Efficient Mode-Matching Based on Closed Form Integrals of Pridmore-Brown Modes

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\[(P_\mu, \Psi_\nu) \rightarrow \langle F_\mu, \Psi_\nu \rangle\]
APU: Auxiliary Power Unit
- produces power when main engines are switched off
- to start main engines, AC, ...
- major source of ramp noise

APU on an Airbus A380
Background / motivation

- APU: Auxiliary Power Unit
  - produces power when main engines are switched off
  - to start main engines, AC, ...
  - major source of ramp noise

⇒ Need for sound propagation modeling

APU on an Airbus A380
APU exhaust duct geometry

Study sound propagation / attenuation
Study sound propagation / attenuation
- straight and circular cylindrical duct
Study sound propagation / attenuation

- straight and circular cylindrical duct
- non-uniform parallel mean flow (axially varying)
Study sound propagation / attenuation

- straight and circular cylindrical duct
- non-uniform parallel mean flow (axially varying)
- strong temperature gradients (axially varying)
Study sound propagation / attenuation

- straight and circular cylindrical duct
- non-uniform parallel mean flow (axially varying)
- strong temperature gradients (axially varying)
- segments with different lining (BCs)
  ⇒ mode-matching
From ‘classical’ to a new mode-matching method

‘classical’ \[\rightarrow\] new mode-matching

\((P_\mu, \Psi_\nu)\) \[\rightarrow\] \(\langle F_\mu, \Psi_\nu \rangle\)
From ‘classical’ to a new mode-matching method

‘classical’ (CMM) → new (BLM) mode-matching

\[(P_\mu, \Psi_\nu) \rightarrow \langle F_\mu, \Psi_\nu \rangle\]

with \(\Psi_\nu = J_m(\alpha_\nu r)\)

with \(\Psi_\nu = F_\nu, \quad F = [P, U, V, W]\)
From ‘classical’ to a new mode-matching method

‘classical’ (CMM) → new (BLM) mode-matching

\((P_\mu, \Psi_\nu)\) → \(\langle F_\mu, \Psi_\nu \rangle\)

\[= \int_0^d P_\mu \Psi_\nu r \, dr \quad \rightarrow \quad = \int_0^d \left[ w_1 P_\mu P_\nu + w_2 U_\mu P_\nu \\
+ w_3 (V_\mu V_\nu + W_\mu W_\nu) \right] r \, dr \]

with \(\Psi_\nu = J_m(\alpha_\nu r)\)  
with \(\Psi_\nu = F_\nu\),  \(F = [P, U, V, W]\)
From ‘classical’ to a new mode-matching method

\[
\text{‘classical’ (CMM)} \quad \rightarrow \quad \text{new (BLM) mode-matching}
\]

\[
(P_\mu, \Psi_\nu) \quad \rightarrow \quad \langle F_\mu, \Psi_\nu \rangle
\]

\[
= \int_0^d P_\mu \Psi_\nu r \, dr \quad \rightarrow \quad = \int_0^d \left[ w_1 P_\mu P_\nu + w_2 U_\mu P_\nu + w_3 (V_\mu V_\nu + W_\mu W_\nu) \right] r \, dr
\]

quadrature \quad \rightarrow \quad = \frac{id}{k_\mu - k_\nu} \left[ \frac{P_\nu V_\mu - V_\nu P_\mu}{\Omega_\nu} \right]_{r=d}

with \( \Psi_\nu = J_m(\alpha_\nu r) \) \quad with \( \Psi_\nu = F_\nu, \quad F = [P, U, V, W] \)
From ‘classical’ to a new mode-matching method

‘classical’ (CMM) → new (BLM) mode-matching

\[(P_\mu, \Psi_\nu) \rightarrow \langle F_\mu, \Psi_\nu \rangle\]

\[= \int_0^d P_\mu \Psi_\nu r \, dr \rightarrow = \int_0^d \left[ w_1 P_\mu P_\nu + w_2 U_\mu P_\nu + w_3 (V_\mu V_\nu + W_\mu W_\nu) \right] r \, dr\]

quadrature \rightarrow \frac{id}{k_\mu - k_\nu} \left[ \frac{P_\nu V_\mu - V_\nu P_\mu}{\Omega_\nu} \right]_{r=d} \rightarrow \text{cheap} \text{ accurate}
1 Problem formulation

2 ‘Classical’ mode-matching (CMM)

3 New ‘BLM’ mode-matching
   - Closed-form integrals of Helmholtz modes
   - Closed-form integrals of parallel flow modes
   - Closed-form integrals of radial Pridmore-Brown modes
   - Bilinear map based mode-matching

4 Numerical results

5 Conclusions
Outline

1. Problem formulation

2. ‘Classical’ mode-matching (CMM)

3. New ‘BLM’ mode-matching
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4. Numerical results

5. Conclusions
Inviscid, non-heat-conducting, ideal gas
⇒ Euler equations
⇒ gas laws: $p = \rho RT$, $c^2 = \gamma RT$
Modeling Assumptions

- Inviscid, non-heat-conducting, ideal gas
  \[ \Rightarrow \text{Euler equations} \]
  \[ \Rightarrow \text{gas laws: } p = \rho R T, \ c^2 = \gamma R T \]

- Acoustics: interested in small perturbations \( p_1, \rho_1, v_1 \)

\[
p(x, y, z, t) = p_0(x, y, z) + p_1(x, y, z, t)
\]

\[ \Rightarrow \text{Linearized Euler equations} \]
Modeling Assumptions

- Inviscid, non-heat-conducting, ideal gas
  → Euler equations
  → gas laws: \( p = \rho RT, \quad c^2 = \gamma RT \)

- Acoustics: interested in small perturbations \( p_1, \rho_1, \nu_1 \)

  \[ p(x, y, z, t) = p_0(x, y, z) + p_1(x, y, z, t) \]

