about 76% for slaughtering. Because the company is organized as a co-operative, it cannot refuse a request from one of its members for slaughtering animals. It is free to choose which slaughterhouse should serve the request. However, there exists an animal welfare restriction, forbidding animals to be on transport in a truck for more than 8 hours. The aim of the study was to suggest the optimal size and location of slaughterhouses and an allocation of animals to the slaughterhouses, given today's geographical distribution of the animal population.

The company faces a trade off between the number of slaughterhouses it owns and its transportation costs. The problem is an uncapacitated facility location problem. Fundamental parameters of such a problem are the costs of operating and owning the facilities and the unit transportation cost between customers and the facilities. The objective is to minimize total costs.

In the standard uncapacitated facility location problem (see e.g. Hax & Candea (1984)), facility costs are just fixed set-up costs. In our case the facilities have cost functions with economies of scale: unit slaughtering costs are decreasing as the number of animals allocated to the slaughterhouse increases (see for example Mathis & Koscianski (2002) for a definition). The total cost curve of each of the slaughterhouses has a typical S-shape, often found in long run cost curves, see Figure 2: the function is concave in the first part, when marginal costs are decreasing, and convex towards the end, when marginal costs are increasing. At all points the marginal costs are lower than the average costs, leading to economies of scale. The transportation costs are best described by a cost function that is linear in the distance between the farmers and the slaughterhouses. As a result, the objective function is non-linear, non-convex and non-concave.

There is an extensive research literature on facility location problems. Such models have been used since the 50's and early 60's (see for example Baumol & Wolfe (1958) and Cooper (1963)), but recent advances in optimization technology and the integration into information systems with decent user interfaces have made their dissemination wider. In location theory it is useful to separate the literature into two classes: Firstly we have optimization problems where the purpose is to minimize the cost or maximize the profit of locating a set of facilities under constraints like capacities, number of facilities, distance to customers and so on. A good survey of research with focus on solution methods can be found in Labbé & Loveaux (1997). Secondly, we have models where location is modelled for competitive companies in the tradition of Hotelling. An overview of such models with different market assumptions and objectives can be found in Eiselt, Laporte & Thisse (1993) and Eiselt & Laporte (2000).

Our paper is within the first class. In structure the problem resembles the facility location problem with staircase costs (FLSC), see for example Holmberg (1984), Holmberg (1985), Holmberg (1994), Holmberg, Rönnqvist & Yuan (1999), or Harkness & ReVelle (2002), or the modular capacitated plant location problem (MCPL), see for example Correia & Captivo (2003) or Correia & Captivo (2004). Our problem diverges from these by having a continuous and differentiable objective function. However, the solution strategy that we employed to tackle the problem brings us within the framework of these papers, still exploiting the continuity, as we will explain later in this section.

The problem of the Norwegian Meat Co-operative was first investigated in Borgen, Schea, Rømo & Tomasgard (2000) in 2000. The problem instance has 435 possible locations and 435 demand points. In Borgen et al. (2000) it was formulated as a mixed integer program with a piecewise linear objective, and a standard branch-and-bound method was used in the solution procedure (using xpress-mp). It was

not possible to solve the problem to optimality due to the size of the branching tree and weak lower bounds from the LP-relaxation. In fact, the best results in Borgen et al. (2000) were obtained when using the commercial software in combination with a simple heuristic which reduced the number of possible locations to 45. Even this reduced problem instance was only solved within 10% of optimality in 12 hours. Comparing this solution value with the best lower bound found on the original problem showed a gap of 27%. The purpose of the paper we present here is twofold. Firstly, we reduced the gap between the lower bound and the upper bound on the problem instance mentioned above and thereby find a good enough solution to the real life problem. Secondly, we find a more efficient solution method in terms of the time spent.

The poor performance of the approach in Borgen et al. (2000) is due to the piecewise linear approximation of the total cost curve in the facilities, which leads to a weak LP-relaxation, as the authors indeed mention. We therefore propose decomposing the problem using Lagrangean relaxation. The relaxation makes the problem separable in the facilities, and we use an efficient algorithm based on a solution method for continuous knapsack to solve the subproblems.