  → Linearized Euler equations

- Time-harmonic modal perturbations

  \[ p(x, y, z, t) = p_0(y, z) + \text{Re}\{P(y, z)e^{-i\omega t + ikx}\} \]

  of a parallel mean flow

  \( \vec{v}_0 = u_0(y, z)e_x, \quad \rho_0 = \rho_0(y, z), \quad T_0 = T_0(y, z), \quad c_0 = c_0(y, z) \)
For perturbations $p_1, \rho_1, \mathbf{v}_1$ of a parallel mean flow

$$\mathbf{v}_0 = u_0(y,z)e_x, \quad \rho_0 = \rho_0(y,z), \quad T_0 = T_0(y,z), \quad c_0 = c_0(y,z)$$
For perturbations $p_1$, $\rho_1$, $v_1$ of a parallel mean flow

$$v_0 = u_0(y, z)e_x, \quad \rho_0 = \rho_0(y, z), \quad T_0 = T_0(y, z), \quad c_0 = c_0(y, z)$$

the Linearized Euler equations can be reduced to:

\[
\nabla \cdot \left( \frac{c_0^2}{\Omega^2} \nabla P \right) + \left( 1 - \frac{k^2 c_0^2}{\Omega^2} \right) P = 0 \quad \text{on } \mathcal{A}
\]

with \( \Omega = \omega - ku_0 \)
For perturbations $p_1, \rho_1, v_1$ of a parallel mean flow

$$v_0 = u_0(y, z) e_x, \quad \rho_0 = \rho_0(y, z), \quad T_0 = T_0(y, z), \quad c_0 = c_0(y, z)$$

the Linearized Euler equations can be reduced to:

‘Generalized Pridmore-Brown’ equation (arbitrary cross-section $A$)

For modes of the form $p_1(x, y, z, t) = P(y, z) e^{i k x - i \omega t}$:

$$\nabla \cdot \left( \frac{c_0^2}{\Omega^2} \nabla P \right) + \left( 1 - \frac{k^2 c_0^2}{\Omega^2} \right) P = 0 \quad \text{on } A$$

Ingard-Myers boundary condition for slipping flow

$$-i \omega (v_1 \cdot n) = (-i \omega + v_0 \cdot \nabla) \frac{p_1}{Z} \quad \text{on } \partial A$$

Note: $Z$ different for each segment

with $\Omega = \omega - ku_0$
Boundary value problem

Pridmore-Brown equation (circular cross-section)

\[ P'' + \left( \frac{1}{r} + \frac{T'_0}{T_0} + 2 \frac{k u'_0}{\Omega} \right) P' + \left( \frac{\Omega^2}{c_0^2} - k^2 - \frac{m^2}{r^2} \right) P = 0 \]

Boundary conditions

\[ i \omega Z P' + \rho_0 \Omega^2 P = 0 \text{ at } r = d \quad P \text{ is regular at } r = 0 \]

Boundary Value Problem (with free \(k\)) = Eigenvalue Problem

Multiple solutions: modes of the form \( P(r) e^{i k x - i \omega t + i m \theta} \)

- eigenfunction \( P_{m \mu}(r) \)
- ‘eigenvalue’ (axial wavenumber): \( k_{m \mu} \)
Boundary value problem

Pridmore-Brown equation (circular cross-section)

\[ P'' + \left( \frac{1}{r} + \frac{T'_0}{T_0} + 2\frac{k u'_0}{\Omega} \right) P' + \left( \frac{\Omega^2}{c_0^2} - k^2 - \frac{m^2}{r^2} \right) P = 0 \]

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Multiple solutions: modes of the form \( P(r) e^{ikx-i\omega t+im\theta} \)

- eigenfunction \( P_{m\mu}(r) \)
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Non-uniform parallel flow: modes are found numerically
Some examples of modes

Axial wavenumbers $k$ (‘eig.vals’)

Right-running eig.funcs $P_\mu (r)$

Pressure field for mode $\mu = 1$
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Total field in segment $l$: sum of left- and right-running waves

$$p_l(x, r) = \sum_{\mu=1}^{\infty} \left( a_{l,\mu}^+ P_{l,\mu}^+(r) e^{ik_{l,\mu}^+ (x-x_{l-1})} + a_{l,\mu}^- P_{l,\mu}^-(r) e^{ik_{l,\mu}^- (x-x_l)} \right)$$
‘Classical’ mode-matching

At the interface at \( x = x_l \):

\[ p_l(r) = \sum_{\mu=1}^{\mu_{\text{max}}} \left( b_{l,\mu}^+ P_{l,\mu}^+(r) + a_{l,\mu}^- P_{l,\mu}^-(r) \right). \]
Continuity of pressure at $x = x_l$

$$p_l(x_l, r) = p_{l+1}(x_l, r)$$
‘Classical’ mode-matching

Continuity of pressure at $x = x_l$

$$
\sum_{\mu=1}^{\mu_{\text{max}}} \left( b_{l,\mu}^+ P_{l,\mu}^+ + a_{l,\mu}^- P_{l,\mu}^- \right) = \sum_{\mu=1}^{\mu_{\text{max}}} \left( a_{l+1,\mu}^+ P_{l+1,\mu}^+ + b_{l+1,\mu}^- P_{l+1,\mu}^- \right)
$$

$2\mu_{\text{max}}$ unknowns: outgoing amplitudes $a_{l,\mu}^-$, $a_{l+1,\mu}^+$
‘Classical’ mode-matching

Projection onto hard-wall eigenfunctions \( \Psi_\nu = J_m(\alpha_\nu r) \)

\[
\sum_{\mu=1}^{\mu_{\text{max}}} \left( b_{l,\mu}^+ (P_{l,\mu}^+, \Psi_\nu) + a_{l,\mu}^- (P_{l,\mu}^-, \Psi_\nu) \right) \\
= \sum_{\mu=1}^{\mu_{\text{max}}} \left( a_{l+1,\mu}^+ (P_{l+1,\mu}^+, \Psi_\nu) + b_{l+1,\mu}^- (P_{l+1,\mu}^-, \Psi_\nu) \right)
\]

\( \nu = 1, \ldots, \nu_{\text{max}} \)
‘Classical’ mode-matching

Similarly for continuity of axial velocity.
Similarly for continuity of axial velocity. Resulting system of $2\mu^\text{max}$ equations:

$$\begin{bmatrix} A^+ & A^- \\ C^+ & C^- \end{bmatrix} \begin{bmatrix} b^+_l \\ a^-_l \end{bmatrix} = \begin{bmatrix} B^+ & B^- \\ D^+ & D^- \end{bmatrix} \begin{bmatrix} a^+_l+1 \\ b^-_{l+1} \end{bmatrix}.$$
Similarly for continuity of axial velocity.

Resulting system of $2\mu_{\text{max}}$ equations:

$$\begin{bmatrix} A^+ & A^- \\ C^+ & C^- \end{bmatrix} \begin{bmatrix} b_{l+1}^- \\ a_l^- \end{bmatrix} = \begin{bmatrix} B^+ & B^- \\ D^+ & D^- \end{bmatrix} \begin{bmatrix} a_{l+1}^+ \\ b_{l+1}^- \end{bmatrix}.$$ 

$S$-matrix formalism to compute amplitudes of all segments (numerically stable)
Matrix entries are inner products

\[ A^{\pm}_{\nu \mu} = (P^{\pm}_{l, \mu}, \Psi_{\nu}) = \int_{0}^{d} P^{\pm}_{l, \mu}(r) \Psi_{\nu}(r) r \, dr \]
Computing inner products

Matrix entries are inner products

\[ A_{\nu\mu}^{\pm} = (P_{l,\mu}^{\pm}, \Psi_{\nu}) = \int_{0}^{d} P_{l,\mu}^{\pm}(r)\Psi_{\nu}(r) r \, dr \]

Note that for non-uniform flow:

- \( P_{l,\mu}^{\pm} \) is determined numerically
- \( P_{l,\mu}^{\pm} \) and \( \Psi_{\nu} \) are oscillatory \( \sim \) Bessel functions
- All inner-products have to be determined at all interfaces by quadrature
Computing inner products

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Problem

Computing inner products numerically is expensive / less accurate
Computing inner products

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- All inner-products have to be determined at all interfaces by quadrature

**Problem**
Computing inner products numerically is expensive / less accurate

**Million euro question**
Can we find closed-form expressions for the inner-products?
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Prototype example of ‘Generalized Prid-Brown’ : Helmholtz eqn

$$\nabla^2 \psi + \beta^2 \psi = 0$$

on arbitrarily shaped cross-section $\mathcal{A}$
Prototype example of ‘Generalized Prid-Brown’ : Helmholtz eqn

\[ \nabla^2 \psi + \beta^2 \psi = 0 \]
\[ \nabla^2 \phi + \alpha^2 \phi = 0 \]

on arbitrarily shaped cross-section \( \mathcal{A} \)
Prototype example of ‘Generalized Prid-Brown’ : Helmholtz eqn

\[ \phi \left( \nabla^2 \psi + \beta^2 \psi \right) = 0 \]

\[ \psi \left( \nabla^2 \phi + \alpha^2 \phi \right) = 0 \]

on arbitrarily shaped cross-section \( A \)
Prototype example of ‘Generalized Prid-Brown’ : Helmholtz eqn

\[ \phi \left( \nabla^2 \psi + \beta^2 \psi \right) = 0 \]
\[ \psi \left( \nabla^2 \phi + \alpha^2 \phi \right) = 0 \]

on arbitrarily shaped cross-section \( \mathcal{A} \)

Subtract and integrate over \( \mathcal{A} \)

\[
(\alpha^2 - \beta^2) \iint_{\mathcal{A}} \phi \psi \, dS = \iint_{\mathcal{A}} \left( \phi \nabla^2 \psi - \psi \nabla^2 \phi \right) dS
\]
Prototype example of ‘Generalized Prid-Brown’ : Helmholtz eqn

\[ \phi \left( \nabla^2 \psi + \beta^2 \psi \right) = 0 \]

\[ \psi \left( \nabla^2 \phi + \alpha^2 \phi \right) = 0 \]

on arbitrarily shaped cross-section \( \mathcal{A} \)

Subtract and integrate over \( \mathcal{A} \)

\[ (\alpha^2 - \beta^2) \int \int_{\mathcal{A}} \phi \psi \, dS = \int \int_{\mathcal{A}} \nabla \cdot (\phi \nabla \psi - \psi \nabla \phi) \, dS \]
Prototype example of ‘Generalized Prid-Brown’ : Helmholtz eqn

\[
\phi \left( \nabla^2 \psi + \beta^2 \psi \right) = 0 \\
\psi \left( \nabla^2 \phi + \alpha^2 \phi \right) = 0
\]

on arbitrarily shaped cross-section \( \mathcal{A} \)

Subtract and integrate over \( \mathcal{A} \)

\[
(\alpha^2 - \beta^2) \int \int_{\mathcal{A}} \phi \psi \, dS = \int_{\Gamma} (\phi \nabla \psi \cdot \mathbf{n} - \psi \nabla \phi \cdot \mathbf{n}) \, d\ell
\]
Closed form integrals of 2D eigenmodes

Prototype example of ‘Generalized Prid-Brown’ : Helmholtz eqn

\[ \phi \left( \nabla^2 \psi + \beta^2 \psi \right) = 0 \]
\[ \psi \left( \nabla^2 \phi + \alpha^2 \phi \right) = 0 \]

on arbitrarily shaped cross-section \( \mathcal{A} \)

Subtract and integrate over \( \mathcal{A} \)

\[ (\alpha^2 - \beta^2) \iint_{\mathcal{A}} \phi \psi \, dS = \int_{\Gamma} (\phi \nabla \psi \cdot \mathbf{n} - \psi \nabla \phi \cdot \mathbf{n}) \, d\ell \]

2D inner-product for Helmholtz eigenfunctions

\[ \langle \phi, \psi \rangle = \frac{1}{\alpha^2 - \beta^2} \int_{\Gamma} (\phi \nabla \psi \cdot \mathbf{n} - \psi \nabla \phi \cdot \mathbf{n}) \, d\ell \]