The technique of Lagrangean relaxation together with a heuristic to turn the Lagrangean relaxation solutions into feasible solutions for the original problem has been successfully applied already in the past on ordinary capacitated and uncapacitated facility location problems. We refer to the celebrated paper by Cornuejols, Fisher & Nemhauser (1977) and for other examples to Holmberg et al. (1999), Shetty (1990) or Nemhauser & Wolsey (1988). The piecewise linear relaxation brings the problem within the framework of staircase costs or modular costs mentioned above. The continuity of the objective function allows us to use another model for representing the choice of the capacities of the facilities than those used in Holmberg & Ling (1997) and in Correia & Captivo (2003). In Holmberg & Ling (1997) Lagrangean relaxation on different constraints is employed, in fact leading to a rather weak relaxation as noticed also in Correia & Captivo (2003). Our Lagrangean relaxation resembles the one in Correia & Captivo (2003). For deriving a feasible solution from infeasible optimal solutions of the Lagrangean relaxations, we have another method than the one in the latter paper. Our method always produces a feasible solution, whereas the one in Correia & Captivo (2003) may fail to do so. Implementing this method (which we did in 2001, indeed independently from Correia & Captivo (2003)), led to satisfactory results for the Norwegian Meat Co-operative. We managed to reach a solution which is provably within 1% of optimal in 95 minutes of computing time on a PC.

In Section 2 we give a detailed description of the problem we have solved for the Norwegian Meat Co-operative. We formulate the problem as a mixed integer program in Section 3. In Section 4 we present the Lagrangean relaxation, efficient solution of the subproblems, and a simple heuristic to generate feasible solutions from the infeasible solution of the Lagrangean relaxation. In Section 5 we show computational results. Apart from the results on the practical problem, we show additional computational tests on variations of the problem instance.

2 The problem data

In this section we describe in detail the cost components of the problem and specify how we incorporate them in our model. We used unit cost data for slaughterhouses based on a German study (Kern 1994). The costs include fixed costs (capital cost, personal, insurance) and variable costs (energy, personal, water, cleaning, repairs, classification, material, waste management). The average cost function, cf. Figure 1, is close to convex and monotonically decreasing with volume, representing a

Figure 1: Unit cost curve for slaughterhouses as a function of volume.

situation with economies of scale. In Figure 2 the total cost curve of a facility is depicted. These functions are equal for all facilities. We approximate each of them by a piecewise linear function.

The transportation time is defined as the total duration from loading the first animal on the truck, until the last animal has left the truck at the slaughterhouse. It can be split in two parts:

- The collecting time: the time consumed while collecting the animals within the municipality, including driving time, stopping time and expected waiting time at the slaughterhouse before the truck is unloaded;
- The travelling time: the time of going from the municipality centre of the region where the animals are located to the municipality centre where the slaughterhouse is located.

The collection time is approximated by the average time of filling up the car on a collection round-trip. As there was no indication of differences between the different regions we assumed equal velocity of the cars in all regions and no differences in collecting times or costs based on regions. The transportation operator is paid by the travelling distance, and has additional payment linear in the number of animals on the truck. Thus, the transportation costs are linear in the distance to the slaughterhouse and linear in the number of animals transported in the truck.

The choice between different car types is included in the model through pre-processing. We include a large car and a medium sized car in our analysis. It turns out that with our available cost data, large cars are always preferred to small cars, if feasible. The increased cost in driving from the slaughterhouse to the municipality and back is more than outweighed by the benefit of increasing the number of animals transported per trip. Still, the range of the large cars is limited. A larger car has to pick up more animals and thereby the collecting time increases. The 8 hour rule must be satisfied. This means that within some radius around a slaughterhouse

Figure 2: Total cost curve for slaughterhouses as a function of volume.

the transportation costs are lower due to the ability to use larger trucks. Outside this range smaller cars must be used and the transportation costs increase slightly. We assume that cars pick up animals from one municipality only, and for each combination of slaughterhouse and demand municipality the best car size is found by preprocessing. This is reflected in the transportation cost and time matrices. In Norway there are 435 municipalities, hence the travel cost matrix and the travel time matrix will both have 435^2 elements defining the cost and the time needed to go between each pair of municipality centres.

3 The mathematical programming model

In this section the problem is formulated as a mixed integer linear programming problem. The model resembles the model for the uncapacitated facility location problem (see e.g. Nemhauser & Wolsey (1988)). The main difference is in the definition of the facility costs, as mentioned in the previous section. We approximate these by piecewise-linear non-convex, non-concave functions, modelled in a standard way by special ordered sets of type 2 (Williams 1999): using an ordered set of continuous variables, one for each breakpoint of the function, with value between 0 and 1. In a feasible solution at most two variables corresponding to neighbouring breakpoints have positive values adding up to 1. They specify a slaughterhouse volume as the convex combination of the two breakpoint volumes.