For arbitrary boundary conditions on \( \phi \) and \( \psi \)
Prototype example of ‘Generalized Prid-Brown’ : Helmholtz eqn

\[ \phi \left( \nabla^2 \psi + \beta^2 \psi \right) = 0 \]
\[ \psi \left( \nabla^2 \phi + \alpha^2 \phi \right) = 0 \]

on arbitrarily shaped cross-section \( \mathcal{A} \)

Subtract and integrate over \( \mathcal{A} \)

\[ (\alpha^2 - \beta^2) \int\int_{\mathcal{A}} \phi \psi \, dS = \int_{\Gamma} (\phi \nabla \psi \cdot \mathbf{n} - \psi \nabla \phi \cdot \mathbf{n}) \, d\ell \]

2D inner-product for Helmholtz eigenfunctions

\[ \langle \phi, \psi \rangle = \frac{1}{\alpha^2 - \beta^2} \int_{\Gamma} (\phi \nabla \psi \cdot \mathbf{n} - \psi \nabla \phi \cdot \mathbf{n}) \, d\ell \]

For arbitrary boundary conditions on \( \phi \) and \( \psi \)

What if \( \alpha = \beta \) and \( \phi = \psi \)?
Replace first equation with inhomogeneous version

\[ \nabla^2 \chi + \alpha^2 \chi = f \]
Replace first equation with inhomogeneous version

\[ \nabla^2 \chi + \alpha^2 \chi = f \]
\[ \nabla^2 \phi + \alpha^2 \phi = 0 \]
Replace first equation with inhomogeneous version

\[
\phi \left( \nabla^2 \chi + \alpha^2 \chi \right) = f \phi \\
\chi \left( \nabla^2 \phi + \alpha^2 \phi \right) = 0
\]
Replace first equation with inhomogeneous version

\[ \phi \left( \nabla^2 \chi + \alpha^2 \chi \right) = f \phi \]

\[ \chi \left( \nabla^2 \phi + \alpha^2 \phi \right) = 0 \]

Subtract and integrate over \( \mathcal{A} \)

\[ \int \int_{\mathcal{A}} f \phi \, dS = \int \int_{\mathcal{A}} \left( \phi \nabla^2 \chi - \chi \nabla^2 \phi \right) \, dS \]
Replace first equation with inhomogeneous version
\[
\phi \left( \nabla^2 \chi + \alpha^2 \chi \right) = f \phi \\
\chi \left( \nabla^2 \phi + \alpha^2 \phi \right) = 0
\]

Subtract and integrate over \( \mathcal{A} \)
\[
\iint_{\mathcal{A}} f \phi \, dS = \iint_{\mathcal{A}} \nabla \cdot (\phi \nabla \chi - \chi \nabla \phi) \, dS
\]
Replace first equation with inhomogeneous version

\[
\phi \left( \nabla^2 \chi + \alpha^2 \chi \right) = f \phi \\
\chi \left( \nabla^2 \phi + \alpha^2 \phi \right) = 0
\]

Subtract and integrate over \( A \)

\[
\iint_A f \phi \, dS = \int_\Gamma \left( \phi \nabla \chi \cdot \hat{n} - \chi \nabla \phi \cdot \hat{n} \right) \, d\ell
\]
Replace first equation with inhomogeneous version

\[ \phi \left( \nabla^2 \chi + \alpha^2 \chi \right) = f \phi \]
\[ \chi \left( \nabla^2 \phi + \alpha^2 \phi \right) = 0 \]

Subtract and integrate over \( A \)

\[ \int \int_A f \phi dS = \int_\Gamma (\phi \nabla \chi \cdot n - \chi \nabla \phi \cdot n) \, dl \]

2D inner-product for Helmholtz eigenfunctions

\[ \langle \phi, \phi \rangle = \int_\Gamma (\phi \nabla \chi \cdot n - \chi \nabla \phi \cdot n) \, dl. \]
Closed form integrals of 2D eigenmodes

Replace first equation with inhomogeneous version

\[
\phi \left( \nabla^2 \chi + \alpha^2 \chi \right) = f \phi \\
\chi \left( \nabla^2 \phi + \alpha^2 \phi \right) = 0
\]

Subtract and integrate over \( \mathcal{A} \)

\[
\iint_{\mathcal{A}} f \phi \, dS = \int_{\Gamma} \left( \phi \nabla \chi \cdot \mathbf{n} - \chi \nabla \phi \cdot \mathbf{n} \right) \, d\ell
\]

2D inner-product for Helmholtz eigenfunctions

\[
\langle \phi, \phi \rangle = \int_{\Gamma} \left( \phi \nabla \chi \cdot \mathbf{n} - \chi \nabla \phi \cdot \mathbf{n} \right) \, d\ell.
\]

For almost arbitrary boundary conditions on \( \phi \) and \( \chi \)
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5. Conclusions
Define vector of shape functions $F = [P, U, V, W]$

$P$ solution of Generalized PB equation, $U, V, W$ follow from $P$
Closed form integrals for parallel flow modes

- Define vector of shape functions \( \mathbf{F} = [P, U, V, W] \)
- \( P \) solution of Generalized PB equation, \( U, V, W \) follow from \( P \)

Similarly to 2D Helmholtz ex., it can be found:

Closed form integral of parallel flow modes

\[
\langle \langle \mathbf{F}, \tilde{\mathbf{F}} \rangle \rangle = \int \int_A \tilde{\Omega} \left[ \left( \frac{u_0}{\rho_0 c_0^2} + \frac{\tilde{k}}{\rho_0 \tilde{\Omega}} \right) \tilde{P}P + \frac{\omega}{\tilde{\Omega}} \tilde{P}U - \rho_0 u_0 (\tilde{V}V + \tilde{W}W) \right] dS
\]

\[
= \frac{i}{k - \tilde{k}} \int_{\Gamma} \frac{\tilde{P}(V n_y + W n_z) - (\tilde{V} n_y + \tilde{W} n_z)P}{\tilde{\Omega}} d\ell,
\]
Closed form integrals for parallel flow modes

- Define vector of shape functions $\mathbf{F} = [P, U, V, W]$
- $P$ solution of Generalized PB equation, $U, V, W$ follow from $P$

Similarly to 2D Helmholtz ex., it can be found:

Closed form integral of parallel flow modes

$$\langle \mathbf{F}, \mathbf{\bar{F}} \rangle =$$

$$\iint_{A} \frac{1}{\Omega} \left[ \left( \frac{u_0}{\rho_0 c_0^2} + \frac{k}{\rho_0 \tilde{\Omega}} \right) \tilde{P}P + \frac{\omega}{\tilde{\Omega}} \tilde{P}U - \rho_0 u_0 (\tilde{V}V + \tilde{W}W) \right] dS$$

$$= \frac{i}{k - \tilde{k}} \int_{\Gamma} \frac{\tilde{P}(Vn_y + Wn_z) - (\tilde{V}n_y + \tilde{W}n_z)P}{\tilde{\Omega}} d\ell,$$

- Weighted products of parallel flow eigenfunctions
Closed form integrals for parallel flow modes

- Define vector of shape functions \( \mathbf{F} = [P, U, V, W] \)
- \( P \) solution of Generalized PB equation, \( U, V, W \) follow from \( P \)

Similarly to 2D Helmholtz ex., it can be found:

Closed form integral of parallel flow modes

\[
\left\langle \mathbf{F}, \mathbf{\tilde{F}} \right\rangle = \int_{\tilde{\Omega}} \int_{\mathcal{A}} \frac{1}{\tilde{k}} \left[ \left( \frac{u_0}{\rho_0 c_0^2} + \frac{\tilde{k}}{\rho_0 \tilde{\Omega}} \right) \tilde{P} P + \frac{\omega}{\tilde{\Omega}} \tilde{P} U - \rho_0 u_0 (\tilde{V} V + \tilde{W} W) \right] dS
\]

\[
= \frac{i}{k - \tilde{k}} \int_{\Gamma} \frac{\tilde{P} (V n_y + W n_z) - (\tilde{V} n_y + \tilde{W} n_z) P}{\tilde{\Omega}} d\ell,
\]

- Weighted products of parallel flow eigenfunctions
- \( \left\langle \mathbf{F}, \mathbf{F} \right\rangle \): soln of inhomogeneous ‘generalized PB’ eqn required
Closed form integrals for parallel flow modes

- Define vector of shape functions $F = [P, U, V, W]$
- $P$ solution of Generalized PB equation, $U, V, W$ follow from $P$

Similarly to 2D Helmholtz ex., it can be found:

Closed form integral of parallel flow modes

$$
\langle\langle F, \tilde{F} \rangle\rangle = 
\int \int_A \frac{1}{\tilde{\Omega}} \left[ \left( \frac{u_0}{\rho_0 c_0^2} + \frac{\tilde{k}}{\rho_0 \tilde{\Omega}} \right) \tilde{P} P + \frac{\omega}{\tilde{\Omega}} \tilde{P} U - \rho_0 u_0 (\tilde{V} V + \tilde{W} W) \right] dS
$$

$$
= \frac{i}{k - \tilde{k}} \int_{\Gamma} \frac{\tilde{P} (V n_y + W n_z) - (\tilde{V} n_y + \tilde{W} n_z) P}{\tilde{\Omega}} d\ell,
$$

- Weighted products of parallel flow eigenfunctions
- $\langle\langle F, F \rangle\rangle$: soln of inhomogeneous ‘generalized PB’ eqn required
- $\langle\langle F, F \rangle\rangle$ not positive definite
  $\Rightarrow$ “bilinear form” (not inner-product)
1 Problem formulation

2 ‘Classical’ mode-matching (CMM)

3 New ‘BLM’ mode-matching
   - Closed-form integrals of Helmholtz modes
   - Closed-form integrals of parallel flow modes
   - Closed-form integrals of radial Pridmore-Brown modes
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4 Numerical results

5 Conclusions
Circularly symmetric: modes of the form $F(r) e^{im\theta}$
Circularly symmetric: modes of the form $F(r)e^{im\theta}$

$F(r) = [P(r), U(r), V(r), W(r)]$ where

- $P$ solution of the radial Pridmore-Brown equation
- $U, V, W$ follow from $P$
Circularly symmetric: modes of the form $F(r) e^{im\theta}$

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- $P$ solution of the radial Pridmore-Brown equation
- $U, V, W$ follow from $P$

**Exact integrals of radial Pridmore-Brown modes**

$$\langle F, \tilde{F} \rangle =$$

$$\int_0^d \frac{1}{\tilde{\Omega}} \left[ \left( \frac{u_0}{\rho_0 c_0^2} + \frac{\tilde{k}}{\rho_0 \tilde{\Omega}} \right) P\tilde{P} + \frac{\omega}{\tilde{\Omega}} U\tilde{P} - \rho_0 u_0 (V\tilde{V} + W\tilde{W}) \right] r \, dr$$

$$= \frac{id}{k - \tilde{k}} \left[ \frac{\tilde{P}V - \tilde{V}P}{\tilde{\Omega}} \right]_{r=d}$$
Closed form integrals for radial Pridmore-Brown modes

Circularly symmetric: modes of the form $\mathbf{F}(r) e^{im\theta}$

$\mathbf{F}(r) = [P(r), U(r), V(r), W(r)]$ where

- $P$ solution of the radial Pridmore-Brown equation
- $U, V, W$ follow from $P$

Exact integrals of radial Pridmore-Brown modes

$$\langle \mathbf{F}, \tilde{\mathbf{F}} \rangle = \int_0^d \frac{1}{\tilde{\Omega}} \left[ \left( \frac{u_0}{\rho_0 c_0^2} + \frac{\tilde{k}}{\rho_0 \tilde{\Omega}} \right) P\tilde{P} + \frac{\omega}{\tilde{\Omega}} U\tilde{P} - \rho_0 u_0 (V\tilde{V} + W\tilde{W}) \right] r \, dr$$

$$= \frac{id}{k - \tilde{k}} \left[ \frac{\tilde{P} V - \tilde{V} P}{\tilde{\Omega}} \right]_{r=d}$$