Let n be the number of points (municipalities) in the problem. We denote by K the number of breakpoints in the facility cost functions, by $f_1, \dots, f_K$ the breakpoint volumes, and by $p_1, \dots, p_K$ the unit slaughter costs at the breakpoints. The decision variables employed for representing the function in the special ordered sets are $y_{j1}, \dots, y_{jK}$ for a facility at location j , $j = 1, \dots, n$. We define $f_1 = 0$. In this way $y_{j1} = 1$ signifies that no facility is located at point j . To make sure that the fixed cost of installing a facility is represented, f_2 is chosen small. Hence p_2 becomes high

as the fixed costs are shared only by these few units. Let $y_j, j = 1, \dots, n$, be the K -dimensional vectors made up by all $y_{jk}, k = 1, \dots, K$. We refer to Williams (1999) for a precise formulation of the constraints to represent a special ordered set of type 2. We represent the feasible set of values for y_j by $Y_j, j = 1, \dots, n$. Let d_i denote the supply (number of animals to be slaughtered) in point i , and w the average weight of one unit of supply (one animal). By t_{ij} we denote the unit transportation cost from point i to point j . We define parameters a_{ij} as $a_{ij} = 1$ if the transportation time from i to j is less than 8 hours (including the collecting time), and $a_{ij} = 0$, otherwise. We use the decision variables x_{ij} to denote the number of supply units transported from point i to point j . Although by its definition x_{ij} should be integer we allow it to take any non-negative real value, as we will discuss after giving the model. We call the following model MIP:

$$\begin{aligned}
\min \quad & \sum_{i=1}^n \sum_{j=1}^n t_{ij} x_{ij} + \sum_{j=1}^n \sum_{k=1}^K f_k p_k y_{jk} \\
\text{s.t.} \quad & \sum_{j=1}^n x_{ij} = d_i, \quad i = 1, \dots, n, \tag{1} \\
& \sum_{k=1}^K y_{jk} f_k = w \sum_{i=1}^n x_{ij}, \quad j = 1, \dots, n, \tag{2} \\
& 0 \leq x_{ij} \leq a_{ij} d_i, \quad i, j = 1, \dots, n, \tag{3} \\
& y_j \in Y_j, \quad j = 1, \dots, n, \tag{4} \\
& x_{ij} \in \mathbb{R}, \quad i, j = 1, \dots, n. \tag{5}
\end{aligned}$$

In the objective the term $\sum_{k=1}^K f_k p_k y_{jk}$ is the slaughter costs at location j . The restrictions (1) enforce all supply from point i to be allocated to a facility. Restrictions (2) define the total volume allocated to the facilities. They also prohibit allocation to points without a facility. Restrictions (3) prohibit transportation on infeasible links. Restrictions (4) have been explained above. According to restrictions (5) x may take continuous values. This choice in the model is justified by the fact that, for any given y , the remaining problem is a bipartite transportation problem. Hence, if $d_i, i = 1, \dots, n$, are integer and the breakpoints are integer multiples of w , then x will take integer values in any optimal solution.

There are no explicit binary variables in the model to indicate where facilities are to be located. This information is obtained from the values of the x -variables directly: no facility is located at point j if and only if $x_{ij} = 0$ for all i . The information can also be derived from the values of the y -variables: no facility is located at point j if and only if $y_{j1} = 1$.

4 Lagrangean Relaxation

We define the Lagrangean relaxation by relaxing the supply constraints (1) in MIP. Introducing the vector $\lambda = (\lambda_1, \dots, \lambda_n)$ of multipliers gives the Lagrangean subproblem:

$$\begin{aligned}
LR(\lambda) = \min \quad & \sum_{i=1}^n \sum_{j=1}^n t_{ij} x_{ij} + \sum_{j=1}^n \sum_{k=1}^K f_k p_k y_{jk} + \sum_{i=1}^n \lambda_i (d_i - \sum_{j=1}^n x_{ij}) \\
\text{s.t.} \quad & \sum_{k=1}^K y_{jk} f_k = w \sum_{i=1}^n x_{ij}, \quad j = 1, \dots, n, \\
& 0 \leq x_{ij} \leq a_{ij} d_i, \quad i, j = 1, \dots, n, \\
& y_j \in Y_j, \quad j = 1, \dots, n, \\
& x_{ij} \in \mathbb{R}, \quad i, j = 1, \dots, n.
\end{aligned}$$