Weighted products of Pridmore-Brown eigenfunctions
For $k = \tilde{k}$ and $P = \tilde{P}$: soln to inhomogeneous PB eqn required

\[
\frac{\Omega^2}{rc_0^2} \left( \frac{rc_0^2}{\Omega^2} \tilde{P}' \right)' + \left( \frac{\Omega^2}{c_0^2} - k^2 - \frac{m^2}{r^2} \right) \tilde{P} = 2 \frac{\omega u_0'}{\Omega^2} P' - 2 \left( \frac{u_0 \Omega}{c_0^2} + k \right) P
\]
For $k = \tilde{k}$ and $P = \tilde{P}$: soln to inhomogeneous PB eqn required

$$\frac{\Omega^2}{rc_0^2} \left( \frac{r c_0^2}{\Omega^2} \hat{P}' \right)' + \left( \frac{\Omega^2}{c_0^2} - k^2 - \frac{m^2}{r^2} \right) \hat{P} = 2 \frac{\omega u_0'}{\Omega^2} P' - 2 \left( \frac{u_0 \Omega}{c_0^2} + k \right) P$$

No free parameters $\Rightarrow$ numerically cheaper than eigenvalue probl.
Closed form integrals for radial Pridmore-Brown modes

For \( k = \tilde{k} \) and \( P = \tilde{P} \): soln to inhomogeneous PB eqn required

\[
\frac{\Omega^2}{r c_0^2} \left( \frac{r c_0^2}{\Omega^2} \hat{P}' \right)' + \left( \frac{\Omega^2}{c_0^2} - k^2 - \frac{m^2}{r^2} \right) \hat{P} = 2 \frac{\omega u_0'}{\Omega^2} P' - 2 \left( \frac{u_0 \Omega}{c_0^2} + k \right) P
\]

No free parameters ⇒ numerically cheaper than eigenvalue probl.

Exact integrals of radial Pridmore-Brown modes (\( k = \tilde{k} \))

\[
\langle F, F \rangle = \int_0^d r \frac{1}{\tilde{\Omega}} \left[ \left( \frac{u_0}{\rho_0 c_0^2} + \frac{k}{\rho_0 \Omega} \right) P^2 + \frac{\omega}{\Omega} U P - \rho_0 u_0 (V^2 + W^2) \right] \, dr
\]

\[= i d \left[ \hat{P} V - \hat{V} P \frac{\Omega}{\hat{\Omega}} \right]_{r=d} \]
Some special cases

With Ingard-Myers condition (slipping flow)

\[ \langle \mathbf{F}, \tilde{\mathbf{F}} \rangle = \left[ \frac{\text{id}\tilde{P}\tilde{P}}{(k - \tilde{k})\tilde{\Omega}\omega} \left( \frac{\Omega}{Z} - \frac{\tilde{\Omega}}{\tilde{Z}} \right) \right]_{r=d} \]
Some special cases

With Ingard-Myers condition (slipping flow)

\[ \langle F, \tilde{F} \rangle = \left\langle \frac{\text{id} \tilde{PP}}{(k - \tilde{k})\tilde{\Omega}} \left( \frac{\Omega}{Z} - \frac{\tilde{\Omega}}{\tilde{Z}} \right) \right\rangle_{r=d} \]

For hard walls:

"orthogonal":

\[
\begin{cases}
\langle F, \tilde{F} \rangle = 0 \\
\langle F, F \rangle \neq 0
\end{cases}
\]
Some special cases

With Ingard-Myers condition (slipping flow)

\[
\langle \mathbf{F}, \mathbf{\tilde{F}} \rangle = \left[ \frac{i d \mathbf{\tilde{P}P}}{(k - \bar{k})\bar{\Omega} \omega} \left( \frac{\Omega}{Z} - \frac{\bar{\Omega}}{\bar{Z}} \right) \right]_{r=d}
\]

For hard walls:

“orthogonal”:

\[
\begin{align*}
\langle \mathbf{F}, \mathbf{\tilde{F}} \rangle &= 0 \\
\langle \mathbf{F}, \mathbf{F} \rangle &\neq 0
\end{align*}
\]

In case of no-slip flow:

\[
\langle \mathbf{F}, \mathbf{\tilde{F}} \rangle = \left[ \frac{i d \mathbf{\tilde{P}P}}{(k - \bar{k})\omega} \left( \frac{1}{Z} - \frac{1}{\bar{Z}} \right) \right]_{r=d}
\]
Some special cases

With Ingard-Myers condition (slipping flow)

\[
\langle F, \tilde{F} \rangle = \left[ \frac{\text{id}\tilde{P}P}{(k - \tilde{k})\tilde{\Omega}\omega} \left( \frac{\Omega}{Z} - \frac{\tilde{\Omega}}{\tilde{Z}} \right) \right]_{r=d}
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For hard walls:

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In case of no-slip flow:

\[
\langle F, \tilde{F} \rangle = \left[ \frac{\text{id}\tilde{P}P}{(k - \tilde{k})\omega} \left( \frac{1}{Z} - \frac{1}{\tilde{Z}} \right) \right]_{r=d}
\]

For different modes \((k \neq \tilde{k})\) with same impedance \(Z = \tilde{Z}\):

“orthogonal”:

\[
\begin{cases}
\langle F, \tilde{F} \rangle = 0 \\
\langle F, F \rangle \neq 0
\end{cases}
\]
1. Problem formulation

2. ‘Classical’ mode-matching (CMM)

3. New ‘BLM’ mode-matching
   - Closed-form integrals of Helmholtz modes
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   - Closed-form integrals of radial Pridmore-Brown modes
   - Bilinear map based mode-matching

4. Numerical results

5. Conclusions
Bilinear map-based mode-matching

Classic mode-matching (CMM)

\[
\sum_{\mu=1}^{\mu_l} b_{l,\mu}^+ (P_{l,\mu}^+, \Psi_\nu) + a_{l,\mu}^- (P_{l,\mu}^-, \Psi_\nu)
\]

\[
= \sum_{\mu=1}^{\mu_{l+1}} a_{l+1,\mu}^+ (P_{l+1,\mu}^+, \Psi_\nu) + b_{l+1,\mu}^- (P_{l+1,\mu}^-, \Psi_\nu)
\]

With test functions (for example)

\[
\Psi_\nu = J_m(\alpha_\nu r)
\]
Bilinear map-based mode-matching

Classic mode-matching (CMM)

$$\sum_{\mu=1}^{\mu_l} b_{l,\mu}^+ (P_{l,\mu}^+, \Psi_\nu) + a_{l,\mu}^- (P_{l,\mu}^-, \Psi_\nu)$$

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With test functions (for example)

$$\Psi_\nu = J_m(\alpha_\nu r)$$

Quadrature required for $$(P_\mu, \Psi_\nu)$$ terms (non-uniform flow)
Bilinear map-based (BLM) mode-matching

\[ \sum_{\mu=1}^{\mu_l} b_{l,\mu}^+ \langle F_{l,\mu}^+, \Psi_\nu \rangle + a_{l,\mu}^- \langle F_{l,\mu}^-, \Psi_\nu \rangle \]

\[ = \sum_{\mu=1}^{\mu_{l+1}} a_{l+1,\mu}^+ \langle F_{l+1,\mu}^+, \Psi_\nu \rangle + b_{l+1,\mu}^- \langle F_{l+1,\mu}^-, \Psi_\nu \rangle \]

With test functions (for example)

\[ \Psi_\nu = F_{l,\nu} \]
Bilinear map-based (BLM) mode-matching

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\sum_{\mu=1}^{\mu_l} b_{l,\mu}^+ \langle F_{l,\mu}^+, \Psi_\nu \rangle + a_{l,\mu}^- \langle F_{l,\mu}^-, \Psi_\nu \rangle
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\]

With test functions (for example)

\[\Psi_\nu = F_{l,\nu}\]

Closed form expressions for \(\langle F_{\mu}, \Psi_\nu \rangle\) terms
Bilinear map-based (BLM) mode-matching

\[
\begin{bmatrix}
A^+ & A^- \\
C^+ & C^-
\end{bmatrix}
\begin{bmatrix}
b_{l+1}^+ \\
a_{l}^-
\end{bmatrix} = \begin{bmatrix}
B^+ & B^- \\
D^+ & D^-
\end{bmatrix}
\begin{bmatrix}
a_{l+1}^+ \\
b_{l+1}^-
\end{bmatrix}.
\]
Bilinear map-based (BLM) mode-matching

\[
\begin{bmatrix}
A^+ & A^- \\
C^+ & C^- \\
\end{bmatrix}
\begin{bmatrix}
b^+_l \\
a^-_l \\
\end{bmatrix}
= 
\begin{bmatrix}
B^+ & B^- \\
D^+ & D^- \\
\end{bmatrix}
\begin{bmatrix}
a^+_{l+1} \\
b^-_{l+1} \\
\end{bmatrix}.
\]

Non-slipping flow or hard wall, and \( \Psi_{\nu} = F_{l,\nu} \):

\[
\begin{bmatrix}
A^+ & A^- \\
C^+ & C^- \\
\end{bmatrix}
\text{is diagonal}
\]
Outline

1. Problem formulation

2. ‘Classical’ mode-matching (CMM)

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4. Numerical results

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### Numerical results — test configurations

![Diagram showing test configurations](image)

<table>
<thead>
<tr>
<th>Configuration</th>
<th>I</th>
<th>III</th>
</tr>
</thead>
<tbody>
<tr>
<td>Helmholtz &amp; azi.</td>
<td>$\omega = 13.86, m = 5$</td>
<td>$\omega = 15, m = 5$</td>
</tr>
<tr>
<td>Temperature</td>
<td>$T = 1$</td>
<td>$T = 2 \log(2) \left(1 - \frac{r^2}{2}\right)$</td>
</tr>
<tr>
<td>Mean flow</td>
<td>$M = 0.5(1 - r^2)$</td>
<td>$M = 0.3 \cdot \tanh(10(1 - r))$</td>
</tr>
<tr>
<td>Soft-wall impedance</td>
<td>$Z = 1 - 1i$</td>
<td>$Z = 1 - 1i$</td>
</tr>
<tr>
<td>Incident rad. mode nr.</td>
<td>$\mu = 1$</td>
<td>$\mu = 2$</td>
</tr>
</tbody>
</table>
Numerical results — Conf I: no-slip flow, uniform temp

Real part of pressure

(a) Classical mode-matching.

(b) Bilinear map-based mode-matching.

Perfect match between BLM and CMM results
Numerical results — Conf I: no-slip flow, uniform temp

Pressure at radial locations: \( r = \{0.035, 0.075, 0.15\} \) m.

Perfect match between BLM and CMM results
Axial velocity at radial locations: \( r = \{0.035, 0.075, 0.15\} \) m.

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Perfect match between BLM and CMM results
Energy balance (Myers’ Energy Corollary) vs $\mu^{\text{max}}$ for conf. 1

Energy balance better for higher $\mu^{\text{max}}$
BLM performs better than CMM
Energy balance (Myers’ Energy Corollary) vs $\mu^{\text{max}}$ for conf. 1

Energy balance better for higher $\mu^{\text{max}}$
Energy balance (Myers’ Energy Corollary) vs $\mu^{\text{max}}$ for conf. 1

Energy balance better for higher $\mu^{\text{max}}$

BLM performs better than CMM
Assume that for modal amplitudes holds:

\[ A_n = O(n^p) \quad \text{for } n \to \infty \]

so \( \log |A_n| = p \log n + O(1) \).
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so \( \log |A_n| = p \log n + O(1) \). Then for \( p_n \) defined as

\[ p_n = \frac{\log |A_n|}{\log n} \]

it holds: \( p_n - p = O(1/ \log(n)) \)
Assume that for modal amplitudes holds:

$$A_n = O(n^p) \quad \text{for } n \to \infty$$

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Near the interface, at the wall ('edge'): boundary cond. undefined

⇒ Energy must be finite (edge condition)
Numerical results — Convergence of modal amplitudes

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it holds: \( p_n - p = O(1/\log(n)) \)

Near the interface, at the wall (‘edge’): boundary cond. undefined
⇒ Energy must be finite (edge condition)

It can be shown that:

\[ p < -1 \Rightarrow \text{uniform convergence of Fourier series} \]
⇒ edge condition satisfied
Numerical results — Convergence of modal amplitudes

Do we have $p < -1$ for numerical solutions?
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Convergence of amplitudes (BLM and CMM), for conf. I and III

$p_n \approx -2 \Rightarrow$ edge condition satisfied

BLM amplitudes smoother than CMM as $n \to \infty$: no quadrature inaccuracies for BLM
Numerical results — Convergence of modal amplitudes

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Do we have \( p < -1 \) for numerical solutions?