We rewrite the objective function as

$$\min \sum_{j=1}^n \left(\sum_{i=1}^n t_{ij} x_{ij} + \sum_{k=1}^K f_k p_k y_{jk} - \sum_{i=1}^n \lambda_i x_{ij} \right) + \sum_{i=1}^n \lambda_i d_i.$$

Given λ , the last term is a constant, and therefore the objective is separable in j . We define $LR(\lambda) = \sum_{j=1}^n g_j(\lambda) + \sum_{i=1}^n \lambda_i d_i$ with $g_j(\lambda)$ the optimal value of the Lagrangean subproblem for location j :

$$\begin{aligned} g_j(\lambda) = \min \quad & \sum_{i=1}^n (t_{ij} - \lambda_i) x_{ij} + \sum_{k=1}^K f_k p_k y_{jk} \\ \text{s.t.} \quad & \sum_{k=1}^K y_{jk} f_k = w \sum_{i=1}^n x_{ij}, \\ & 0 \leq x_{ij} \leq a_{ij} d_i, \quad i = 1, \dots, n, \\ & y_j \in Y_j, \\ & x_{ij} \in \mathbb{R}, \quad i = 1, \dots, n. \end{aligned}$$

4.1 Solving the subproblem

We describe how the subproblem for each location j can be solved. The total slaughter cost is strictly monotonically increasing in the total weight. The unit increase in cost between the breakpoints k and $k+1$ is denoted by α_k . We define the breakpoints of the cost function in terms of the number of animals as $F_k = f_k/w$, $k = 1, \dots, K$, and we define $F_{K+1} = \infty$. These breakpoints may be fractional values, even if they are defined as a number of animals. The total slaughter costs in $g_j(\lambda)$ at slaughterhouse j in these terms is then $\sum_{k=1}^K w F_k p_k y_{jk}$. The slope of the linear segment of the cost function between breakpoint k and $k+1$ is denoted by α_k : i.e., $\alpha_k = \frac{w(F_{k+1} p_{k+1} - F_k p_k)}{F_{k+1} - F_k}$, $k = 1, \dots, K-1$, and $\alpha_K = w p_K$.

Let us consider the subproblem for location j . Related to this location we define another K subproblems, one for each $k = 1, \dots, K$. For each $i = 1, \dots, n$ let $q_{ik} = t_{ij} - \lambda_i + w \alpha_k$ be the extra cost for bringing one more animal from location i to location j . For $k = 1, \dots, K$ define

$$\begin{aligned} g_{jk}(\lambda) = \min \quad & \sum_{i=1}^n q_{ik} x_{ij} \\ \text{s.t.} \quad & F_k \leq \sum_{i=1}^n x_{ij} \leq F_{k+1}, \\ & 0 \leq x_{ij} \leq a_{ij} d_i, \quad i = 1, \dots, n, \\ & x_{ij} \in \mathbb{R}, \quad i = 1, \dots, n. \end{aligned}$$

Thus, $g_j(\lambda) = \min_k g_{jk}(\lambda)$, $j = 1, \dots, n$. To find $g_{jk}(\lambda)$ a method similar to the one for solving continuous knapsack problems is applied (Martello & Toth 1990). That method has to be adapted for the *lower* bounds on total capacity.

We order the points i in order of increasing q_{ik} : $q_{1k} \leq \dots \leq q_{nk}$. Start setting x_{ij} 's equal to their maximum value $a_{ij} d_i$ in that order until for some order index (i_1) for the first time $\sum_{l=1}^{i_1} a_{lj} d_l \geq F_k$. If $q_{(i_1)k} < 0$ set $x_{(i_1)j} = a_{(i_1)j} d_{(i_1)}$. Otherwise set $x_{(i_1)j} = \lceil F_k \rceil - \sum_{l=1}^{i_1-1} a_{lj} d_l$. In the latter case $x_{lj} = 0$ for all $l = i_1 + 1, \dots, n$ and the optimal solution is found. In the former case continue until for some index (i_2) for the first time either $q_{(i_2)k} \geq 0$, in which case we set $x_{(i_2)j} = 0$, or $\sum_{l=1}^{i_2} a_{lj} d_l > F_{k+1}$, in which case we set $x_{(i_2)j} = \lfloor F_{k+1} \rfloor - \sum_{l=1}^{i_2-1} a_{lj} d_l$. In both cases set $x_{lj} = 0$, for all $l = i_2 + 1, \dots, n$. Non-existence of an index (i_1) means that $\sum_{i=1}^n a_{ij} d_i < F_k$. In that case the problem is infeasible and we set $g_{jk}(\lambda) = \infty$.