Convergence of amplitudes (BLM and CMM), for conf. I and III

\[
\begin{align*}
p_n &\approx -2 \Rightarrow \text{edge condition satisfied} \\
\text{BLM amplitudes smoother than CMM as } n &\to \infty: \text{no quadrature inaccuracies for BLM}
\end{align*}
\]
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4. Numerical results

5. Conclusions
Conclusions

Classic mode-matching (CMM):

- Uniform flow & temp: Mode shapes are Bessel functions. Inner products are available in closed form.
- Parallel (non-uniform) flow & temp: Mode shapes are Pridmore-Brown solutions (determined numerically). Inner products require numerical quadrature → expensive & less accurate.

Bilinear map-based mode-matching (BLM):

- Parallel (non-uniform) flow & temp: Mode shapes are Pridmore-Brown solutions (determined numerically). Closed form expressions for 'inner-products' → cheaper & more accurate. Solutions in very good agreement with CMM.
Conclusions

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  - Solutions in very good agreement with CMM
A computational mode-matching approach for propagation in three-dimensional ducts with flow.  

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L.M. Delves and J.N. Lyness.
A numerical method for locating the zeros of an analytic function.

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A modification of the delves-lyness method for locating the zeros of analytic functions.

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On the acoustic boundary condition in the presence of flow.

Mike K. Myers.
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D.C. Pridmore-Brown.  
Sound propagation in a fluid flowing through an attenuating duct.  
Sjoerd W. Rienstra.
A classification of duct modes based on surface waves.

P. Sijtsma and H. van der Wal.
Modelling a spiralling type of non-locally reacting liner.
Backup slide from here...
S-matrix formalism

Take into account propagation (decaying exponentials)
consider outgoing waves unknown

⇒ Segment scattering matrix:

\[
\begin{bmatrix}
  a^+_l \\
  a^-_l
\end{bmatrix}
= \begin{bmatrix}
  S^1_{11} & S^1_{12} \\
  S^2_{21} & S^2_{22}
\end{bmatrix}
\begin{bmatrix}
  a^+_l \\
  a^-_l
\end{bmatrix}
\]

Segment S-matrices can be combined to compute all amplitudes
**S-matrix formalism**

![Diagram showing the propagation of waves through segments](image)

- Take into account propagation (decaying exponentials)
- Consider outgoing waves unknown

⇒ Segment scattering matrix:

\[
\begin{bmatrix}
   a_{l+1}^+ \\
   a_l^-
\end{bmatrix}
= \begin{bmatrix}
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   S_{21}^l & S_{22}^l
\end{bmatrix}
\begin{bmatrix}
   a_l^+ \\
   a_{l+1}^-
\end{bmatrix}
\]

Segment S-matrices can be combined to compute all amplitudes

**S-matrix formalism**: numerically stable due to decaying exponentials
Solution of
\[ Lx = f \quad \text{with} \quad L = A - \lambda I \]
exists if \( \lambda \) satisfies
\[ \det(L) = \det(A - \lambda I) \neq 0 \]
Solution of

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exists if \( \lambda \) satisfies

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Similarly: solution of

\[ \nabla^2 \chi + \alpha^2 \chi = \phi, \quad a\chi = b\nabla \chi \cdot n \text{ on } \Gamma \]

exists if \( \alpha \) is not an eigenvalue of homogeneous problem
Arbitrary boundary conditions?

Solution of

\[ Lx = f \quad \text{with} \quad L = A - \lambda I \]

exists if \( \lambda \) satisfies

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Similarly: solution of

\[ \nabla^2 \chi + \alpha^2 \chi = \phi, \quad a\chi = b\nabla \chi \cdot n \quad \text{on} \quad \Gamma \]

exists if \( \alpha \) is not an eigenvalue of homogeneous problem

\[ \Rightarrow \quad \text{Boundary conditions on} \ \chi \ \text{and} \ \phi \ \text{must be different} \]
Myers’ Energy Corollary (exact for non-uniform flow & temp):

\[ \frac{\partial E}{\partial t} + \nabla \cdot I = -D \]

with

\[ E = \frac{p_1^2}{2\rho_0 c_0^2} + \frac{1}{2} \rho_0 |\mathbf{v}_1|^2 + \rho_1 \mathbf{v}_0 \cdot \mathbf{v}_1 + \frac{\rho_0 T_0 s_1^2}{2C_p}, \]

\[ I = (\rho_0 \mathbf{v}_1 + \rho_1 \mathbf{v}_0) \left( \frac{p_1}{\rho_0} + \mathbf{v}_0 \cdot \mathbf{v}_1 \right) + \rho_0 \mathbf{v}_0 T_1 s_1, \]

\[ D = -\rho_0 \mathbf{v}_0 \cdot (\mathbf{\omega}_1 \times \mathbf{v}_1) - \rho_1 \mathbf{v}_1 \cdot (\mathbf{\omega}_0 \times \mathbf{v}_0) \]
\[ + s_1 (\rho_0 \mathbf{v}_1 + \rho_1 \mathbf{v}_0) \cdot \nabla T_0 - s_1 \rho_0 \mathbf{v}_0 \cdot \nabla T_1 \]

Flux through walls equals source/sink contributions

\[ \int_{\partial V} \mathbf{I} \cdot \mathbf{n} \, dA + \int_V \mathbf{D} \, dV = 0 \]