Initialise: Choose values for $\epsilon_1 > 0$, $\epsilon_2 > 0$, V , V_1 and η_0 .
Set UB equal to the value of some approximate solution. Set $LB \leftarrow -\infty$.
Set $v \leftarrow 1$, $v_1 \leftarrow 1$, choose starting point $\lambda^{(1)}$, and set $\eta = \eta_0$.
Iterate: Until $v = V$,

1. Determine $LR(\lambda^{(v)})$.
If $LR(\lambda^{(v)}) > LB$, set $LB \leftarrow LR(\lambda^{(v)})$ and $v_1 \leftarrow 0$;
Else, set $v_1 \leftarrow v_1 + 1$. If $v_1 = V_1$ set $\eta \leftarrow \frac{\eta}{2}$ and set $v_1 \leftarrow 0$;
2. Derive a feasible solution x^G from $x(\lambda^{(v)})$, yielding value $G^{(v)}$. If $G^{(v)} \leq UB$, set $UB \leftarrow G^{(v)}$, $x^* \leftarrow x^G$ and set $\eta \leftarrow \eta_0$. If $UB - LB < 1$, stop: UB is the optimal solution value.
3. Calculate the gradient $s^{(v)} = \nabla LR(\lambda^{(v)})$, set step length $t^{(v)} = \eta \frac{UB - LR(\lambda^{(v)})}{\|s^{(v)}\|^2}$, and $\lambda^{(v+1)} \leftarrow \lambda^{(v)} + t^{(v)} s^{(v)}$.
4. If $\|s^{(v)}\| \leq \epsilon_1$ or $\|\lambda^{(v+1)} - \lambda^{(v)}\| \leq \epsilon_2$, stop;
Else, set $v \leftarrow v + 1$.

Output: LB; UB; x^* .

Figure 3: Subgradient algorithm.

4.2 The Lagrangean dual

The best lower bound on the optimal solution value of the original problem will be found by solving the Lagrangean dual problem (LD):

$$\max_{\lambda} LR(\lambda).$$

We do so using sub-gradient optimization (e.g. see Nemhauser & Wolsey (1988)). The sub-gradient optimization routine that we have used is standard and often used for facility location problems. For example, it can be found in Holmberg et al. (1999).

The partial derivative of LR is given by

$$\delta_i = \frac{\partial LR(\lambda)}{\partial \lambda_i} = d_i - \sum_{j=1}^n x_{ij}(\lambda),$$

with $x_{ij}(\lambda)$ the optimal solution of the Lagrangean relaxation with multipliers λ . Hence, the gradient of LR is given by $\nabla LR(\lambda) = (\delta_1, \dots, \delta_n)$. For sake of completeness we present the subgradient algorithm in Figure 3 in pseudo-code. The algorithm involves determination of an upper bound on the optimal solution, the description of which is given below.

For given λ , $LR(\lambda)$ yields a lower bound on the optimal solution value, but in general the optimal solution of the Lagrangean relaxation is not a feasible solution to MIP. In Step 2 in the algorithm we use a heuristic to find an upper bound $G^{(v)}$ by finding a feasible solution to MIP based on the solution of the Lagrangean relaxation in iteration v . In the next section we present this heuristic.

4.3 An approximate solution based on Lagrangean relaxation

In general the optimal solution of the Lagrangean relaxation is not feasible to MIP. Here we show how to turn an optimal solution $x_{ij}(\lambda)$ of the Lagrangean relaxation, given λ , into a feasible solution of MIP:

1. We start by setting $x_{ij}^1 = d_i$ if $x_{ij}(\lambda) > 0$, $\forall i, j$. We also introduce artificial variables z_j^1 , which get value 1 if there exists an i such that $x_{ij}^1 = d_i$, and 0 otherwise. (These variables indicate if a facility is located at point j or not.)
2. If for i there are indices j such that $x_{ij}^1 = d_i$, then from among those j choose the one with minimum t_{ij} , index j^* say, and set $x_{ij^*}^2 = d_i$ and $x_{ij}^2 = 0$, $\forall j \neq j^*$.
3. If for i , $x_{ij}^1 = 0 \forall j$ and there exists indices j for which $z_j^1 = 1$ and $a_{ij} = 1$, then choose from the latter the one with minimum t_{ij} , index j^0 say, and set $x_{ij^0}^2 = d_i$ and $x_{ij}^2 = 0$, $\forall j \neq j^0$.
4. If for i , $x_{ij}^1 = 0$, $\forall j$ and no index j exists for which $z_j^1 = 1$ and $a_{ij} = 1$, then set $x_{ij}^2 = 0$, $\forall j$.
5. Set $z_j^2 = 0$ if and only if $x_{ij}^2 = 0$, $\forall i$. Otherwise set $z_j^2 = 1$.
6. We finish by greedily locating facilities for the set of $I = \{i \mid x_{ij}^2 = 0, \forall j\}$: For every j with $z_j^2 = 0$ determine $I_j = \{i \in I \mid a_{ij} = 1\}$. Choose the set $I_{j'}$ with highest cardinality (ties are broken arbitrarily), locate a facility in the corresponding point j' , i.e., change $z_{j'}^2 = 0$ into $z_{j'}^2 = 1$ and reset $x_{ij'}^2 = a_{ij'} d_i$, $\forall i \in I_{j'}$. Repeat this step as long as the set I is not empty.

Given this approximate solution, a set of facilities is opened (those with $z_j^2 = 1$), and supply is allocated as suggested by $x_{ij}^2, \forall i, j$. At iteration v in the algorithm from Figure 3 we denote this solution value by $G^{(v)}$ and define $G = \min_v G^{(v)}$.

This solution can be improved by fixing the locations of the facilities installed according to z^2 and using a general purpose branch-and-bound code to get the best possible allocation given these locations, yielding a solution value denoted by G_S .

5 The computational results

We present here the most important results from solving the problem instance with 435 location points and 435 demand points. In addition to the original problem instance, we ran the algorithm with several related cases. A case is described by a combination of scenarios for demand (D) and transportation costs (C). The dataset $(D, C) = (0, 0)$ represents the original problem instance. The demand scenarios $D = 1 \dots 10$ are generated randomly from a multivariate normal distribution with expectation equal to the original animal population and without correlation between the different regions. The cost scenarios consider an increase in transportation costs of 20% and 40% as well as a decrease of 20% and 40% with respect to the original cost data. The results from the calculations are shown in Table 1 below. The costs are given in NOK 1000. The gap is defined as $\frac{UB(G_S) - LB}{UB(G_S)}$.

In the computational experiments we use 6 breakpoints to approximate the facility cost curve. The points given in (*tons/year*, *kr/kilo*) are (rounded from the nearest multiple of w , the average weight of an animal): (0, 0), (1, 8000), (1000, 8.03), (3000, 4.07), (9000, 2.18), (40000, 1.10). The number and the values of the breakpoints have been chosen in direct consultation with the Norwegian Meat Co-operative, already by Borgen et al. (2000). Choosing the same ones also facilitates comparison to the first attempt to tackle the slaughterhouse problem in Borgen et al. (2000).

To set parameters in the algorithm, we performed some first test runs, which indicated that the best choice for η_0 was around 2, and for both ϵ_1 and ϵ_2 it was $1.0 \cdot 10^{-20}$. We set the maximum number of iterations to $V = 25000$. Test runs showed that the initial values for the dual multipliers did not have much influence

Dataset		LB	UB(G)	UB(G_S)	runtime	gap
D	C					
0	0	241467	244162	243787	1:33:55	1.0%
0	+20%	259672	262013	261918	1:32:00	0.9%
0	+40%	277727	280067	279975	1:31:21	0.8%
0	-20%	222826	228464	226375	1:34:32	1.6%
0	-40%	204036	209641	207998	1:33:16	1.9%
1	0	238487	238595	238571	1:31:49	0.0%
2	0	234211	246894	242547	1:30:54	3.4%
3	0	244099	252300	250484	1:32:59	2.5%
4	0	241412	252578	248077	1:32:24	2.7%
5	0	243834	247560	247176	1:31:45	1.2%
6	0	230981	246829	243451	1:20:25	5.1%
7	0	248278	258753	256339	1:26:23	3.1%
8	0	244406	254396	252288	1:33:35	3.1%
9	0	243119	254074	250583	1:30:43	3.0%
10	0	245017	250707	249476	1:33:46	1.8%

Table 1: Computational results.

on the performance. We used $V_1 = 225$ to limit the number of iterations without improvement, and report the best results in Table 1.

The best results from the previous attempt to solve the original problem instance in Borgen et al. (2000) was a solution with 11 slaughterhouses, a solution value of 258101 and an optimality gap of 27%. The best lower bound found for the original problem instance using our approach is 241467. The best feasible solution value(G) found by the 6-step method described in Section 4.3 is 244162. The best solution found after improvement as described at the end of Section 4.3 has value $G_S = 243787$. Hence, within running times of approximately 95 minutes the gap can be reduced to around 1%. Compared to previous results, our method shows substantial improvements both in running time, lower bound and upper bound.

Equally important is the effect on the solution itself. The method is able to reduce the optimal number of slaughterhouses from 11 to 9 for all problem instances based on the original animal population¹ ($D = 0$ -instances). The best feasible solutions for demand scenarios $D = 1, 5, 10$ have 9 slaughterhouses, whereas the method finds solutions with 10 slaughterhouses for the remaining datasets.

The transportation costs account for 37.8% of the total costs in the original problem instance. A raise in the costs of transportation of 40% results in a share of about 45% of the total costs. Decreasing the transportation costs by 40 % reduces the share to approximately 27% of the total costs. We also analyzed the impact of changing the transportation costs when it comes to the sizes and locations of slaughterhouses. This is shown in Table 2 where the best solutions are given in terms of the chosen locations and their size. Locations are numbered from 1 to 435 and the sizes of the slaughterhouses are given in tons. The level of the transportation costs has only a small influence on the geographical location of the slaughterhouses and the allocation of animals is also changing only slightly. Solving

¹In fact, the best solution also has a tenth slaughterhouse, where only a single animal was allocated. This comes from the fact that the animal is located on a remote island. The facility cost of installing a single animal facility is about 2 million kroner in the model. Still this is lower than the alternative cost of shifting the whole slaughterhouse structure along the cost to bring this animal within the 8 hour limit for transportation to the closest facility. In practice, of course, this is solved by a local barbecue.

the transportation problem given the locations from the original problem instance, but with the alternative transportation costs, results in similar solutions in terms of costs. All our analyses show that there are many alternative solutions with almost the same cost level. What matters seems to be the number of slaughterhouses, which should be as low as possible.

Case (0, 0)		Case (0, +20%)		Case (0, +40%)		Case (0, -20%)		Case (0, -40%)	
Loc.	Size	Loc.	Size	Loc.	Size	Loc.	Size	Loc.	Size
79	17386	79	17382	79	17382	79	18577	79	18577
184	12930	184	12930	184	12930	184	13108	184	13108
238	6972	238	7031	238	7,031	225	7401	225	7401
291	16097	291	14563	291	14563	280	7272	280	7272
339	12741	323	12477	323	12477	330	17566	330	17647
363	1872	363	3615	363	3615	363	4074	363	3992
395	1385	395	2008	395	2008	395	1385	395	1385
415	1585	415	962	415	962	415	1585	415	1585
432	400	432	400	432	400	432	400	432	400

Table 2: Slaughterhouse location and size (in tons).

6 Conclusions

We have shown how to model economies of scale in uncapacitated facility location problems. Our problem instance is motivated from an application where the purpose is to decide location and size of a set of slaughterhouses. Still the formulation is general and can be viewed as an extension of the classical uncapacitated problem.

When the non-convex and non-concave objective function is approximated by a piecewise linear function using specially ordered sets, we seem to get a weak LP-formulation. By using Lagrangean relaxation we are able to improve the lower bound. Also by implementing a simple greedy heuristic we manage to find feasible solutions from the infeasible Lagrangean solutions. The use of Lagrangean relaxation reduces solution time for the problem and improves the quality of the solutions dramatically.

There is reason to believe that the approach shown in this paper will perform even better if the bounds are used in a branch-and-bound scheme. However, our main interest was to solve this real life problem close to optimality to make the company happy. This happened far before the gap was reduced to 1.0%, and results from this research have been important input in the restructuring process of the company.

The various runs also showed that a lot of different solutions exist to the problem with little difference in solution values, but with different locations in the solution.

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